Tohoku Math. J.
53 (2001), 337–367

# PRINCIPAL BUNDLES OVER PROJECTIVE MANIFOLDS WITH PARABOLIC STRUCTURE OVER A DIVISOR

Dedicated to the memory of Krishnamurty Guruprasad

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(Received May 25, 1999, revised January 15, 2001)

**Abstract.** Principal $G$-bundles with parabolic structure over a normal crossing divisor are defined along the line of the interpretation of the usual principal $G$-bundles as functors from the category of representations, of the structure group $G$, into the category of vector bundles, satisfying certain axioms. Various results on principal bundles are extended to the more general context of principal bundles with parabolic structures, and also to parabolic $G$-bundles with Higgs structure. A simple construction of the moduli space of parabolic semistable $G$-bundles over a curve is given, where $G$ is a semisimple linear algebraic group over $\boldsymbol{C}$.

**1. Introduction.** The isomorphism classes of principal $G$-bundles over a scheme $X$, where $G$ is an affine algebraic group, are in bijective correspondence with a certain class of functors from the category, $\mathrm{Rep}(G)$, of left $G$-modules, to the category of locally free coherent sheaves on $X$ [No1]. The class of functors in question share the abstract properties of the construction of an associated vector bundle to a principal $G$-bundle for each left representation of $G$, and in particular, the above bijective correspondence maps a principal $G$-bundle $P$ to the functor $\hat{P}$ which associates to a left $G$-module $V$ the vector bundle associated to $P$ for $V$.

Nori in [No1] exploited this observation to classify the finite vector bundles over a curve $X$, that is, vector bundles $E$ with $p(E)$ isomorphic to $q(E)$, where $p, q$ are some pair of distinct polynomials with nonnegative integral coefficients, as the vector bundles whose pullback to some étale Galois cover of $X$ is trivializable. This way of looking at principal bundles has been further utilized in later works, which include [DM], [Si3].

A vector bundle $E$ with parabolic structure over a divisor $D$ is, loosely speaking, a weighted filtration of the restriction of $E$ to $D$. The notion of parabolic vector bundles was first introduced in a work of Mehta and Seshadri [MS], in the context of their investigation of the unitary representations of the fundamental group of a punctured Riemann surface. However, it has evolved into a topic of intrinsic interest in the study of vector bundles, with generalizations by Maruyama and Yokogawa [MY], to the higher dimensional varieties.

A straightforward generalization of the notion of the parabolic structure on vector bundles to $G$-bundles has some inherent difficulties stemming primarily from the fact that the weights of the flag defining a parabolic structure are required to be in the interval $[0, 1)$. There are some working definitions suitable for different purposes (cf. [LS], [BR]), but none

2000 *Mathematics Subject Classification*. 14F05, 14L30.

of them seem to coincide with the usual definition of parabolic vector bundles when $G$ is taken to be $GL(n, \boldsymbol{C})$. (It seems that the definition of a parabolic $G$-bundle given in [BR] is not even complete; in [LS] the object of study is the moduli stack and therefore, the ad hoc definition given there suffices.) In particular, these definitions do not suffice in addressing the counterpart of the following fundamental result for the case of principal bundles with more general structure groups, namely, the analogue of the Narasimhan-Seshadri correspondence. More precisely, the principal bundle analogue of *the existence of a bijective correspondence between the set of the isomorphism classes of polystable parabolic vector bundles of rank $n$ on a Riemann surface $X$ with parabolic structure over the divisor $D \subset X$ and the set of equivalence classes of homomorphisms from $\pi_1(X - D)$ to $U(n)$*, remained unanswered (this is the main result of [MS]).

We recall that given any $\rho \in \mathrm{Hom}(\pi_1(X-D), U(n))/U(n)$, the corresponding parabolic vector bundle is given by the extension due to Deligne (cf. [D1]) of the unitary flat vector bundle over $X - D$, associated to $\rho$, to a vector bundle over $X$ equipped with a logarithmic singular connection; the parabolic structure over an irreducible component of $D$ is obtained from the residue of the logarithmic singular connection along that component.

Instead of that we interchange the rôles of a principal bundle and its earlier mentioned interpretation as a functor by Nori, and simply adapt a natural reformulation of the functor to the parabolic context as the definition of a $G$-bundle with parabolic structure. This is indeed a principal bundle analog of parabolic vector bundles since parabolic $GL(n, \boldsymbol{C})$-bundles are exactly the parabolic vector bundles of rank $n$ (Proposition 2.6).

We generalize various aspects of $G$-bundles to the parabolic $G$-bundles, which include establishing a relationship between unitary flat connections and parabolic polystable $G$-bundles.

The main tool used here is a relationship between parabolic vector bundles and vector bundles equipped with an action of a finite group, which was established in [Bi2].

A consequence of the identification between the space of all parabolic principal $G$-bundles over $X$ with the space of all $\Gamma$-linearized principal $G$-bundles over a suitable Galois cover $Y$ over $X$, with Galois group $\Gamma$, is a simple construction of the moduli space of parabolic semistable $G$-bundles over a smooth projective curve $X$. More precisely, this identification reduces the problem of constructing a moduli space of parabolic $G$-bundles over $X$ to the problem of constructing a moduli space of $\Gamma$-linearized principal $G$-bundles over $Y$. We describe a construction of a moduli space of $\Gamma$-linearized principal $G$-bundles. In the absence of any parabolic structure, our method of construction gives an alternative and shorter approach than the earlier one due to Ramanathan for the construction of the moduli space of usual $G$-bundles (cf. also [BS]).

The paper is organized as follows. In Section 2 we define $G$-bundles with parabolic structure, where $G$ is an affine algebraic group, and in Proposition 2.6 it is proved that parabolic $GL(n, \boldsymbol{C})$-bundles are precisely the parabolic vector bundles of rank $n$. In Section 3 the semistability and polystability of parabolic $G$-bundles have been defined. In Section 4 parabolic $G$-bundles are related to the flat unitary connections (Theorem 4.8). In Section 5, the moduli space of parabolic semistable $G$-bundles over a curve is constructed. In the final section the case of parabolic Higgs $G$-bundles is briefly discussed.

We are very grateful to C. S. Seshadri for some useful discussions. The present work was carried out during a visit of the first and third named authors to the Tata Institute of Fundamental Research and a subsequent visit of the second named author to the Institute of Mathematical Sciences. We are thankful to these two institutes for their hospitality.

**2. The parabolic analog of principal bundles.** Let $X$ be a connected smooth projective variety over $\boldsymbol{C}$. Denote by $\mathrm{Vect}(X)$ the category of vector bundles over $X$. The category $\mathrm{Vect}(X)$ is equipped with an algebra structure defined by the tensor product operation

$$\mathrm{Vect}(X) \times \mathrm{Vect}(X) \to \mathrm{Vect}(X),$$

which sends any pair $(E, F)$ to $E \otimes F$, and the direct sum operation $\oplus$, making it an additive tensor category in the sense of [DM, Definition 1.15].

Let $G$ be an affine algebraic group over $\boldsymbol{C}$. A *principal $G$-bundle $P$* over $X$ is a smooth surjective morphism

$$\pi : P \to X \tag{2.1}$$

together with a right action of the group $G$

$$\rho : P \times G \to P$$

satisfying the following two conditions:

1. $(\pi \circ \rho)(p, g) = \pi(p)$;
2. the map $P \times G \to P \times_X P$, defined by $(p, g) \mapsto (p, \rho(p, g))$, is actually an isomorphism.

In [No1] and [No2] an alternative description of principal $G$-bundles was obtained, which will be briefly recalled. It may be noted that in [No1] and [No2] $X$ is allowed to be a much more general space. However we restrict ourselves to the situation where $X$ is a smooth variety, since the applications here will be in this generality.

Let $\mathrm{Rep}(G)$ denote the category of all finite dimensional complex left representations of the group $G$, or equivalently, left $G$-modules. By a $G$-module (or representation) we shall always mean a left $G$-module (or a left representation).

Given a principal $G$-bundle $P$ over $X$ and a left $G$-module $V$, the associated fiber bundle $P \times_G V$ has a natural structure of a vector bundle over $X$. Consider the functor

$$F(P) : \mathrm{Rep}(G) \to \mathrm{Vect}(X), \tag{2.2}$$

which sends any $V$ to the vector bundle $P \times_G V$ and sends any homomorphism between two $G$-modules to the naturally induced homomorphism between the two corresponding vector bundles. The functor $F(P)$ enjoys several natural abstract properties. For example, it is compatible with the algebra structures of $\mathrm{Rep}(G)$ and $\mathrm{Vect}(X)$ defined using direct sum and tensor product operations. Furthermore, $F(P)$ takes an exact sequence of $G$-modules to an exact sequence of vector bundles, it also takes the trivial $G$-module $\boldsymbol{C}$ to the trivial line bundle on $X$, and the dimension of $V$ also coincides with the rank of the vector bundle $F(P)(V)$.

In Proposition 2.9 of [No1] (also Proposition 2.9 of [No2]) it has been established that the collection of principal $G$-bundles over $X$ are in bijective correspondence with the collection of functors from Rep$(G)$ to Vect$(X)$ satisfying the abstract properties that the functor $F(P)$ in (2.2) enjoys. The four abstract properties are described in page 31 of [No1] where they are marked F1–F4. The bijective correspondence sends a principal bundle $P$ to the functor $F(P)$ defined in (2.2).

We shall now see that the above result of [No1] easily extends to the equivariant set-up.

Let $\Gamma \subseteq \mathrm{Aut}(Y)$ be a finite subgroup of the group of automorphisms of a connected smooth projective variety $Y/\mathbf{C}$. The natural action of $\Gamma$ on $Y$ is encoded in a morphism

$$\mu : \Gamma \times Y \to Y .$$

Denote the projection of $\Gamma \times Y$ to $Y$ by $p_2$. The projection of $\Gamma \times \Gamma \times Y$ to the $i$-th factor will be denoted by $q_i$. A *$\Gamma$-linearized vector bundle* on $Y$ is a vector bundle $V$ over $Y$ together with an isomorphism

$$\lambda : p_2^* V \to \mu^* V$$

over $\Gamma \times Y$ such that the following diagram of vector bundles over $\Gamma \times \Gamma \times Y$ is commutative:

$$\begin{tikzcd} q_3^* V \arrow[r, "(q_2, q_3)^*\lambda"] \arrow[dr, "(m \times \mathrm{Id}_Y)^*\lambda"'] & (\mu \circ (q_2, q_3))^* V \arrow[d, "(\mathrm{Id}_\Gamma \times \mu)^*\lambda"] \\ & (\mu \circ (m, \mathrm{Id}_Y))^* V \end{tikzcd}$$

where $m$ is the multiplication operation on $\Gamma$.

The above definition of $\Gamma$-linearization is equivalent to giving isomorphisms of vector bundles

$$\bar{g} : V \to (g^{-1})^* V$$

for all $g \in \Gamma$, satisfying the condition that $\overline{gh} = \bar{g} \circ \bar{h}$ for any $g, h \in \Gamma$.

A *$\Gamma$-homomorphism* between two $\Gamma$-linearized vector bundles is a homomorphism between the two underlying vector bundles which commutes with the $\Gamma$-linearizations. Clearly, the tensor product of two $\Gamma$-linearized vector bundles admits a natural $\Gamma$-linearization; so does the dual of a $\Gamma$-linearized vector bundle. Let $\mathrm{Vect}_\Gamma(Y)$ denote the additive tensor category of $\Gamma$-linearized vector bundles on $Y$ with morphisms being $\Gamma$-homomorphisms.

Imitating the above definition of $\Gamma$-linearization, a *$(\Gamma, G)$-bundle* is defined to be a principal $G$-bundle $P$ as defined in (2.1), together with a lift of the action of $\Gamma$ on $Y$ to a left action of $\Gamma$ on $P$ which commutes with the right action of $G$ on $P$. So a $(\Gamma, GL(n, \mathbf{C}))$-bundle is same as a $\Gamma$-linearized vector bundle of rank $n$.

Now, from the above mentioned result of [No1] it can be deduced that the collection of $(\Gamma, G)$-bundles on $Y$ are in a natural bijective correspondence with the collection of functors from the category Rep$(G)$ to $\mathrm{Vect}_\Gamma(Y)$ satisfying four conditions of [No1] indicated above. We begin the proof of this assertion with the observation that if $P$ is a $(\Gamma, G)$-bundle over $Y$, then for any $G$-module $V$, the vector bundle $P \times_G V$ has a natural $\Gamma$-linearization induced by the left action of $\Gamma$ on $P$. To see the inverse map, first note that if we are given a functor $F$ from $\mathrm{Rep}(G)$ to the category $\mathrm{Vect}_\Gamma(Y)$ satisfying the abstract properties, then by the result mentioned earlier of [No1], the functor $\bar{F}$, which is defined to be the composition of $F$ with the forgetful functor from the category $\mathrm{Vect}_\Gamma(Y)$ to the category $\mathrm{Vect}(Y)$ of vector bundles on $Y$, which forgets the action of $\Gamma$, corresponds to a principal $G$-bundle $P$ over $Y$. For any $g \in \Gamma$ we have a self-equivalence of the category $\mathrm{Vect}(Y)$ given by the functor which sends any vector bundle $W$ to $(g^{-1})^*W$. Now, for a $\Gamma$-linearized vector bundle $E$, the $\Gamma$-linearization gives an isomorphism between $E$ and $(g^{-1})^*E$. Thus by the result of [No1], the composition of this self-equivalence with the functor $\bar{F}$ corresponds to an automorphism of $P$ over $g$. In other words, we have a lift of the action of $\Gamma$ on $Y$ to an automorphism of the total space of $P$ which commutes with the action of $G$. Evidently, the association of any $g \in \Gamma$ to the above constructed action of $g$ on $P$ defines a $(\Gamma, G)$-bundle structure on $P$. Thus we have a bijective correspondence between the collection of all $(\Gamma, G)$-bundles on $Y$ and the collection of functors from the category $\mathrm{Rep}(G)$ to $\mathrm{Vect}_\Gamma(Y)$ satisfying the abstract conditions.

Our next goal is to define parabolic $G$-bundles along the above lines.

Let $D$ be an effective divisor on $X$. For a coherent sheaf $E$ on $X$ the image of $E \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ in $E$ will be denoted by $E(-D)$. The following definition of parabolic sheaf was introduced in [MY].

DEFINITION 2.3. Let $E$ be a torsionfree $\mathcal{O}_X$-coherent sheaf on $X$. A *quasi-parabolic* structure on $E$ over $D$ is a filtration by $\mathcal{O}_X$-coherent subsheaves

$$E = F_1(E) \supset F_2(E) \supset \cdots \supset F_l(E) \supset F_{l+1}(E) = E(-D).$$

The integer $l$ is called the *length of the filtration.* A *parabolic structure* is a quasi-parabolic structure, as above, together with a system of *weights* $\{\alpha_1, \dots, \alpha_l\}$ such that

$$0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_{l-1} < \alpha_l < 1,$$

where the weight $\alpha_i$ corresponds to the subsheaf $F_i(E)$.

We shall denote the parabolic sheaf defined above by $(E, F_*, \alpha_*)$. When there is no scope of confusion, it will be denoted by $E_*$.

For a parabolic sheaf $(E, F_*, \alpha_*)$, define the following filtration $\{E_t\}_{t \in \boldsymbol{R}}$ of coherent sheaves on $X$ parameterized by $\boldsymbol{R}$:

$$E_t := F_i(E)(-[t]D), \tag{2.4}$$

where $[t]$ is the integral part of $t$ and $\alpha_{i-1} < t - [t] \le \alpha_i$, with the convention that $\alpha_0 = \alpha_l - 1$ and $\alpha_{l+1} = 1$.

A *homomorphism* from the parabolic sheaf $(E, F_*, \alpha_*)$ to another parabolic sheaf $(E', F'_*, \alpha'_*)$ is a homomorphism from $E$ to $E'$ which sends any subsheaf $E_t$ into $E'_t$, where $t \in [0, 1]$ and the filtrations are as above.

If the underlying sheaf $E$ is locally free, then $E_*$ will be called a parabolic vector bundle. *Henceforth, all parabolic sheaves will be assumed to be parabolic vector bundles.*

The class of parabolic vector bundles that are dealt with in the present work satisfy certain conditions which will be explained now. The first condition is that all parabolic divisors are assumed to be *divisors with normal crossings*. In other words, any parabolic divisor is assumed to be reduced, its each irreducible component is smooth, and furthermore the irreducible components intersect transversally. The second condition is that all the parabolic weights are *rational numbers*. Before stating the third condition, we remark that quasi-parabolic filtrations on a vector bundle can be defined by giving filtrations by subsheaves of the restriction of the vector bundle to each component of the parabolic divisor. The third and final condition states that on each component of the parabolic divisor the filtration is given by *subbundles*. The precise formulation of the last condition is given in ([Bi2], Assumptions 3.2(1)). *Henceforth, all parabolic vector bundles will be assumed to satisfy the above three conditions.*

Let $E_*$ and $E'_*$ be two parabolic vector bundles on $X$ with the same parabolic divisor $D$. Let

$$\tau : X - D \to X$$

be the natural inclusion. Define $\mathcal{E} := \tau_*\tau^*(E \otimes E')$, which is a quasi-coherent sheaf on $X$. For any $t \in \boldsymbol{R}$ the subsheaf of $\mathcal{E}$ generated by all $E_a \otimes E'_b$, where $a + b \geq t$, will be denoted by $\mathcal{E}_t$. The filtration $\{\mathcal{E}_t\}_{t\in\boldsymbol{R}}$ defines a parabolic structure on the coherent sheaf $\mathcal{E}_0$, which is easily seen to be locally free.

In [Yo], Yokogawa defined the *parabolic tensor product* $E_* \otimes E'_*$ to be the parabolic vector bundle $\mathcal{E}_*$ constructed above.

Let PVect$(X, D)$ denote the category whose objects are parabolic vector bundles over $X$ with parabolic structure over the divisor $D$ satisfying the above three conditions, and the morphisms of the category are homomorphisms of parabolic vector bundles (which was defined earlier). For any two $E_*, V_* \in$ PVect$(X, D)$, their parabolic tensor product $E_* \otimes V_*$ is also an element of PVect$(X, D)$. The trivial line bundle with the trivial parabolic structure (this means that the length of the parabolic flag is zero) acts as the identity element for the parabolic tensor multiplication. *The parabolic tensor product operation on* PVect$(X, D)$ *has all the abstract properties enjoyed by the usual tensor product operation of vector bundles.*

The direct sum of two vector bundles with parabolic structures has an obvious parabolic structure. Evidently, PVect$(X, D)$ is closed under the operation of taking direct sum. The category PVect$(X, D)$ is an additive tensor category with the direct sum and the parabolic tensor product operation. It is straightforward to check that PVect$(X, D)$ is also closed under the operation of taking the parabolic dual defined in [Yo].

For an integer $N \geq 2$, let PVect$(X, D, N) \subseteq$ PVect$(X, D)$ denote the subcategory consisting of all parabolic vector bundles all of whose parabolic weights are multiples of $1/N$. It is straightforward to check that PVect$(X, D, N)$ is closed under all the above operations, namely parabolic tensor product, direct sum and taking the parabolic dual.

As before, let $G$ be an affine algebraic group over $\boldsymbol{C}$. Let $D$ be a normal crossing divisor on $X$. The content of the following definition is clearly an imitation of the Proposition 2.9 of [No1].

DEFINITION 2.5. A *parabolic principal $G$-bundle* with parabolic structure over $D$ is a *functor* $F$ from the category $\mathrm{Rep}(G)$ to the category $\mathrm{PVect}(X, D)$ satisfying the four conditions of [No1] mentioned earlier. The functor is further required to satisfy the condition that there is an integer $N$, which depends on the functor, such that the image of the functor is contained in $\mathrm{PVect}(X, D, N)$.

A justification of the above definition will be provided by showing that the class of parabolic principal $GL(n, \boldsymbol{C})$-bundles with parabolic structure over $D$ is naturally isomorphic to the subclass of $\mathrm{PVect}(X, D)$ consisting of all parabolic vector bundles such that the rank of the underlying vector bundle is $n$. This is carried out in the following proposition.

PROPOSITION 2.6. *Let $D$ be a normal crossing divisor on $X$. The collection of parabolic $GL(n, \boldsymbol{C})$-bundles on $X$ with parabolic structure over $D$ is identified, in a bijective fashion, with the subclass of* $\mathrm{PVect}(X, D)$ *consisting of parabolic vector bundles of rank $n$. Under this identification, a parabolic $GL(n, \boldsymbol{C})$-bundle is identified with the parabolic vector bundle associated to it for the standard representation of $GL(n, \boldsymbol{C})$ on $\boldsymbol{C}^n$.*

PROOF. Let $D = \sum_{i=1}^{c} D_i$ be the decomposition of the divisor $D$ into its irreducible components.

Take any $E_* \in \mathrm{PVect}(X, D)$ such that all the parabolic weights of $E_*$ are multiples of $1/N$; that is, $E_* \in \mathrm{PVect}(X, D, N)$.

The "Covering Lemma" of Kawamata (Theorem 1.1.1 of [KMM], Theorem 17 of [K]) says that there is a connected smooth projective variety $Y$ over $\boldsymbol{C}$ and a Galois covering morphism

$$p : Y \to X \tag{2.7}$$

such that the reduced divisor $\tilde{D} := (p^* D)_{\mathrm{red}}$ is a normal crossing divisor on $Y$ and furthermore, $p^* D_i = k_i N.(p^* D_i)_{\mathrm{red}}$, where $k_i$, $1 \leq i \leq c$, are positive integers. Let $\Gamma$ denote the Galois group for the covering map $p$.

As before, $\mathrm{Vect}_\Gamma(Y)$ denotes the category of all $\Gamma$-linearized vector bundles on $Y$. The isotropy group of any point $y \in Y$, for the action of $\Gamma$ on $Y$, will be denoted by $\Gamma_y$.

Let $\mathrm{Vect}_\Gamma^D(Y, N)$ denote the subcategory of $\mathrm{Vect}_\Gamma(Y)$ consisting of all $\Gamma$-linearized vector bundles $W$ over $Y$ satisfying the following two conditions:

1. for a general point $y$ of an irreducible component of $(p^* D_i)_{\mathrm{red}}$, the action of the isotropy group $\Gamma_y$ on the fiber $W_y$ is of order a divisor of $N$, which is equivalent to the condition that for any $g \in \Gamma_y$, the action of $g^N$ on $W_y$ is the trivial action;
2. for a general point $y$ of an irreducible component of a ramification divisor for $p$ not contained in $(p^* D)_{\mathrm{red}}$, the action of $\Gamma_y$ on $W_y$ is the trivial action.

We note that $\mathrm{Vect}_\Gamma^D(Y, N)$ is also an additive tensor category.

In [Bi2] an identification between the objects of PVect$(X, D, N)$ and the objects of $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$ has been constructed. Given a $\Gamma$-homomorphism between two $\Gamma$-linearized vector bundles, there is a naturally associated homomorphisms between the corresponding vector bundles, and this identifies, in a bijective fashion, the space of all $\Gamma$-homomorphisms between two objects of $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$ and the space of all homomorphisms between the corresponding objects of PVect$(X, D, N)$. An equivalence between the two additive tensor categories, namely PVect$(X, D, N)$ and $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$, is obtained in this way. Since the description of this identification is already given in [Bi1], [Bi2], [Bi3] and [BN], it will not be repeated here.

We observe that an earlier assertion that the parabolic tensor product operation enjoys all the abstract properties of the usual tensor product operation of vector bundles, is a consequence of the fact that the above equivalence of categories indeed preserves the tensor product operation.

The above equivalence of categories has the further property that it takes the parabolic dual of a parabolic vector bundle to the usual dual of the corresponding $\Gamma$-linearized vector bundle.

Let $W \in \mathrm{Vect}_{\Gamma}^{D}(Y, N)$ be the $\Gamma$-linearized vector bundle of rank $n$ on $Y$ that corresponds to the given parabolic vector bundle $E_*$. The fiber bundle

$$\pi : P \to Y\,,$$

whose fiber $\pi^{-1}(y)$ is the space of all $\boldsymbol{C}$-linear isomorphisms from $\boldsymbol{C}^n$ to the fiber $W_y$, has a the structure of a $(\Gamma, GL(n, \boldsymbol{C}))$-bundle over $Y$.

We have established earlier the existence of a natural one-to-one identification between the collection of all $(\Gamma, GL(n, \boldsymbol{C}))$-bundles on $Y$ and the collection of all functors from the category $\mathrm{Rep}(GL(n, \boldsymbol{C}))$ to $\mathrm{Vect}_{\Gamma}(Y)$ satisfying certain properties. Let $F(P)$ denote the functor corresponding to the $(\Gamma, G)$-bundle $P$.

Now, given any $V \in \mathrm{Rep}(GL(n, \boldsymbol{C}))$, consider the parabolic vector bundle on $X$ that corresponds to the $\Gamma$-linearized vector bundle $F(P)(V)$ over $Y$ which is also an object in $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$. Let

$$F(E_*) : \mathrm{Rep}(GL(n, \boldsymbol{C})) \to \mathrm{PVect}(X, D) \tag{2.8}$$

denote the functor obtained in this way. It is straightforward to check that $F(E_*)$ does not depend on the choice of the covering $p$ in (2.7). Indeed, for another such covering $p'$, choose a covering $\bar{p}$ as in (2.7) such that $\bar{p}$ factors through both $p$ and $p'$. It is straightforward to check that the map in (2.8) for the covering $\bar{p}$ coincides with that for both $p$ and $p'$.

Since all the parabolic weights of $E_*$ are multiples of $1/N$, we conclude that all the parabolic weights of the parabolic dual $E_*^*$ and also those of any $k$-fold parabolic tensor powers $\bigotimes^k E_*$ are all multiples of $1/N$. Consequently, all the parabolic weights of any subbundle of the underlying vector bundle for $(\bigotimes^k E_*) \otimes (\bigotimes^l E_*^*)$ with the induced parabolic structure are also multiples of $1/N$. Any irreducible $GL(n, \boldsymbol{C})$-module is isomorphic to a submodule of some $(\bigotimes^i \boldsymbol{C}^n) \otimes (\bigotimes^j (\boldsymbol{C}^n)^*)$, where $\boldsymbol{C}^n$ is the standard $GL(n, \boldsymbol{C})$-module. Thus the image of $F(E_*)$ in (2.8) is contained in PVect$(X, D, N)$.

We now note that from the property mentioned earlier of the correspondence between the two categories, namely $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$ and $\mathrm{PVect}(X, D, N)$, that it takes the usual tensor product to the parabolic tensor product, it follows immediately that the map $F(E_*)$ satisfies all the abstract properties needed in Definition 2.5 in order to define a parabolic principal $GL(n, \boldsymbol{C})$-bundle.

This completes the construction of a parabolic principal $GL(n, \boldsymbol{C})$-bundle from $E_*$.

We can now simply trace back the steps to construct a parabolic vector bundle from a parabolic principal $GL(n, \boldsymbol{C})$-bundle.

Let $P_*$ be a parabolic principal $GL(n, \boldsymbol{C})$-bundle on $X$ with parabolic structure over $D$. Assume that the image of the functor $P_*$ is contained in $\mathrm{PVect}(X, D, N)$.

Take a covering $p$ as in (2.7). To any $V \in \mathrm{Rep}(GL(n, \boldsymbol{C}))$, associate the $\Gamma$-linearized vector bundle on $Y$ that corresponds to the parabolic vector bundle $P_*(V)$ by the correspondence between parabolic vector bundles and $\Gamma$-linearized vector bundles constructed in [Bi2]. Let

$$\bar{P} : \mathrm{Rep}(GL(n, \boldsymbol{C})) \to \mathrm{Vect}_{\Gamma}(Y) \tag{2.9}$$

be the map constructed in this way. We note that the image of $\bar{P}$ is contained in $\mathrm{Vect}_{\Gamma}^{D}(Y, N)$. The functor $\bar{P}$ satisfies all the conditions needed to define a $(\Gamma, G)$-bundle. Let $P$ denote the $(\Gamma, GL(n, \boldsymbol{C}))$-bundle defined by $\bar{P}$. The vector bundle $W'$ associated to $P$ for the standard action of $GL(n, \boldsymbol{C})$ on $\boldsymbol{C}^n$ has a natural $\Gamma$-linearization. Let $E'_*$ be the parabolic vector bundle on $X$ corresponding to the $\Gamma$-linearized vector bundle $W'$.

It is easy to check that the parabolic principal $GL(n, \boldsymbol{C})$-bundle corresponding to $E'_*$ by the earlier construction actually coincides with $P_*$. Conversely, the composition of the two constructions is also the identity map on $\mathrm{PVect}(X, D)$.

The $\Gamma$-linearized vector bundle $W'$ is simply the image of the standard representation of $GL(n, \boldsymbol{C})$ by the functor $\bar{P}$ constructed in (2.9). This implies that the parabolic vector bundle $E'_*$ is indeed the image of the standard representation of $GL(n, \boldsymbol{C})$ on $\boldsymbol{C}^n$ by the functor $P_*$. This completes the proof of the proposition. □

Let $P_*$ be a parabolic $G$-bundle with parabolic structure over $D$. For a homomorphism $G \to H$, the corresponding map $\mathrm{Rep}(H) \to \mathrm{Rep}(G)$ composes with the functor $P_*$ to give a functor from $\mathrm{Rep}(H)$ to $\mathrm{PVect}(X, D)$. This composition of maps defines a parabolic $H$-bundle $P_*^H$, with a parabolic structure over $D$. This construction coincides with the extension of the structure group of a principal $G$-bundle to $H$.

In the next section we shall define the notion of semistability for parabolic principal bundles.

**3. Semistability for parabolic principal bundles.** Fix an ample line bundle $L$ over $X$, which is a connected smooth projective variety over $\boldsymbol{C}$ of dimension $d$. For a coherent sheaf $F$ over $X$, the degree $\deg(F)$ is defined as follows:

$$\deg(F) := \int_X c_1(F) \cup c_1(L)^{d-1}.$$

Note that if two sheaves $F_1$ and $F_2$ are isomorphic outside a subvariety of codimension two, then $\deg(F_1) = \deg(F_2)$. Hence the degree of a coherent sheaf defined over the complement of a subvariety $X$, of codimension two or more, is well-defined.

Let $P$ be a principal $G$-bundle over $X$. A reduction of the structure group of $P$ to a subgroup $Q \subset G$ is defined by giving a section of the fiber bundle $P/Q \to X$ with fiber $G/Q$.

DEFINITION 3.1 [RS]). Let $P(Q)$ denote a reduction of the structure group of $P$ to a maximal parabolic subgroup $Q \subset G$ over an open set $U \subseteq X$ with $\operatorname{codim}(X - U) \geq 2$. The principal $G$-bundle $P$ is called *semistable* (resp. *stable*) if for every such situation, the line bundle over $U$ associated to $P(Q)$ for any character of $Q$ dominant with respect to a Borel subgroup contained in $Q$, is of nonpositive degree (resp. strictly negative degree). The principal bundle $P$ is called *polystable* if there is a reduction of the structure group of $P$ to $M$, namely $P(M) \subset P$ where $M \subset G$ is a maximal reductive subgroup of a parabolic subgroup of $G$, such that $P(M)$ is a stable principal $M$-bundle and furthermore, for any character of $M$ trivial on the intersection with the center of $G$, the corresponding line bundle associated to $P(M)$ is of degree zero.

*Unless explicitly stated otherwise, henceforth all groups considered will be assumed to be semisimple and affine algebraic over* $\mathbf{C}$.

The following proposition will be needed for extending the above definition to the parabolic context. The definition of parabolic semistable and parabolic polystable vector bundles is given in [MY] and [MS].

PROPOSITION 3.2. *Let $E_*$, $F_* \in \mathrm{PVect}(X, D)$ be two parabolic semistable (resp. parabolic polystable) vector bundles on $X$. Then the parabolic tensor product $E_* \otimes F_*$ is also parabolic semistable (resp. parabolic polystable), and furthermore the parabolic dual of $E_*$ is also parabolic semistable (resp. parabolic polystable).*

PROOF. Choose $N \in \mathbf{N}$ such that both $E_*$ and $F_*$ have all their parabolic weights as multiples of $1/N$. Fix a Galois covering $p$ with Galois group $\Gamma$ as in (2.7).

Since the covering map $p$ is a finite morphism, the line bundle $\tilde{L} := p^*L$ is ample on $Y$. The degree of a coherent sheaf on $Y$ is defined using $\tilde{L}$.

A $\Gamma$-linearized vector bundle $V'$ over $Y$ is called *$\Gamma$-semistable* (resp. *$\Gamma$-stable*) if for any proper nonzero coherent subsheaf $F' \subset V'$, invariant under the action of $\Gamma$ and with $V'/F'$ being torsionfree, the following inequality is valid:

$$\frac{\deg(F')}{\operatorname{rank}(F')} \leq \frac{\deg(V')}{\operatorname{rank}(V')} \quad \left(\text{respectively, } \frac{\deg(F')}{\operatorname{rank}(F')} < \frac{\deg(V')}{\operatorname{rank}(V')}\right).$$

The $\Gamma$-linearized vector bundle $V'$ is called *$\Gamma$-polystable* if it is a direct sum of $\Gamma$-stable vector bundles of same slope (:= degree/rank).

Now, $\Gamma$-invariant subsheaves of $V'$ are in bijective correspondence with the subsheaves of the parabolic vector bundle corresponding to $V'$, and furthermore, the degree of a $\Gamma$-invariant subsheaf is simply $\#\Gamma$-times the parabolic degree of the corresponding subsheaf with the induced parabolic structure ([Bi2], (3.12)). ($\#\Gamma$ is the order of the group $\Gamma$.)

Let $V$ and $W$ be the $\Gamma$-linearized vector bundles corresponding to the parabolic semistable (resp. parabolic polystable) vector bundles $E_*$ and $F_*$, respectively. From the above remarks it follows that in order to complete the proof of the proposition it suffices to show that $V \otimes W$ is $\Gamma$-semistable (resp. $\Gamma$-polystable) and $V^*$ is $\Gamma$-semistable (resp. $\Gamma$-polystable).

It is immediate that if $V^*$ is $\Gamma$-semistable (resp. $\Gamma$-polystable) if and only if $V$ is so.

The Lemma 3.13 of [Bi2] says that a $\Gamma$-linearized vector bundle $V'$ is $\Gamma$-semistable if and only if it is semistable in the usual sense. From a theorem of [MR] which says that a vector bundle is semistable if and only if its restriction to the general complete intersection curve of sufficiently high degree is semistable, it immediately follows that the tensor product of two semistable vector bundles is again semistable. Thus the $\Gamma$-linearized vector bundle $V \otimes W$ must be $\Gamma$-semistable if $V$ and $W$ are both individually $\Gamma$-semistable.

The main theorem of [Si1] (Theorem 1, page 878) implies that a $\Gamma$-linearized vector bundle $V'$ admits a Hermitian-Yang-Mills connection with respect to a $\Gamma$-invariant Kähler metric representing $c_1(\tilde{L})$ if and only if $V'$ is $\Gamma$-polystable. If $\nabla^V$ and $\nabla^W$ are Hermitian-Yang-Mills connections on $V$ and $W$ respectively, then $\nabla^V \otimes \mathrm{Id}_W + \mathrm{Id}_V \otimes \nabla^W$ is a Hermitian-Yang-Mills connections on $V \otimes W$. Thus $V \otimes W$ is $\Gamma$-polystable if $V$ and $W$ are both individually $\Gamma$-polystable. This completes the proof of the proposition. □

Given a homomorphism $G \to H$ and a principal $G$-bundle $P$, the quotient space

$$P \times_G H = \frac{P \times H}{G}$$

has a natural structure of a principal $H$-bundle. This construction of the principal bundle $H$-bundle from the principal $G$-bundle $P$ is called the *extension of the structure group* of $P$ to $H$. From [RR] (also [RS], Theorem 3) we know that if $P$ is semistable (resp. polystable) $G$-bundle, then $P \times_G H$ is a semistable (resp. polystable) $H$-bundle. Note that the above theorem of [RS] applies, since by our assumption $G$ is semisimple and hence the connected component of the center of $G$ is trivial.

The following definitions of semistability and polystability of a parabolic principal $G$-bundle are motivated by the above result of [RR].

DEFINITION 3.3. Let $P_*$ be a functor from the category $\mathrm{Rep}(G)$ to the category $\mathrm{PVect}(X, D)$ defining a parabolic $G$-bundle as in Definition 2.5. This functor $P_*$ will be called a *parabolic semistable* (resp. *parabolic polystable*) *principal $G$-bundle* if and only if the image of the functor is contained in the category of parabolic semistable (resp. parabolic polystable) vector bundles.

We observe that from Proposition 3.2 it follows that the subcategory of $\mathrm{PVect}(X, D)$ consisting of parabolic semistable (resp. parabolic polystable) vector bundles is closed under tensor product. Furthermore, to check parabolic semistability (resp. parabolic polystability) it is not necessary to check the criterion for $V_1 \otimes V_2 \in \mathrm{Rep}(G)$ if it has been checked for $V_1$ and $V_2$ individually.

PROPOSITION 3.4. *A parabolic $G$-bundle $P_*$ is parabolic semistable (resp. parabolic polystable) if and only if there is a faithful representation*

$$\rho : G \to GL(V)$$

*such that the corresponding parabolic vector bundle $P_*(\rho)$ is parabolic semistable (resp. parabolic polystable). Consequently, if for one faithful representation $\rho$ the parabolic vector bundle $P_*(\rho)$ is a parabolic semistable (resp. parabolic polystable), then for any representation $\rho'$, the parabolic vector bundle $P_*(\rho')$ is parabolic semistable (resp. parabolic polystable).*

PROOF. If $P_*$ is a parabolic semistable (resp. parabolic polystable) $G$-bundle, then for any faithful representation $\rho$, the parabolic bundle $P_*(\rho)$ is by definition parabolic semistable (resp. parabolic polystable).

To prove the converse, suppose that $P_*(\rho)$ is a parabolic semistable (resp. parabolic polystable) vector bundle, where $\rho$ is a faithful representation of $G$ on $V$. We note that since $\rho$ is faithful, any $W \in \mathrm{Rep}(G)$ is isomorphic to a $G$-submodule of a direct sum of $G$-modules of the form $(\bigotimes^k V) \otimes (\bigotimes^l V^*)$; this follows from Proposition 3.1(a) of [D2].

Choose an integer $N$ such that both $P_*(\rho)$ and $P_*(W)$ are in $\mathrm{PVect}(X, D, N)$. Fix a Galois covering $p$ as in (2.7) with Galois group $\Gamma$.

Let $V_\Gamma^\rho$ be the $\Gamma$-linearized vector bundle corresponding to $P_*(\rho)$. It has been established in the proof of Proposition 3.2 that $V_\Gamma^\rho$ is $\Gamma$-semistable (resp. $\Gamma$-polystable). This, as we saw in the proof of Proposition 3.2, implies that the tensor product $(\bigotimes^k V_\Gamma^\rho) \otimes (\bigotimes^l (V_\Gamma^\rho)^*)$ is $\Gamma$-semistable (resp. $\Gamma$-polystable).

Since $W$ is isomorphic to a direct sum of $G$-submodules of $G$-modules of the type $(\bigotimes^k V) \otimes (\bigotimes^l V^*)$, the $\Gamma$-linearized vector bundle corresponding to the parabolic vector bundle $P_*(W)$ must be $\Gamma$-semistable (resp. $\Gamma$-polystable). This completes the proof of the proposition. □

Propositions 2.6 and 3.4 together have the following corollary:

COROLLARY 3.5. *A parabolic principal $SL(n, \mathbf{C})$-bundle $P_*$ is parabolic semistable (resp. parabolic polystable) if and only if the parabolic vector bundle associated to $P_*$ for the standard representation of $SL(n, \mathbf{C})$ on $\mathbf{C}^n$ is parabolic semistable (resp. parabolic polystable).*

Proposition 3.4 suggests an alternative definition of parabolic semistability (resp. parabolic polystability) of parabolic $G$-bundles, namely a functor $P_*$ is a parabolic semistable (resp. parabolic polystable) $G$-bundle if for some faithful representation $\rho$, the parabolic vector bundle $P_*(\rho)$ is parabolic semistable (resp. parabolic polystable). This definition is close in spirit to the definition of semistability of principal Higgs bundles made by Simpson in Section 8 (page 49) of [Si3]. This definition has the advantage that it extends to reductive groups as opposed to the set-up of semisimple groups in Definition 3.3. However, it needs an extra assumption of the vanishing of the first Chern class. We shall not go further into the fine distinction between this definition and Definition 3.3 adapted in the present work.

REMARK 3.6. For a parabolic semistable vector bundle $E_*$ a certain inequality involving Chern classes of $E$ and the parabolic data was established in [Bi2], which is a generalization to the parabolic context of the Bogomolov inequality involving the first two Chern classes of a semistable vector bundle. Now, if $P_*$ is a parabolic semistable $G$-bundle then for any $V \in \mathrm{Rep}(G)$, we have the parabolic analog of the Bogomolov inequality for the parabolic vector bundle $P_*(V)$. In particular, an inequality is obtained for the adjoint representation of $G$, which in the absence of a parabolic structure says that for a semistable principal $G$-bundle $P$,

$$c_2(\mathrm{ad}(P)) \cup c_1(L)^{d-2} \geq 0,$$

where $\mathrm{ad}(P) = P \times_G \mathfrak{g}$ is the adjoint bundle associated to $P$ for the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$. The analog of Bogomolov inequality for parabolic semistable vector bundles has a simple interpretation in terms of the analog of the Chern classes for parabolic vector bundles, namely it is the usual Bogomolov inequality with the Chern classes being replaced by their parabolic analogs [Bi3]. □

In Section 2 the notion of the extension of the structure group of a parabolic principal bundle was defined. The following proposition, which is immediate from Definition 3.3, is a parabolic analog of Theorem 3 of [RS].

PROPOSITION 3.7. *If $G \to H$ is a homomorphism of groups and if $P_*$ is a parabolic semistable (resp. parabolic polystable) $G$-bundle, then the parabolic $H$-bundle, obtained by the extension of structure group of $P_*$, is also parabolic semistable (resp. parabolic polystable).*

In the next section we shall relate a space of equivalence classes of homomorphisms from the fundamental group $\pi_1(X - D)$ to a maximal compact subgroup of $G$ with the space of parabolic polystable $G$-bundles on $X$ with parabolic structure over $D$.

**4. Representations of $\pi_1(X - D)$ and parabolic principal bundles.** Let $P$ be a $(\Gamma, G)$-bundle on $Y$. It is called *semistable* (resp. *polystable*) if and only if the underlying $G$-bundle is semistable (resp. polystable) according to Definition 3.1.

On the other hand, $P$ is called *$\Gamma$-semistable* (resp. *$\Gamma$-polystable*) if and only if $P$ satisfies the condition of semistability (resp. polystability) in Definition 3.1 with all reductions of structure group being $\Gamma$-equivariant.

The following simple proposition identifies the above two definitions.

PROPOSITION 4.1. *A $(\Gamma, G)$-bundle $P$ is $\Gamma$-semistable (resp. $\Gamma$-polystable) if and only if $P$ is semistable (resp. polystable).*

PROOF. If $P$ is semistable (resp. polystable), then obviously $P$ is $\Gamma$-semistable (resp. $\Gamma$-polystable).

To prove the converse we first remark that it is known that the $G$-bundle $P$ is semistable (resp. polystable) if and only if the adjoint vector bundle $\mathrm{ad}(P)$ is semistable (resp. polystable) [ABi, Proposition 2.10], [AB].

Now, if $P$ is $\Gamma$-semistable (resp. $\Gamma$-polystable), then it is easy to deduce that $\mathrm{ad}(P)$ is $\Gamma$-semistable (resp. $\Gamma$-polystable).

Thus it suffices to show that any $\Gamma$-linearized vector bundle is semistable (resp. polystable) in the usual sense if it is $\Gamma$-semistable (resp. $\Gamma$-polystable). But this has been established in Lemma 3.13 of [Bi2]. This completes the proof of the proposition. □

For an integer $N \geq 1$, let $PG(X, D, N)$ denote the collection of all parabolic $G$-bundles $P_*$ on $X$ with parabolic structure over the divisor $D$ and satisfying the condition that for any $V \in \mathrm{Rep}(G)$, the parabolic vector bundle $P_*(V)$ has all its parabolic weights as multiples of $1/N$.

The category $\mathrm{PVect}(X, D, N)$ is closed under the operations of taking the parabolic dual and the parabolic tensor product, and furthermore given any faithful $G$-module $V$, any irreducible $G$-module is isomorphic to a submodule of $(\bigotimes^k V) \otimes (\bigotimes^l V^*)$, for some $k$ and $l$. These two facts combine together to imply that $P_* \in PG(X, D, N)$ if and only if $P_*(V) \in \mathrm{PVect}(X, D, N)$ for some faithful $G$-module $V$.

Fix a Galois covering $p : Y \to X$ as in (2.7). For any irreducible component $D_i$ of $D$, a general point

$$y \in (p^* D_i)_{\mathrm{red}}$$

of the reduced divisor $(p^* D_i)_{\mathrm{red}}$ has a cyclic subgroup $\Gamma_y \subseteq \Gamma$ of order $k_i N$ as the isotropy group. For a $(\Gamma, G)$-bundle $P$, the isotropy subgroup $\Gamma_y \subseteq \Gamma$ for any point $y \in Y$ has an action on the fiber $P_y$ of $P$ over $y$.

Let $[\Gamma, G, N]$ denote the collection of $(\Gamma, G)$-bundles on $Y$ satisfying the following two conditions:

1. for a general point $y$ of an irreducible component of $(p^* D_i)_{\mathrm{red}}$, the action of $\Gamma_y$ on $P_y$ is of order $N$; in other words, for any $g \in \Gamma_y$, the action of $g^N$ on $P_y$ is the trivial action;
2. for a general point $y$ of an irreducible component of a ramification divisor for $p$ not contained in $(p^* D)_{\mathrm{red}}$, the action of $\Gamma_y$ on $P_y$ is the trivial action.

So $[\Gamma, GL(n, C), N]$ coincides with the collection of rank $n$ vector bundles in the category $\mathrm{Vect}_\Gamma^D(Y, N)$ defined in Section 2.

For any $P_* \in PG(X, D, N)$, consider the composition

$$\mathrm{Rep}(G) \xrightarrow{P_*} \mathrm{PVect}(X, D, N) \to \mathrm{Vect}_\Gamma^D(Y, N)\,, \tag{4.2}$$

where $\mathrm{Vect}_\Gamma(Y)$ is the category of $\Gamma$-linearized vector bundles on $Y$ and the right-hand-side map in (4.2) is the identification of the parabolic vector bundles with the $\Gamma$-linearized vector bundles. It is straightforward to check that the composition of functors in (4.2) satisfies the conditions in Section 2 to define a $(\Gamma, G)$-bundle $P$ on $Y$.

THEOREM 4.3. *The $(\Gamma, G)$-bundle $P$ constructed above is in $[\Gamma, G, N]$. The map from $PG(X, D, N)$ to $[\Gamma, G, N]$, which sends any $P_*$ to $P$ constructed above, is actually* *bijective. Furthermore, $P_*$ is parabolic semistable (resp. parabolic polystable) if and only if $P$ is $\Gamma$-semistable (resp. $\Gamma$-polystable), or in view of Proposition* 4.1, *if and only if $P$ is semistable (resp. polystable).*

PROOF. Let $\rho : G \to GL(V)$ be a faithful representation of $G$. Let $P_{GL}$ be the principal $(\Gamma, GL(V))$-bundle on $Y$ obtained by extending the structure group of $P$ using $\rho$. Thus we have a $\Gamma$-equivariant embedding

$$f : P \to P_{GL} \tag{4.4}$$

of the total spaces of principal bundles.

In the correspondence between parabolic vector bundles and $\Gamma$-linearized vector bundles, the parabolic vector bundle $P_*(\rho)$ corresponds by definition to the $\Gamma$-linearized vector bundle $P(\rho)$ associated to $P_{GL}$ using the standard representation of $GL(V)$.

We are given that $P_*(\rho) \in \mathrm{PVect}(X, D, N)$. From the construction of the correspondence between parabolic vector bundles and $\Gamma$-linearized vector bundles in [Bi2] it is immediate that for any $g \in \Gamma_y$, where $y$ is a general point of a component of $(p^*D_i)_{\mathrm{red}}$, the action of $g^N$ on the fiber $P(\rho)_y$ is actually the trivial action. Furthermore, the action of $\Gamma_y$ on $P(\rho)_y$, where $y$ is a general point of a component of the ramification divisor for $p$ not contained in $(p^*D)_{\mathrm{red}}$, is the trivial action.

Since $f$ in (4.4) is $\Gamma$-equivariant embedding, we conclude that $P \in [\Gamma, G, N]$.

Let $F : PG(X, D, N) \to [\Gamma, G, N]$ denote the map which assigns to $P_*$ the $(\Gamma, G)$-bundle $P$ by the above construction.

To construct the inverse of $F$, take any $P' \in [\Gamma, G, N]$. Consider the composition

$$\mathrm{Rep}(G) \xrightarrow{[P']} \mathrm{Vect}^D_\Gamma(Y, N) \longrightarrow \mathrm{PVect}(X, D)\,, \tag{4.5}$$

where $[P']$ denotes the functor which associates to a $G$-module the $\Gamma$-linearized vector bundle obtained by the extension of the structure group of $P'$, and the right-hand side map is the correspondence between parabolic vector bundles and $\Gamma$-linearized vector bundles. It is straightforward to check that this composition satisfies all the conditions needed to define a parabolic $G$-bundle. Let $P'_*$ denote the parabolic $G$-bundle defined by the composition (4.5).

To show that $P'_*$ is actually in $PG(X, D, N)$, take a faithful representation $\rho$ of $G$ on $V$. It is easy to derive from the given condition, namely $P' \in [\Gamma, G, N]$, that the inclusion $P'_*(\rho) \in \mathrm{PVect}(X, D, N)$ is valid. Now from the argument used repeatedly involving that any irreducible $G$-module is isomorphic to some submodule of $(\bigotimes^k V) \otimes (\bigotimes^l V^*)$, it follows that $P'_* \in PG(X, D, N)$.

Evidently, $F(P'_*) = P'$. Similarly, the composition of the two constructions is the identity map on $PG(X, D, N)$.

Since the $\Gamma$-linearized vector bundle corresponding to a given parabolic vector bundle is $\Gamma$-semistable (resp. $\Gamma$-polystable) if and only if the original parabolic vector bundle is parabolic semistable (resp. parabolic polystable), the second part of the theorem follows. □

Recall the earlier remark that a principal $G$-bundle over $Y$ is semistable (resp. polystable) if and only if the adjoint vector bundle is semistable (resp. polystable). Therefore, Proposition

4.1 implies that a $(\Gamma, G)$-bundle $P$ over $Y$ is $\Gamma$-semistable (resp. $\Gamma$-polystable) if and only if $\mathrm{ad}(P)$ is $\Gamma$-semistable (resp. $\Gamma$-polystable). Now Theorem 4.3 has the following corollary:

COROLLARY 4.6. *A parabolic $G$-bundle $P_* \in PG(X, D)$ over $X$ is parabolic semistable (resp. parabolic polystable) if and only if the parabolic vector bundle $P_*(\mathrm{ad})$ is parabolic semistable (resp. parabolic polystable), where* ad *is the adjoint representation of $G$ on the Lie algebra $\mathfrak{g}$ of $G$.*

A *$\Gamma$-connection* on a $(\Gamma, G)$-bundle $P$ on $Y$ is defined to be a $C^\infty$-connection on the principal bundle $P$ which is preserved by the action of $\Gamma$ on $P$. Fix a maximal compact subgroup $K$ of $G$. A *unitary* $\Gamma$-connection on $P$ is defined to be a $\Gamma$-connection $\nabla$ satisfying the condition that there is a point $y \in Y$ and an element $z \in P_y$ in the fiber, such that the end point of the horizontal lift, based at $z$, of any loop on $Y$, based at $y$, is contained in the orbit of $z$ for the action of $K$ on $P$. This condition is equivalent to the following: there is a $C^\infty$ reduction of the structure group, say $P_K \subset P$, of the $G$-bundle $P$ to $K$ and a connection on $P_K$ whose extension is $\nabla$. By a *flat $\Gamma$-connection* $\nabla$ on a $(\Gamma, G)$-bundle $P$ we shall mean that the connection $\nabla$ on $P$ satisfies the following conditions:

1. $\nabla$ is a $\Gamma$-connection;
2. the curvature of $\nabla$ vanishes identically;
3. the (local) horizontal sections of $P$ are holomorphic.

The following proposition, which is rather easy to prove, gives a criterion for the existence of a $\Gamma$-connection which is both unitary and flat.

PROPOSITION 4.7. *A $(\Gamma, G)$-bundle $P$ admits a unitary flat $(\Gamma, G)$-connection if and only if the following two conditions hold:*

1. *$P$ is $\Gamma$-polystable;*
2. $c_2(\mathrm{ad}(P)) = 0$, *where $c_2$ is the rational second Chern class.*

*Furthermore, a principal $(\Gamma, G)$-bundle satisfying the above two conditions admits a unique unitary flat $\Gamma$-connection.*

PROOF. Let $P$ be a $(\Gamma, G)$-bundle admitting a unitary flat connection $\nabla$. From Theorem 1 (page 24) of [RS] it follows that $P$ is $\Gamma$-polystable. Also, $c_2(\mathrm{ad}(P)) = 0$, since $\nabla$ induces a flat connection on $\mathrm{ad}(P)$.

To prove the converse, let $P$ be a $(\Gamma, G)$-bundle satisfying the two conditions in the statement of the proposition.

Proposition 4.1 says that $P$ is polystable. Now Theorem 1 of [RS] implies that $P$ has a unique Hermitian-Einstein connection which we shall denote by $\nabla$. Let $\bar{\nabla}$ be the Hermitian-Einstein connection on the adjoint bundle $\mathrm{ad}(P)$ induced by $\nabla$. Since

$$c_1(\mathrm{ad}(P)) = 0 = c_2(\mathrm{ad}(P))$$

Theorem 1 (page 19) of [Si2] implies that $\bar{\nabla}$ is a flat connection. As $G$ is semisimple, its Lie algebra does not have a nontrivial center, and hence $\nabla$ must be a flat connection. Since $P$ has a unique Hermitian-Einstein connection (Theorem 1 of [RS]), $\nabla$ must be invariant under the action of $\Gamma$ on $P$. This completes the proof of the proposition. □

Let $P_*$ be a parabolic $G$-bundle on $X$ with parabolic structure over $D$. Let $P_*(\mathrm{ad})$ be the parabolic vector bundle associated to $P_*$ for the adjoint representation of $G$. Let $P$ be the $\Gamma$-linearized $G$-bundle associated to $P_*$ by Theorem 4.3. Since $P_*(\mathrm{ad})$ corresponds to the adjoint bundle $\mathrm{ad}(P)$, the fibers of the vector bundle underlying the parabolic vector bundle $P_*(\mathrm{ad})$ over $X - D$ have a structure of a Lie algebra isomorphic to $\mathfrak{g}$, the Lie algebra of $G$. Let $P_*(\mathrm{ad})_0$ denote underlying vector bundle for the parabolic vector bundle $P_*(\mathrm{ad})$.

A *unitary connection* on $P_*$ is a connection $\nabla$ on the restriction of $P_*(\mathrm{ad})_0$ to $X - D$ satisfying the following two conditions:

1. $\nabla$ preserves the Lie algebra structure of the fibers;
2. there is a $C^\infty$ Lie algebra subbundle $W$ of $P_*(\mathrm{ad})_0$ over $X - D$ such that the fibers of $W$ are isomorphic to the Lie algebra of $K$, and furthermore, $W$ is left invariant by the connection $\nabla$.

The second condition is equivalent to the one that there is a reduction of the structure group of $P_*(\mathrm{ad})|_{X-D}$ to $K$ and a connection $\nabla'$ on this reduction such that $\nabla$ is the extension of $\nabla'$.

A *unitary flat connection* on $P_*$ is defined to be a connection $\nabla$ on $P_*(\mathrm{ad})_0$ over $X - D$ satisfying the following five conditions:

1. $\nabla$ is a unitary connection as defined above;
2. the curvature of $\nabla$ vanishes identically;
3. (local) flat sections are holomorphic sections;
4. the connection $\nabla$ extends across $D$ as a *logarithmic connection* on the vector bundle $P_*(\mathrm{ad})_0$;
5. for any irreducible component $D_i$ of $D$, the weighted filtration of the vector bundle $P_*(\mathrm{ad})_0|_{D_i}$ over $D_i$, defined by the residue of $\nabla$ along $D_i$, coincides with the parabolic structure of $P_*(\mathrm{ad})$ over $D_i$.

The notions of logarithmic connections and their residues can be found in [D1]. The residue of $\nabla$ along $D_i$ is a section

$$\mathrm{Res}(\nabla, D_i) \in H^0(D_i, P_*(\mathrm{ad})_0|_{D_i})$$

over $D_i$. The vector bundle $P_*(\mathrm{ad})_0|_{D_i}$ decomposes as a direct sum of eigenspaces of the residue $\mathrm{Res}(\nabla, D_i)$. The fifth condition means that $\alpha$ is a parabolic weight for the parabolic structure of $P_*(\mathrm{ad})$ over $D_i$ if and only if $-2\pi\sqrt{-1}\alpha$ is an eigenvalue for $\mathrm{Res}(\nabla, D_i)$, and furthermore, the decreasing filtration of $P_*(\mathrm{ad})_0|_{D_i}$, which to any $t \in [0, 1]$ assigns the direct sum of all the eigenspaces of $\mathrm{Res}(\nabla, D_i)$ with $1/(2\pi\sqrt{-1})$ times the eigenvalue less than or equal to $-t$, coincides with the quasi-parabolic filtration over $D_i$.

The following theorem can be easily derived from Theorem 4.3 and Proposition 4.7.

THEOREM 4.8. *A parabolic $G$-bundle $P_*$ admits a unitary flat connection if and only if the following two conditions hold*:

1. *$P_*$ is parabolic polystable*;
2. *$c_2^*(P_*(\mathrm{ad})) = 0$, where $c_2^*$ is the second parabolic Chern class.*

*Furthermore, a parabolic $G$-bundle satisfying the above two conditions admits a unique unitary flat connection.*

PROOF. Let $P_* \in PG(X, D, N)$ be a parabolic $G$-bundle satisfying the above two conditions. Take a Galois covering $p$ as in (2.7). Let $P \in [\Gamma, G, N]$ be the $(\Gamma, G)$-bundle that corresponds to $P_*$.

Using Theorem 4.3, we conclude that $P$ is $\Gamma$-polystable. Since the pullback of the $i$-th parabolic Chern class of a parabolic vector bundle by the morphism $p$ actually coincides with the $i$-th Chern class of the corresponding $\Gamma$-linearized vector bundle ([Bi3]), we have

$$c_2(\mathrm{ad}(P)) = 0\,.$$

Thus by Proposition 4.7 the $G$-bundle $P$ has a unitary flat connection. Using the $\Gamma$-invariance property of the induced connection on $\mathrm{ad}(P)$, a connection on $P_*(\mathrm{ad})_0$ over $X - D$ is obtained. It is straightforward to check that this connection satisfies all the conditions needed to define a unitary flat connection on $P_*$.

Conversely, if $P_*$ has a flat unitary connection $\nabla$, then we pull back the connection $\nabla$ on $P_*(\mathrm{ad})_0$ over $X - D$ using the projection $p$. This is a connection on $\mathrm{ad}(P)$ over $Y - p^{-1}(D)$. The conditions on $\nabla$ ensure that this connection over $Y - p^{-1}(D)$ extends across $p^{-1}(D)$ to produce a regular connection on $\mathrm{ad}(P)$. This connection on $\mathrm{ad}(P)$ gives a unitary flat connection on $P$. Hence $P$ must be $\Gamma$-polystable by Proposition 4.7. This in turn implies that $P_*$ is parabolic polystable. From the earlier remark on the pullback of parabolic Chern classes it follows that $c_2^*(P_*(\mathrm{ad})) = 0$, since $c_2(\mathrm{ad}(P)) = 0$, as $\mathrm{ad}(P)$ admits a flat connection. The uniqueness statement in the theorem is equivalent to that in Proposition 4.7. This completes the proof of the theorem. □

For a connected smooth subvariety $X' \subset X$ such that $X' \cap D$ is a normal crossing divisor on $X$, there is an obvious restriction functor from $\mathrm{PVect}(X, D)$ to $\mathrm{PVect}(X', X' \cap D)$. The gives a restriction map from $PG(X, D)$ to $PG(X', X' \cap D)$ simply by composing a functor with the above restriction functor. The present section is closed by making a remark on the restrictions of parabolic semistable $G$-bundles.

REMARK 4.9. In [MR] it was proved that given a vector bundle $E$ over a connected smooth projective variety $X/\boldsymbol{C}$, the restriction of $E$ to the general complete intersection curve of sufficiently large degree is semistable if and only if $E$ itself is semistable. In [Bh] this result of Mehta and Ramanathan was extended to the parabolic context. On the other hand, it is a straightforward task to extend the above theorem of [MR] to the set-up of $\Gamma$-linearized vector bundles. So now applying Theorem 4.3, we obtain a very simple proof of the theorem proved in [Bh], which states that the restriction of a parabolic vector bundle to the general complete intersection curve of sufficiently large degree is parabolic semistable if and only if so is the original parabolic vector bundle, for the particular class of parabolic vector bundles considered here. Now Proposition 3.4 implies that the restriction of a parabolic $G$-bundle $P_*$ to the general complete intersection curve of sufficiently large degree is parabolic semistable if and only if $P_*$ itself is parabolic semistable.

In the next section we shall give a construction of the moduli space of parabolic $G$-bundles over a curve.

**5. Construction of the moduli space.** Let $Y$ be a connected smooth projective curve over $\boldsymbol{C}$, equipped with a faithful action of a finite group $\Gamma$. Let $H$ be a semisimple linear algebraic group over $\boldsymbol{C}$. We say a $\Gamma$-linearized semistable principal $H$-bundle $P$ over $Y$ is of *degree zero* if, the associated vector bundle $P(V)$, for every finite dimensional representation $H \to GL(V)$, is of *degree zero*.

For every $z \in Y$, we fix once and for all the isomorphism class of the action of the isotropy subgroup $\Gamma_z$ on the fiber over $z$ of a $\Gamma$-linearized principal $H$-bundle. Let

$$F_H : (\text{Schemes})/\boldsymbol{C} \to (\text{Sets})$$

be the functor defined by

$$F_H(T) = \left\{\begin{matrix}\text{isomorphism classes of } \Gamma\text{-semistable } H\text{-bundles} \\ \text{of degree 0 on } Y \text{ parameterized by } T\end{matrix}\right\}.$$

Let $\rho : H \hookrightarrow G$ be a faithful representation of $H$ in $G = SL(n)$. Let $y \in Y$ be a *marked point* on $Y$. Let $E$ be a principal $G$-bundle on $Y \times T$ or equivalently, family of principal $G$-bundles on $Y$ parametrised by a scheme $T$, and let $E(G/H)_y$ denote the restriction of $E(G/H)$ to $y \times T \approx T$. Let $F_{H,y}$ be the functor defined as follows:

$$F_{H,y}(T) = \left\{\begin{matrix}\text{isomorphism classes of pairs } (E, \sigma_y), \text{ where } E = \{E_t\}_{t\in T} \\ \text{is a family of semistable principal } (\Gamma, G)\text{-bundles over } Y \\ \text{parameterized by } T \text{ and } \sigma_y : T \to E(G/H)_y \text{ is a section}\end{matrix}\right\}.$$

Notice that, by reduction of structure group, the functor $F_H$ can also be realised as:

$$F_H(T) = \left\{\begin{matrix}\text{isomorphism classes of pairs } (E, s), \text{ where} E = \{E_t\}_{t\in T} \\ \text{is a family of semistable } (\Gamma, G)\text{-bundles and } s = \{s_t\}_{t\in T} \text{ is a} \\ \text{family of } \Gamma\text{-invariant sections of } \{E(G/H)_t\}_{t\in T}.\end{matrix}\right\}.$$

(That the reduced $H$-bundle $P$ also has *degree zero* can be seen as follows: any representation $V$ of $H$ can be realised as a sub-module of direct sums of the tensor representations, $T^{m,n}(\rho) = (\bigotimes^m \rho) \otimes (\bigotimes^n \rho^*)$. Since $P$ is semistable as a $(\Gamma, H)$-bundle, being a reduction of structure group of $E$, it follows that $P(V)$ is also semistable. Moreover, it is a semistable sub-bundle of the semistable vector bundle $\bigoplus P(T^{m,n}(\rho))$ of degree zero, and hence $P(V)$ is of degree zero.)

With this description we have the following proposition.

PROPOSITION 5.1. *Let $\alpha_y : F_H \to F_{H,y}$ be the morphism obtained by evaluating sections at $y$. Then $\alpha_y$ is a proper morphism of functors.*

PROOF. We begin by remarking that "properness of morphism" in our sense does not include "separatedness". We use the valuation criterion for properness. Let $T$ be an affine smooth curve and let $p \in T$. Then by the valuation criterion, we need to show the following:

Let $E$ be a family of $\Gamma$-semistable principal $G$-bundles on $Y \times T$ together with a section $\sigma_y : T \to E(G/H)_y$, such that for every $t \in T - p$, we are given a family of $(\Gamma, H)$-reductions, that is, a family $\{s_t\}_{t \in T-p}$ of $\Gamma$-invariant sections $s_t : Y \to E(G/H)_t$, with the property that the equality $s_t(y) = \sigma_y(t)$ is valid for every $t \in T - p$. Then we need to extend the family $s_{T-p}$ to $s_T$ as a $\Gamma$-invariant section of $E(G/H)$ on $X \times T$ such that $s_p(y) = \sigma_y(p)$ as well.

Observe that, since $G/H$ is affine, there exists a $G$-module $W$ such that $G/H \overset{i}{\hookrightarrow} W$ is a *closed G-embedding* and $0 \notin G/H$. Thus we get a closed embedding

$$E(G/H) \hookrightarrow E(W)$$

and a family of $\Gamma$-semistable vector bundles $\{E(W)_t\}_{t \in T}$ together with a family of $\Gamma$-invariant sections $s_{T-p}$ and evaluations $\{\sigma_y(t)\}_{t \in T}$ such that $s_t(y) = \sigma_y(t)$ for all $t \neq p$.

For the section $s_{T-p}$, viewed as a $\Gamma$-invariant section of $E(W)_{T-p}$, we have two possibilities:

(a) it extends as a regular section $s_T$;

(b) it has a pole along $Y \times p$.

Observe that in the situation of (a), the section $s_{T-p}$ extends as an usual section not necessarily $\Gamma$-invariant, and then, since it is $\Gamma$-invariant on a Zariski open subset, it will in fact be $\Gamma$-invariant on the whole of $Y \times T$. Thus if (a) holds, then we have

$$s_T(Y \times (T - p)) \subset E(G/H) \subset E(W),$$

and since $E(G/H)$ is closed in $E(W)$, it follows that $s_T(Y \times p) \subset E(G/H)$. Thus we have $s_p(Y) \subset E(G/H)_p$. Further by continuity, $s_p(y) = \sigma_y(p)$ as well, and this completes the proof of the proposition if the situation (a) holds.

Therefore, to complete the proof it suffices to check that the possibility (b) does not occur. Suppose it does occur. For our purpose, we could take the local ring $A$ of $T$ at $p$, which is a discrete valuation ring, with a uniformizer $\pi$. Let $K$ be its quotient field. The section $s_{T-p} = s_K$ is a section of $E(W)_K$, i.e., it is a *rational section* of $E(W)$, and we have supposed that it has poles on the divisor $Y \times p \subset Y \times T$, *say of order* $k \geq 1$.

Thus, by multiplying $s_{T-p}$ by $\pi^k$, where $\pi$ is the uniformizer, we get a regular section $s'_T$ of $E(W)$ on $Y \times T$. If $s'_T = \{s'_t\}_{t \in T}$, then we have the following:

1. $s'_t = \lambda(t) \cdot s_t$ for every $t \in T - p$, where $\lambda : T \to \boldsymbol{C}$ is a function given by $\pi^k$, vanishing of order $k$ at $p$.

2. $s'_p$ is a nonzero section of $E(W)_p$.

Notice that, $s'_p$ is a section of $E(W)_p$, and $E(W)_p$ is a *semistable vector bundle* of degree 0, since $E(W)$ is a family of semistable $\Gamma$-linearized vector bundles of degree 0. Therefore, a nonzero section of $E(W)_p$ is nowhere vanishing. So, from (2) it follows that

$$(*) \qquad s'_p(z) \neq 0 \quad \text{for any } z \in Y\,.$$

By assumption, $s_t(y) = \sigma_y(t)$ for every $t \in T - p$, and hence

$$s'_t(y) = \lambda(t) \cdot \sigma_y(t) \quad t \in T - p\,.$$

Therefore, by using continuity, and since $\sigma_y(p)$ is well-defined, we conclude that $\lambda(t) \cdot \sigma_y(t)$ tends to $\lambda(p) \cdot \sigma_y(p) = 0$, as $t \to p$. Also,

$$s'_t(y) \to s'_p(y)$$

as $t \to p$. Hence by continuity it follows that $s'_p(y) = 0$, which contradicts the assertion $(*)$.

Therefore, we conclude that the possibility (b) does not occur. This completes the proof of the proposition. □

Let $H \subset G = SL(n)$ as above. We recall very briefly the definition of Grothendieck Quot scheme used in the construction of the moduli space of vector bundles (cf. [Se2]). Let $\mathcal{F}$ be a $\Gamma$-coherent sheaf on $Y$ and $\mathcal{F}(m)$ be $\mathcal{F} \otimes \mathcal{O}_Y(m)$ (following the usual notation). Choose an integer $m_0 = m_0(n, d)$ (depending on $n$ and $d$, where $n =$ rank and $d =$ degree) such that for any $m \geq m_0$ and any semistable $\Gamma$-linearized vector bundle $V$ of rank $n$ and degree $d$ on $Y$ we have $h^i(V(m)) = 0$ for $i \geq 1$ and $V(m)$ is generated by its global sections. *Note that we shall be working throughout only with the situation where the underlying bundles have degree* 0, *since we work with* $SL(n)$*-bundles.*

Fix an integer $m$ as above, and set $N = h^0(V(m))$. Let $P$ denote the linear polynomial $P(x) = nx + n(1 - g)$. Consider the Quot scheme $Q_G$ consisting of coherent sheaves $F$ over $Y$, which are quotients of $\mathcal{E}$ where $\mathcal{E} = \boldsymbol{C}^N \otimes_{\boldsymbol{C}} \mathcal{O}_Y(-m)$ with a fixed Hilbert polynomial $P$. Let $\mathcal{F}$ denote the universal quotient sheaf on $Y \times Q_G$. Note that since $\mathcal{E}$ is also a $\Gamma$ sheaf, the group $\Gamma$ acts on the Quot scheme $Q_G$. Further, the group $\mathcal{G} := \mathrm{Aut}(\mathcal{E})$ acts canonically on $Q_G$ and hence on $Y \times Q_G$ (with the trivial action on $Y$), commuting with the $\Gamma$-action and lifts to an action on $\mathcal{E}$ as well. Let us denote by $\mathcal{G}^\Gamma$ the subgroup of $\mathcal{G}$ consisting of all $\Gamma$-invariant automorphisms of $\mathcal{E}$. We remark that $\mathcal{G}^\Gamma$ is a product of copies of $GL(k)$, and therefore reductive (cf. [Se3]). Let $Q_G^\Gamma$ be the closed subscheme of $Q_G$ of $\Gamma$-invariant points. We shall also fix the *local type* of our $\Gamma$-bundles which gives a connected component of $Q_G^\Gamma$ (cf. [Se3] for the definition of *local type*).

*By an abuse of notation we shall henceforth denote by* $\mathcal{G}$ *the group* $\mathcal{G}^\Gamma$.

Let $R$ denote the $\mathcal{G}$-invariant open subset of $Q_G^\Gamma$ defined by

$$R = \left\{ q \in Q_G^\Gamma \;\middle|\; \begin{array}{l} \mathcal{F}_q = \mathcal{F}|_{Y \times q} \text{ is locally free such that the canonical map} \\ \boldsymbol{C}^N \to H^0(\mathcal{F}_q(m)) \text{ is an } \textit{isomorphism}, \text{ and } \det \mathcal{F}_q \simeq \mathcal{O}_Y \end{array} \right\}.$$

We denote by $R^{\mathrm{ss}}$ the $\mathcal{G}$-invariant open subset of $R$ consisting of semistable $\Gamma$-bundles, and let $\mathcal{F}$ continue to denote the restriction of $\mathcal{F}$ to $Y \times R^{\mathrm{ss}}$.

Fix a base point $y \in Y$. Let $q'' : (\mathrm{Sch})/\boldsymbol{C} \to (\mathrm{Sets})$ be the following functor:

$$q''(T) = \left\{ (V_t, s_t) \;\middle|\; \begin{array}{l} \{V_t\} \text{ is a family of } (\Gamma, G)\text{-bundles in } R^{\mathrm{ss}} \text{ parameterized} \\ \text{by } T \text{ and } s_t \in H^0(Y, V(G/H)_t)^\Gamma \text{ for any } t \in T \end{array} \right\}.$$

Namely, $q''(T)$ consists of all pairs of a $\Gamma$-linearized vector bundle of rank $n$ in $R^{\mathrm{ss}}$ (or, equivalently, $(\Gamma, G)$-bundles) together with a $\Gamma$-invariant reduction of structure group to $H$. By appealing to the general theory of Hilbert schemes, one can show that $q''$ is representable by a $R^{\mathrm{ss}}$-scheme (cf. [Ra2, Lemma 3.8.1]). The $R^{\mathrm{ss}}$-scheme representing $q''$ will be denoted by $Q_H$.

The universal sheaf $\mathcal{F}$ on $Y \times R^{ss}$ is in fact a $\Gamma$-linearized vector bundle. Denoting by the same $\mathcal{F}$ the associated $(\Gamma, G)$-bundle, set $Q' = (\mathcal{F}/H)_y$. Then in our notation $Q' = \mathcal{F}(G/H)_y$, that is, we take the bundle over $Y \times R^{ss}$ associated to $\mathcal{F}$ with fiber $G/H$ and take its restriction to $y \times R^{ss} \approx R^{ss}$. Let $f : Q' \to R^{ss}$ be the natural morphism. Then, since $H$ is reductive, $f$ is an *affine morphism*.

Observe that $Q'$ parameterizes semistable $\Gamma$-linearized vector bundles together with initial values of reductions to $H$.

Define the *evaluation map* of $R^{ss}$-schemes as follows:

$$\phi_y : Q_H \to Q', \quad (V, s) \mapsto (V, s(y)).$$

PROPOSITION 5.2. *The evaluation map $\phi_y : Q_H \to Q'$ is affine.*

PROOF. The Proposition can be proved in two steps, by showing that $\phi_y$ is proper and is injective. That it is proper follows exactly as in the proof of Proposition 5.1.

To see the injectivity we proceed as follows: Let $G/H \hookrightarrow W$ be as in Proposition 5.1. Take two points $(E, s)$ and $(E', s') \in Q_H$ such that $\phi_y(E, s) = \phi_y(E', s')$ in $Q'$, i.e., $(E, s(y)) = (E', s'(y))$. So we may assume that $E \simeq E'$ and that $s$ and $s'$ are two different sections of $E(G/H)$ with $s(y) = s'(y)$.

Using the map $G/H \hookrightarrow W$, we may consider $s$ and $s'$ as sections in $H^0(Y, E(W))^\Gamma$. Observe that $E(W)$ is semistable of degree zero, since by definition $E$ is semistable of degree zero. Recall that a nonzero section of a semistable vector bundle of degree zero is nowhere vanishing. From this it follows immediately that if $E$ and $F$ are semistable vector bundles with $\mu(E) = \mu(F)$, then the evaluation map

$$(*) \qquad \phi_y : \mathrm{Hom}(E, F) \to \mathrm{Hom}(E_y, F_y)$$

is *injective*.

Consider $s$ and $s' \in \mathrm{Hom}(\mathcal{O}_Y, E(W))^\Gamma$. Since $\phi_y(s) = \phi_y(s')$, from $(*)$ it follows that $s = s'$. This proves that $\phi_y$ is injective. This concludes the proof of the Proposition 5.2. □

REMARK 5.3. Observe that the action of $\mathcal{G}$ on $R^{ss}$ lifts to an action on $Q_H$. Indeed, the $\mathcal{G}$ action lifts to an action on the associated bundle $\mathcal{F}(G/H)$, and hence to its space of sections.

Recall the commutative diagram

$$\begin{tikzcd} Q_H \arrow[r, "\phi_y"] \arrow[dr, "\psi"'] & Q' \arrow[d, "f"] \\ & R^{ss} \end{tikzcd}$$

By Proposition 5.2, $\phi_y$ is affine. Further, one knows that $f$ is an affine morphism, since the fiber $G/H$ is affine. Hence, we conclude that $\psi$ is a $\mathcal{G}$ *equivariant affine morphism*.

REMARK 5.4. Let $(E, s)$ and $(E', s')$ be in the same $\mathcal{G}$-orbit of $Q_H$. Then we have $E \simeq E'$. Let $\mathrm{Aut}_G(E)$ be the group of automorphisms of the principal $G$-bundles $E$. Identifying $E'$ with $E$, we see that $s$ and $s'$ lie in the same orbit of $\mathrm{Aut}_G(E)$ on $H^0(Y, E(G/H))^{\Gamma}$. Now it is routine to check that the orbits under this natural action of $\mathrm{Aut}_G(E)$ correspond precisely to the $(\Gamma, H)$ reductions which are isomorphic as $(\Gamma, H)$ bundles. Thus, we see that the reductions $s$ and $s'$ give isomorphic $(\Gamma, H)$-bundles.

Conversely, if $(E, s)$ and $(E', s')$ are such that $E \simeq E'$ and the $\Gamma$-reductions $s, s'$ give isomorphic $(\Gamma, H)$-bundles, then as in the above argument we see that $(E, s)$ and $(E', s')$ lie in the same $\mathcal{G}$-orbit.

Consider the $\mathcal{G}$-action on $Q_H$ with the linearization induced by the *affine $\mathcal{G}$-morphism* $Q_H \to R^{ss}$. By Ramanathan's Lemma (cf. Lemma 5.1 of [Ra2]), since a good quotient of $R^{ss}$ by $\mathcal{G}$ exists, it follows that a good quotient $Q_H//\mathcal{G}$ exists. Moreover, by the universal property of categorical quotients, the canonical morphism

$$\bar{\psi} : Q_H//\mathcal{G} \to R^{ss}//\mathcal{G}$$

is also *affine*.

REMARK 5.6. Following Lemma 10.7 in [Si3] one can show without much difficulty that the dimension of the Zariski tangent space of $Q_H$ at $E$ is $\dim(H^1(Y, \mathrm{ad}(E))^{\Gamma}) - \dim(H^0(Y, \mathrm{ad}(E))^{\Gamma}) + \dim(\mathcal{G})$. (See also [Se3, pp. 214].) Now, since $Y$ is a curve, it follows immediately that the scheme $Q_H$ is smooth.

THEOREM 5.7. *Let $M_Y(H)$ denote the scheme $Q_H//\mathcal{G}$. This scheme $M_Y(H)$ is the coarse moduli scheme of semistable $(\Gamma, H)$-bundles. The scheme $M_Y(H)$ is normal and projective, and furthermore, if $H \hookrightarrow GL(V)$ is a faithful representation, then the canonical morphism $\bar{\psi} : M_Y(H) \to M_Y(GL(V))$ is finite.*

PROOF. Since $Q_H$ is smooth by Remark 5.6, and since $M_Y(H)$ is obtained as a *good quotient*, we conclude that the variety $M_Y(H)$ is normal.

The proof of the projectivity of the moduli space $M_Y(H)$ is given in the subsection below. More precisely, we prove that the moduli $M_Y(H)$ are *topologically compact* and conclude the projectivity. It follows that $\bar{\psi}$ is *proper*. By the remarks above $\bar{\psi}$ is also *affine*, therefore $\bar{\psi}$ is finite. □

The finiteness of $\bar{\psi}$ gives the following:

COROLLARY 5.8. *Let $\Theta$ denote the generalized theta line bundle on $M_Y(GL(V))$. Then the pull-back $\bar{\psi}^*\Theta$ is ample.*

**5a. Projectivity of the $\Gamma$-linearized moduli.** The aim of this subsection is to give a self-contained proof, along the lines of [Se1], that the moduli space $M_Y(H)$ constructed above is topologically compact. For the present, we assume that the group $H$ is *semi simple and of also of adjoint type*. For such groups we have the following well-known property:

LEMMA 5.9. *There exists a faithful* irreducible *representation $H \subset GL(n)$.*

PROOF. We may easily reduce the proof to the case when the group is simple (by taking the tensor product representation for the product group). Then one can simply take any fundamental representation for the simple factors and we are done. □

Fix a maximal compact $K$ of $H$ such that $K \subset U(n)$. Consider the subset $R^{\mathrm{s}} \subset R^{\mathrm{ss}}$ consisting of the stable $(\Gamma, G)$-bundles.

Recall that there is a discrete group $\pi$ which acts discontinuously on the universal cover $\tilde{Y}$ of $Y$, with $X$ as its quotient, and further, $\Gamma$ is a quotient of $\pi$ by a normal subgroup which acts freely on $\tilde{Y}$.

DEFINITION 5.10. We say a principal $(\Gamma, H)$-bundle $E$ is *unitary* if there exists a representation $\rho : \pi \to K$ such that $E$ is isomorphic to the extension of the principal $K$-bundle $V(\rho)$, associated to $\rho$, to a principal $H$ bundle by using the inclusion homomorphism of $K$ into $H$.

LEMMA 5.11. *Let $\phi : Q_H \to R^{\mathrm{ss}}$ be the morphism induced by the representation $\rho$ by extension of structure group. Then, the inverse image of the set of stable points $\phi^{-1}(R^{\mathrm{s}}) = Q^{\mathrm{s}}_H$ consists of unitary $(\Gamma, H)$-bundles.*

PROOF. We consider the adjoint representation $\mathrm{ad} : H \to GL(\mathrm{ad}(H))$. Then, we observe that a principal $H$-bundle $E$ is unitary if and only if the associated adjoint bundle $\mathrm{ad}(E)$ is so (cf. Lemma 10.12, [AB]).

More generally, we claim that, if $\rho : H \to GL(n)$ is an arbitrary finite dimensional representation an $H$-bundle $E$ is unitary if and only if the associated vector bundle $E(\rho)$ is so.

If $E$ is unitary it is obvious that $E(\rho)$ is so. For the converse, assume that $E(\rho)$ is unitary. Then by the earlier remark, it is enough to show that $E(\mathrm{ad})$ is unitary. First observe that, since $\rho$ is a faithful representation of $H$, the adjoint representation can be realized as a $H$-submodule of a direct sum of the tensor representations, $T^{m,n}(\rho) = (\bigotimes^m \rho) \otimes (\bigotimes^n \rho^*)$. Furthermore, since $E(\rho)$ is a unitary vector bundle, the vector bundles $E(T^{m,n}(\rho))$'s are also unitary. Now for vector bundles, one knows that unitarity is equivalent to polystability plus degree zero. Hence, $E(T^{m,n}(\rho))$'s are polystable vector bundle of degree zero. Since $E(\mathrm{ad})$ is a degree zero subbundle of a polystable bundle of degree zero, it follows that $E(\mathrm{ad})$ must be polystable. Therefore, $E(\mathrm{ad})$ is a unitary vector bundle. This proves the claim.

Now by Narasimhan-Seshadri theorem, we see that points of $R^{\mathrm{s}}$ which are stable bundles are all unitary. Hence by the claim above the bundles in the inverse image $\phi^{-1}(R^{\mathrm{s}})$ are also unitary. □

PROPOSITION 5.12. *Let $\rho$ be the faithful irreducible representation of $H$ as obtained above in Lemma* 5.9. *Then the inverse image of $R^{\mathrm{s}}$ by the induced morphism $\phi$ is nonempty.*

PROOF. Let $\pi$ denote the group of holomorphic automorphisms of the universal cover $\tilde{Y}$ of $Y$ which commute with the composition map

$$\tilde{Y} \to Y \to Y/\Gamma .$$

Recall that $\Gamma$ is the quotient of $\pi$ by a normal subgroup which acts freely on $\tilde{Y}$ and by [Se2] a $\Gamma$-vector bundle is polystable (resp. stable) if and only if it arises from a unitary (resp. irreducible unitary) representation of $\pi$. The group $\pi$ can be identified with the free group on the letters $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m$ modulo the following relations:

$$[A_1, B_1] \cdots [A_g, B_g] \cdot C_1 \cdots C_m = \mathrm{id}, \quad C_1^{n_1} = \cdots = C_m^{n_m} = \mathrm{id}.$$

So to prove that the inverse image $\phi^{-1} R^s$ is nonempty, we need to exhibit a representation

$$\chi : \pi \to K$$

such that the composition

$$\rho \circ \chi : \pi \to U(n)$$

is *irreducible.*

Choose elements $h_1, \ldots, h_m \in K$ so that they are elements of order $n_i$, where $i = 1, \ldots, m$ (these correspond to fixing the *local type* of our bundles).

We *claim* that every element of a compact connected real semisimple Lie group is a commutator.

PROOF OF THIS CLAIM. One can proceed as follows. Since $K$ is compact, every element is semisimple and can therefore be put in a maximal torus $T$. Now one proceeds as in [Ra1]. That is, by using the Coxeter-Killing transformation $w$, it can be proved that any $x \in T$ can actually be expressed as $w \cdot y \cdot w^{-1} \cdot y^{-1}$ for some $y \in T$. Indeed, the map $\mathrm{ad}(w) - \mathrm{Id}$ does not have 1 as an eigenvalue, when acting on the Lie algebra $\mathrm{Lie}(T)$ by the adjoint action. Consequently,

$$\mathrm{Ad}(w) : T \to T$$

is surjective, where $\mathrm{Ad}(w)$ is the action on $T$. This proves that any $x \in T$ can actually be expressed as $w \cdot y \cdot w^{-1} \cdot y^{-1}$ for some $y \in T$.

Thus we can solve the equation

$$[a_3, b_3] \cdots [a_g, b_g] = (h_1 \cdots h_m)^{-1}$$

in $K$.

*Observe that, we crucially need that the* $\mathit{genus}(X) \geq 2$.

Since $K$ is a compact connected real semi-simple Lie group, there exists a dense subgroup of $K$ generated by two general elements $(\alpha, \beta)$ (for a proof cf. Lemma 3.1 in [Su]). Now, define the representation

$$\chi : \pi \to K$$

as follows:

$$\chi(A_1) = \alpha, \quad \chi(B_1) = \beta, \quad \chi(A_2) = \beta, \quad \chi(B_2) = \alpha,$$
$$\chi(A_i) = a_i, \quad \chi(B_i) = b_i, \quad \chi(C_j) = h_j, \quad i = 3, \ldots, g \text{ and } j = 1, \ldots, m.$$

It is clear that $\chi$ gives a representation of the group $\pi$. Now, since $\rho$ is irreducible, and the image of $\chi$ contains a dense subgroup, the composition $\rho \circ \chi$ gives an irreducible representation of $\pi$ in the unitary group $U(n)$. Therefore, it gives a *stable* $\Gamma$-linearized vector bundle, which moreover arises as the extension of structure group of a $H$-bundle. This completes the proof of the Proposition. □

COROLLARY 5.13. *In the total family $Q_H$ for the $H$-bundles, there is a non-empty Zariski open subset of unitary bundles.*

PROOF. This follows immediately from the Lemma 5.11 and Proposition 5.12. □

Let $R_H(K)$ denote the space of representations of $\pi$ in $K$. By the local universal property of $Q_H$, for every point of $R_H(K)$ there is an analytic neighborhood and a continuous map into $Q_H$, namely, by taking the unitary $(\Gamma, H)$-bundle associated to a representation of $\pi$ in $K$. These maps patch up modulo the action of the group $\mathcal{G}$, and we get, by the categorical quotient property of $M_Y(H)$, a continuous map $\psi : R(K) \to M_Y(H)$ (cf. [Se1]).

Let $f : Q_H \to M_Y(H)$ be the canonical quotient map.

THEOREM 5.14. *The map $\psi$ is surjective and hence $M_Y(H)$ is compact. In particular, $M_Y(H)$ has a structure of a projective variety when the group $H$ is of adjoint type.*

PROOF. Consider the canonical categorical quotient map $f : Q_H \to M_Y(H)$. Since $f$ is surjective the image $f(Q^s_H)$ in $M_Y(H)$ contains a Zariski open subset $U$ ("Chevalley's lemma").

By the Corollary 5.13 above, the subset $Q^s_H$ is nonempty and consists entirely of unitary bundles. That is, the image $f(Q^s_H)$ is a subset of the image $\psi(R_H(K))$ in $M_Y(H)$. Therefore, it follows that $\psi(R_H(K))$ contains the Zariski open subset $U$, of $M_Y(H)$.

Since the group $H$ is semisimple and the topological type of the bundles is fixed, it follows that the moduli space $M_Y(H)$ is firstly *connected.* It was noted earlier (in the Proof of Theorem 5.7) that $M_Y(H)$ is normal, since $Q_H$ is smooth. Therefore, the connectedness of $M_Y(H)$ implies that it is *irreducible.*

Now, since $R_H(K)$ is compact, the image $\psi(R_H(K))$ is closed in $M_Y(H)$ in the analytic topology and also contains a dense subset. It follows by the irreducibility of $M_Y(H)$, that $\psi(R_H(K))$ is the whole of $M_Y(H)$ and hence $M_Y(H)$ is topologically compact. By the GAGA principle we conclude that $M_Y(H)$ is a projective variety. □

COROLLARY 5.15. *The moduli spaces $M_Y(H)$ are projective for all semisimple groups $H$* (cf. also [BS]).

PROOF. Consider the exact sequence of groups:

$$Z(H) \to H \to H/Z(H)$$

where $Z(H)$ is the centre of $H$. The projectivity of the moduli space $M_Y(H)$ then reduces to that of $M_Y(H/Z(H))$ and of $M_Y(Z(H))$. That $M_Y(H/Z(H))$ is projective is the content of Theorem 5.14. Since $Z(H)$ is abelian, the projectivity of $M_Y(Z(H))$ now follows, since one has only a product of Picard varieties of curves to handle. (Arguments of a similar kind can be found, for example, in [AB].) □

The next section will be devoted to the parabolic analog of Higgs $G$-bundles.

**6. Principal Higgs bundles with parabolic structure.** Let $X/C$ be a connected smooth projective variety of dimension $d$, and $D$ be a normal crossing divisor on $X$. Let $E_* = (E, F_*, \alpha_*)$ be a parabolic vector bundle of rank $n$ on $X$ with parabolic structure over $D$. Define $\mathrm{End}^1(E_*)$ to be the subsheaf of the sheaf of endomorphisms of $E$ that preserves the quasi-parabolic flag of $E_*$. In other words, for a local section $S$ of $\mathrm{End}^1(E_*)$ over an open set $U$, the inclusion

$$S(F_i(E)|_U) \subseteq F_i(E)|_U \tag{6.1}$$

is valid. Clearly, $\mathrm{End}^1(E_*)$ is locally free, and

$$\mathrm{End}(E) \otimes \mathcal{O}_X(-D) \subseteq \mathrm{End}^1(E_*) \subseteq \mathrm{End}(E)\,.$$

Let $\Omega^i_X(\log D)$ denote the vector bundle on $X$ defined by the sheaf of logarithmic differential forms ([D1]). For any irreducible component $D_i$ of $D$, there is a natural residue homomorphism

$$\mathrm{Res}(D_i) : \Omega^1_X(\log D) \to \mathcal{O}_{D_i}$$

which defines a homomorphism

$$\mathrm{Res}(E_*, D_i) : \mathrm{End}^1(E_*) \otimes \Omega^1_X(\log D) \to \mathrm{End}^1(E_*)|_{D_i}\,, \tag{6.2}$$

where $\mathrm{End}^1(E_*)|_{D_i}$ is the restriction of the vector bundle $\mathrm{End}^1(E_*)$ to $D_i$.

DEFINITION 6.3. A *parabolic Higgs vector bundle* is a pair of the form $(E_*, \theta)$, where $E_*$ is a parabolic vector bundle and

$$\theta \in H^0(X, \mathrm{End}^1(E_*) \otimes \Omega^1_X(\log D))$$

satisfying the following two conditions:

1. $\theta \wedge \theta \in H^0(X, \mathrm{End}^1(E_*) \otimes \Omega^2_X(\log D))$ vanishes identically, where the multiplication is defined using the Lie algebra structure of the fibers of $\mathrm{End}(E)$;
2. $\mathrm{Res}(E_*, D_i)(\theta)(F_i(E)|_{D_i}) \subseteq F_{i+1}|_{D_i}$.

A section $\theta$ satisfying the above two conditions is called a *Higgs field* on $E_*$.

For two parabolic Higgs vector bundles $(E_*, \theta)$ and $(E'_*, \theta')$, the section $\theta \oplus \theta'$ is a Higgs field on the parabolic vector bundle $E_* \oplus E'_*$. Similarly, $\theta \otimes \mathrm{Id}_{E'} + \mathrm{Id}_E \otimes \theta'$ is a Higgs field on the parabolic tensor product $E_* \otimes E'_*$. The *direct sum operation* and the *tensor product operation* of parabolic Higgs bundles are defined in this way. Using the natural identification between $\mathrm{End}(E)$ and $\mathrm{End}(E^*)$, a Higgs field $\theta$ on $E$ defines a Higgs field on the parabolic dual $E^*_*$. See [Yo], [Bi3] for the details.

A *homomorphism* from a Higgs bundle $(E_*, \theta)$ to another Higgs bundle $(E'_*, \theta')$ is a parabolic homomorphism $f : E_* \to E'_*$ such that the condition

$$\theta' \circ f = f \circ \theta$$

is valid.

Let $\mathrm{PHVec}(X, D)$ denote the category whose objects are parabolic Higgs vector bundles and whose morphisms are homomorphisms of parabolic Higgs vector bundles. Using the above remarks, $\mathrm{PHVec}(X, D)$ becomes an additive tensor category.

DEFINITION 6.4. A *parabolic Higgs G-bundle* with parabolic structure over $D$ is a functor from $\mathrm{Rep}(G)$ to $\mathrm{PHVec}(X, D)$ satisfying all the conditions in Definition 5.5 with the tensor product operation and the homomorphisms being as defined above.

An equivalent definition of a parabolic Higgs $G$-bundle is the following:

DEFINITION 6.5. A parabolic Higgs $G$-bundle is a pair of the form $(P_*, \theta)$, where $P_*$ is a parabolic $G$-bundle and $\theta$ is a Higgs field on the parabolic vector bundle $P_*(\mathrm{ad})$, where ad is the adjoint representation of $G$.

To show that the two definitions are equivalent, let $\mathcal{P}$ be a functor defining a parabolic Higgs $G$-bundle according to Definition 6.4. Denote by $\mathcal{F}$ the forgetful functor from $\mathrm{PHVec}(X, D)$ to $\mathrm{PVect}(X, D)$, which forgets the Higgs field. The composition $\mathcal{F} \circ \mathcal{P}$ defines a parabolic $G$-bundle $P_*$ according to Definition 5.5. If $\mathcal{P}(\mathrm{ad}) = (E, \theta)$, then associate to $\mathcal{P}$ the pair $(P_*, \theta)$. It is easy to see that $(P_*, \theta)$ is a parabolic Higgs $G$-bundle according to Definition 6.5.

In the reverse direction, let $(P_*, \theta)$ be a parabolic Higgs $G$-bundle according to Definition 6.5. Now given any representation $\rho : G \to \mathrm{Aut}(V)$, consider the induced $G$-equivariant homomorphism

$$\mathrm{ad}(\rho) : \mathfrak{g} \to \mathrm{End}(V)\,.$$

This gives a homomorphism

$$\overline{\mathrm{ad}(\rho)} : P_*(\mathrm{ad})_0 \otimes \Omega^1_X(\log D) \to \mathrm{End}^1(P_*(\rho)) \otimes \Omega^1_X(\log D)\,,$$

where $P_*(\mathrm{ad})_0$ is the underlying vector bundle for the parabolic vector bundle $P_*(\mathrm{ad})$. It is easy to see that $\overline{\mathrm{ad}(\rho)}(\theta)$ is a Higgs field on $P_*(\rho)$. Now, a parabolic Higgs $G$-bundle according to Definition 6.4 is obtained by associating to any $\rho \in \mathrm{Rep}(G)$ the parabolic Higgs bundle $(P_*(\rho), \overline{\mathrm{ad}(\rho)}(\theta))$. It is immediate that the above two constructions are inverses of each other.

A *$\Gamma$-Higgs field* on a $\Gamma$-linearized vector bundle $W$ over $Y$, equipped with an action of $\Gamma$, is an invariant section

$$\phi \in H^0(Y, \mathrm{End}(W) \otimes \Omega^1_Y)^\Gamma$$

satisfying the condition that $\phi \wedge \phi = 0$. In the correspondence between parabolic vector bundles and $\Gamma$-linearized vector bundles that has been used repeatedly here, the space of $\Gamma$-Higgs fields on $W$ is naturally isomorphic to the space of Higgs fields on the parabolic vector bundle that corresponds to $W$ [Bi3]. Therefore, we have a bijective correspondence between the space of parabolic Higgs vector bundles of rank $n$ and the space of $\Gamma$-linearized vector bundles of rank $n$ equipped with a $\Gamma$-Higgs field.

Using this bijective correspondence, it is a simple exercise to extend the proof of Proposition 2.6 to establish a bijective identification between the collection of parabolic Higgs $GL(n, \boldsymbol{C})$-bundles and the collection of parabolic Higgs vector bundles of rank $n$. As before, this bijective identification is defined by using the standard representation of $GL(n, \boldsymbol{C})$. We omit the details.

Before defining parabolic semistable Higgs $G$-bundles, let us first recall the definition of parabolic semistable Higgs vector bundles.

A parabolic Higgs vector bundle $(E_*, \theta)$ is called *parabolic semistable* (resp. *parabolic stable*) if for any nonzero proper coherent subsheaf $V \subset E$ with $E/V$ torsion-free and $\theta(F) \subseteq F \otimes \Omega^1_X(\log D)$, the following inequality is valid:

$$\frac{\text{par-deg}(V_*)}{\text{rank}(V)} \leq \frac{\text{par-deg}(E_*)}{\text{rank}(E)} \quad \left(\text{resp. } \frac{\text{par-deg}(V_*)}{\text{rank}(V)} < \frac{\text{par-deg}(E_*)}{\text{rank}(E)}\right),$$

where $V_*$ is $V$ with the induced structure of a parabolic sheaf. Also, $(E_*, \theta)$ is called *parabolic polystable* if it is direct sum of parabolic stable Higgs vector bundles of same parabolic slope ($:=$ par-deg/rank).

The analog of Proposition 3.2 in the situation of Higgs vector bundles is valid. In other words, parabolic semistable (resp. parabolic polystable) Higgs vector bundles are closed under the operations of tensor product and dual. Indeed, using the above bijective correspondence between parabolic Higgs vector bundles and the $\Gamma$-linearized Higgs bundles, the question reduces to $\Gamma$-linearized Higgs bundles. Clearly, the dual of a semistable (resp. polystable) Higgs bundle is again semistable (resp. polystable). It is easy to see that a $\Gamma$-linearized Higgs bundle is semistable (resp. polystable) if it is semistable (resp. polystable) in the usual sense. Now, from [Si2] we know that the tensor product of two semistable (resp. polystable) Higgs bundles is again semistable (resp. polystable).

DEFINITION 6.6. A parabolic Higgs $G$-bundle $\mathcal{P}$ is called *parabolic semistable* (resp. *parabolic polystable*) if the image of the functor $\mathcal{P}$ is contained in the category of parabolic semistable (resp. parabolic polystable) Higgs vector bundles.

A Higgs field on a $(\Gamma, G)$-bundle $P$ on $Y$ is an invariant section

$$\phi \in H^0(Y, \text{ad}(P) \otimes \Omega^1_Y)^\Gamma$$

such that $\phi \wedge \phi = 0$, where the multiplication is defined by using the Lie algebra structure of the fibers of the vector bundle $\text{ad}(P)$. The $(\Gamma, G)$-Higgs bundle is called *$\Gamma$-semistable* (resp. *$\Gamma$-polystable*) if it satisfies the inequality condition for $\Gamma$-semistability (resp. $\Gamma$-polystability) for the $(\Gamma, G)$-bundle $P$ only for reduction of structure groups such that $\phi$ coincides with the extension of a Higgs field on the reduction.

The bijective correspondence in Theorem 4.3 extends to a bijective correspondence in the context of Higgs bundles. In other words, the collection of $(\Gamma, G)$-Higgs bundles $(P, \phi)$ on $Y$, such that $P \in [\Gamma, G, N]$, is in a natural bijective correspondence with the collection of parabolic Higgs $G$-bundles $(P_*, \theta)$ on $X$ such that $P_* \in PG(X, D, N)$. If $(P, \phi)$ corresponds to $(P_*, \theta)$ by this identification, then $P$ corresponds to $P_*$ in Theorem 4.3. The Higgs fields are related using the adjoint representation.

In [Si1, p. 878, Proposition 3.4] it was shown that the Bogomolov inequality remains valid for semistable $(\Gamma, G)$-Higgs bundles. Thus the analog of Bogomolov inequality for parabolic semistable $G$-bundles, mentioned in Remark 3.6, remains valid for parabolic semistable Higgs $G$-bundles.

Imitating the proof of Corollary 4.6, it can be shown that $\mathcal{P}$ is parabolic semistable (resp. parabolic polystable) if and only if the parabolic Higgs vector bundle $\mathcal{P}(\mathrm{ad})$ is parabolic semistable (resp. parabolic polystable).

As in Section 4, let $K \subset G$ be a maximal compact subgroup. Let $(P, \phi)$ be a $(\Gamma, G)$-Higgs bundle, and $\nabla$ be a unitary $\Gamma$-connection on $P$. (unitary $\Gamma$-connections are defined in Section 4.) Consider the connection

$$\nabla^{\phi} := \nabla + \phi + \phi^*$$

on $P$, where the adjoint $\phi^*$ is the adjoint defined using the real linear involution of $\mathfrak{g}$ that acts as identity on the Lie algebra of $K$ and acts as $-1$ on its $K$-invariant complement. The Yang-Mills equation on $(P, \phi)$ is simply the flatness condition for $\nabla^{\phi}$. In other words, a solution of the Yang-Mills equation is a unitary $\Gamma$-connection $\nabla$ such that the curvature of the connection $\nabla^{\phi}$ vanishes. (See [Si1] for details.)

Consider a pair $(E, \phi)$, where $\phi$ is a $\Gamma$-Higgs field on the $\Gamma$-linearized vector bundle $E$. We know that $(E, \phi)$ admits a connection satisfying the Yang-Mills equation if and only if $(E, \phi)$ is polystable and $c_1(E) = 0 = c_2(E)$ [Si1], [Si2]. Using this result, Proposition 4.7 generalizes to the situation of $(\Gamma, G)$-Higgs bundles. In other words, a $(\Gamma, G)$-Higgs bundle admits a reduction to a maximal compact subgroup satisfying the Yang-Mills equation if and only if the $(\Gamma, G)$-Higgs bundle is polystable and $c_2$ of the adjoint bundle vanishes. Furthermore, such a connection is unique.

Therefore, Theorem 4.8 extends to the situation of parabolic Higgs $G$-bundles. In other words, a bijective correspondence is obtained between the subset of $\mathrm{Hom}(\pi_1(X - D), G)/G$ consisting of flat $G$-connections on $X - D$ with finite order monodromy around each component $D_i$ of $D$ and the set of parabolic polystable Higgs $G$-bundles $(P_*, \theta)$ such that $c_2^*(P_*(\mathrm{ad})) = 0$.

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