Israel Moiseevich Gelfand (1913–2009)

V S Sunder

In this homage to the late I M Gelfand, we try to give an overview of some of his contributions to the theories of Banach algebras in general and C^*-algebras in particular, and briefly elaborate on the celebrated Gelfand–Naimark–Segal construction which gives a way of passing from positivity-preserving linear functionals to representations on Hilbert space.

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When Gelfand passed away on October 5th, 2009, the world might have seen the last of the classical scholars (in the mould of Henri Poincaré or John von Neumann) whose accomplishments/scholarship were not confined by artificial borders. The wikipedia 'paraphrases' his work thus:

'Israel Gelfand is known for many developments including:

- the Gelfand representation in Banach algebra theory;
- the Gelfand–Mazur theorem in Banach algebra theory;
- the Gelfand–Naimark theorem;
- the Gelfand–Naimark–Segal construction;
- Gelfand–Shilov spaces
- the Gelfand–Pettis integral;
- the representation theory of the complex classical Lie groups;

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- contributions to the theory of Verma modules in the representation theory of semisimple Lie algebras (with I.N. Bernstein and S.I. Gelfand);
- contributions to distribution theory and measures on infinite-dimensional spaces;
- the first observation of the connection of automorphic forms with representations (with Sergei Fomin);
- conjectures about the index theorem;
- Ordinary differential equations (Gelfand–Levitan theory);
- work on calculus of variations and soliton theory (Gelfand–Dikii equations);
- contributions to the *philosophy of cusp forms*;
- Gelfand–Fuks cohomology of foliations;
- Gelfand–Kirillov dimension;
- integral geometry;
- combinatorial definition of the Pontryagin class;
- Coxeter functors;
- generalised hypergeometric series;
- Gelfand–Tsetlin patterns;
- and many other results, particularly in the representation theory for the classical groups.'

When I was requested to write something about Gelfand, my first reaction was to say 'how can I hope to do justice to his breadth and depth of mathematical contributions?' and at once realised that no one can. So I agreed to write the piece, and I shall try to say something about what I am familiar with, namely his work related to operator algebras, as paraphrased in the first four items in the awe-inspiring list above.

As a brief advertisement of the sort of thing that these considerations permit us to do, consider the instance of *fractional derivatives*. To see what the meaning of $D^{\frac{1}{2}}f$ or the 'derivative of order half' of f might be, or what D^a might mean for general a, first review the meaning of t^a for a number t: the assignment $t \mapsto t^a = \mathrm{e}^{a \log t}$ is a meaningfully defined, in fact even continuous function for $t > 0$. What Gelfand's theory achieves is to permit us to make the transition from merely polynomial functions to more general functions $f(T)$ of operators, provided the operators are 'good' and the functions are well-behaved on appropriate domains. The reason D^a can be made sense of is a combination of a couple of things, the first of of which we will not be able to go into: (a) D can be viewed as a 'positive self-adjoint, albeit unbounded', operator; and (b) a 'continuous functional calculus' can be defined for self-adjoint operators.

Operator algebras – which arose at the hands of von Neumann, primarily from considerations of the then new quantum mechanics – are far-reaching generalisations of the familiar algebras of continuous functions. They deal typically with 'non-commutative involutive algebras'. A motivating toy example is provided by the class $M_n(\mathbb{C})$ of $n \times n$ matrices $((a_{ij}))$ where, $a_{ij}, 1 \leq i, j \leq n$ are complex numbers, which is an algebra with the usual definition of linear combinations and product of matrices; the involution is the association $((a_{ij})) = A \mapsto A^* = ((\overline{a_{ji}}))$ to a matrix of its *adjoint* or conjugate transpose matrix. A matrix A satisfying $A = A^*$ is said to be *self-adjoint*. If A is self-adjoint, then the set $C^*(A) = \{\sum_{i=1}^{k} c_k A^k : c_k \in \mathbb{C}, k \in \mathbb{N}\}$ of all polynomials in A is a *commutative* $*$-subalgebra of $M_n(\mathbb{C})$. The celebrated *spectral theorem* says that this entire sub-algebra

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can be *diagonalised* meaning that there exists a *unitary matrix* $U \in M_n(\mathbb{C})$[1] such that UCU^* is a diagonal matrix for every $C \in C^*(A)$.

[1]A matrix U is unitary if $U^*U = UU^* = I$; i.e., $U^* = U^{-1}$.

It is true that (i) any matrix Z admits a (unique) *Cartesian decomposition* $Z = X + iY$, with X, Y self-adjoint, as well as a *polar decomposition* $Z = UP$ with U unitary, and P 'positive' (meaning P can be written as A^2 for some self-adjoint matrix); and that (ii) the above two facts and statements which can be proved easily about commutative $*$-subalgebras of $M_n(\mathbb{C})$ lead to a lot of statements which can be deduced about general matrices.

The generalisation of the last sentence from $M_n(\mathbb{C})$ to more general 'operator algebras' is what gives crucial importance to the understanding of commutative operator algebras, and more generally, to the study of *commutative Banach algebras*,[2] and this is where the Gelfand transform makes an appearance. Taking a cue from the mileage obtained from this strategy by commutative algebraists, Gelfand started studying the (proper) maximal ideals of a commutative Banach algebra. The starting point in the analysis is the striking *Gelfand–Mazur theorem* which asserts that the only complex Banach algebra in which every non-zero element is invertible is the one-dimensional $\mathbb{C}$.

[2] A Banach algebra $\mathcal{A}$ is an algebra which comes equipped with a norm such that (i) $\mathcal{A}$ is complete with respect to the distance coming from the norm, and (ii) $\|xy\| \leq \|x\| + \|y\|$, which implies that multiplication is continuous. We shall assume that $\mathcal{A}$ has a multiplicative identity 1 and that $\|1\|=1$.

The *spectrum* sp x of an element x in a not necessarily commutative Banach algebra $\mathcal{A}$ is defined to be

$$\text{sp } x = \{\lambda \in \mathbb{C} : (x - \lambda.1) \text{ is not invertible}\} .$$

Remark 1. A guiding example to keep in mind is $\mathcal{A} = M_m(\mathbb{C})$, in which case sp T is nothing but the set of eigenvalues of the matrix T. Already in this 'small' example of the matrix algebra, this non-emptiness of the spectrum of all matrices amounts to the quite non-trivial fact that all complex polynomials have complex roots.

Using some complex function theory, which is not surprising in view of the above Remark, one proves that sp x is always a non-empty compact set. The next proposition is an almost immediate consequence of this fact and the Gelfand–Mazur theorem.

PROPOSITION 2.

1. *The following conditions on an* $\mathcal{I} \subset \mathcal{A}$ *are equivalent:*
   - $\mathcal{I}$ *is a (proper) maximal ideal in* $\mathcal{A}$ *– meaning* $\{0\} \neq \mathcal{I} \neq \mathcal{A}$*, and* $x, y \in \mathcal{I}, a \in \mathcal{A}, \lambda \in \mathbb{C} \Rightarrow ax, \lambda x + y \in \mathcal{I}$*;*
   - *there exists a homomorphism* $\phi : \mathcal{A} \to \mathbb{C}$ *of unital complex algebras (i.e.,(a)* $\phi((\alpha x + y)z) = (\alpha\phi(x) + \phi(y))\phi(z)$ *for all* $x, y, z \in \mathcal{A}$ *and* $\alpha \in \mathbb{C}$*, and (b)* $\phi(1) = 1$*) such that*

   $$\mathcal{I} = \ker\ \phi = \{x \in \mathcal{A} : \phi(x) = 0\}\ ;$$

   and the correspondence $\mathcal{I} \leftrightarrow \phi$ *is bijective*

2. *For any* $x \in \mathcal{A}$*, the following conditions on a complex number* λ *are equivalent:*
   - $\lambda \in \text{sp } x$
   - *there exists a complex homomorphism* ϕ *as above such that* $\phi(x) = \lambda$*.*

Remark 3. It should be mentioned that commutativity of $\mathcal{A}$ is crucial for this theorem to be valid. For instance, if $\mathcal{A} = M_n(\mathbb{C}), n \geq 2$, then although part 1 of the above proposition is vacuously true, – since there exist neither proper maximal ideals nor complex homomorphisms – part 2 is totally false.

What is true, however, is that knowledge of the commutative theory does lead to deep insights into the non-commutative world.

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A consequence of part 2 of the previous Proposition and the so-called 'spectral radius formula'[3] is that if ϕ is a complex homomorphism on $\mathcal{A}$, then

$$|\phi(x)| \leq \|x\| \ \forall x \in \mathcal{A} \ ,$$

and consequently, $\|\phi\| \leq 1$.

[3] This says that $\sup\{|\lambda| : \lambda \in \text{sp } x\} = \lim_{n\to\infty} ||x^n||^{1/n}$

The set of all unital complex homomorphisms on a unital commutative Banach algebra $\mathcal{A}$ is called the *spectrum* of A and is denoted by $\hat{\mathcal{A}}$. It is not hard to see that complex homomorphism ϕ satisfies $\phi(1) = 1$ precisely when it is not identically zero. Hence, in case $\mathcal{A}$ does not have a unit, one takes $\hat{\mathcal{A}}$ to be the set of all homomorphisms $\phi : \mathcal{A} \to \mathbb{C}$ which are not identically equal to zero. It is a fact that there exists a unital Banach algebra $\mathcal{A}^1$ and a distinguished homomorphism $\phi_0 \in \widehat{\mathcal{A}^1}$ such that $\mathcal{A}$ can be identified with *ker* ϕ_0.

The remarks of the preceding two paragraphs, taken in conjunction with a result of Alaoglu on *weak** topologies, have the following consequence:

Theorem 4. *Define the* Gelfand transform *to be the assignment, to each x in $\mathcal{A}$, of the function $\Gamma(x) = \hat{x} : \hat{\mathcal{A}} \to \mathbb{C}$ defined by*

$$\hat{x}(\phi) = \phi(x) \ .$$

Then:

- *There is a canonical topology on $\hat{\mathcal{A}}$ such that $\hat{\mathcal{A}}$ is a compact Hausdorff space if $\mathcal{A}$ has an identity, and if $\mathcal{A}$ does not have an identity, then $\widehat{\mathcal{A}}$ is a locally compact Hausdorff space, with one-point compactification identifiable with $\widehat{\mathcal{A}^1}$ (and ϕ_0 playing the role of the point at infinity).*

- *$\Gamma(x)$ is a continuous function on $\hat{\mathcal{A}}$ which 'vanishes at infinity' if $\mathcal{A}$ does not have identity; one says $\hat{x} \in C(\hat{\mathcal{A}})$ or $\hat{x} \in C_0(\hat{\mathcal{A}})$ if $\mathcal{A}$ has or does not have an identity.*

- *$\Gamma : \mathcal{A} \to C(\hat{\mathcal{A}})$ (resp., $C_0(\hat{\mathcal{A}})$) is a contractive Banach algebra homomorphism – meaning that $\Gamma((\alpha x + y)z) = (\alpha\Gamma(x) + \Gamma(y))\Gamma(z)$ and $\|\Gamma(x)\| \leq \|x\|$.*

Thus, the Gelfand transform maps every commutative Banach algebra into an algebra of continuous functions.

In the previous theorem, the algebra operations in the spaces of continuous functions are the obvious pointwise ones, while the norm is the 'sup' norm: $\|f\| = \sup\{|f(x)| : x \text{ in the domain of } f\}$.

Thus, the Gelfand transform maps every commutative Banach algebra into an algebra of continuous functions.

Example 5.

1. The space $\ell^1(\mathbb{Z}) = \{\alpha = ((\alpha_n))_{n\in\mathbb{Z}} : \sum_n |\alpha_n| < \infty\}$ is a Banach algebra with respect to 'convolution product' $\alpha * \beta = \gamma$ defined by

$$\gamma_n = \sum_{k\in\mathbb{Z}} \alpha_k \beta_{n-k}$$

and

$$\|\alpha\| = \sum_n |\alpha_n|.$$

Define δ_n to be the sequence whose only non-zero coordinate is a 1 in the n-th place, and notice that

$$\delta_n \delta_m = \delta_{m+n}$$

and in particular δ_0 is the 1 of $\ell^1(\mathbb{Z})$; further,

$$\alpha \in \ell^1(\mathbb{Z}) \Rightarrow \alpha = \sum_{n=-\infty}^{\infty} \alpha_n \delta_n \ .$$

Any locally compact abelian group *G* possesses an intrinsic (left-) *Haar measure* μ which is (left-) translation invariant.

It follows that if $\phi \in \widehat{\ell^1(\mathbb{Z})}$ and $\alpha \in \ell^1(\mathbb{Z})$, then

$$\begin{aligned} \phi(\alpha) &= \phi(\sum_{n=-\infty}^{\infty} \alpha_n \delta_n) \\ &= \sum_{n=-\infty}^{\infty} \alpha_n \phi(\delta_n) \\ &= \sum_{n=-\infty}^{\infty} \alpha_n \phi(\delta_1^n) \\ &= \sum_{n=-\infty}^{\infty} \alpha_n \phi(\delta_1)^n. \end{aligned}$$

So we have an identification $(\widehat{\ell^1(\mathbb{Z})}) \cong \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ via $\widehat{\ell^1(\mathbb{Z})} \ni \phi \leftrightarrow \phi(\delta_1) \in \mathbb{T}$; and we find that $\hat{\alpha}(z) = \sum_n \alpha_n z^n$ for all $\alpha \in \ell^1(\mathbb{Z})$.

2. More generally than in the previous example, any locally compact abelian group G possesses an intrinsic (left-)*Haar measure* μ which is (left-) translation invariant, meaning $\mu(E) = \mu(sE)$ for all $s \in G$ (where $sE = \{st : t \in E\}$; the associated $L^1(G) = \{f : \int_G |f| \mathrm{d}\mu < \infty\}$ is a commutative Banach algebra with respect to convolution product defined by

$$(f * g)(t) = \int_G f(s) g(t - s) \mathrm{d}\mu(t)$$

and norm $\|f\| = \int_G |f| \mathrm{d}\mu$. It is a fact that if Γ denotes the set $\{\gamma : G \to \mathbb{T} | \gamma$ is a continuous group homomorphism then Γ is a group with respect to the product rule

$$(\gamma_1 \gamma_2)(t) = \gamma_1(t) \gamma_2(t) \ ;$$

and that the equation

$$\phi_\gamma(f) = \int_G f(s) \gamma(s) \mathrm{d}\mu(s)$$

defines a bijective correspondence $\Gamma \ni \gamma \to \phi_\gamma \in \widehat{L^1(G)}$. The locally compact topology on $\widehat{L^1(G)}$ equips Γ with the structure of a locally compact group, and the Gelfand transform

$$\hat{f}(\gamma) = \int_G f(s)\gamma(s)\mathrm{d}\mu(s)$$

is essentially nothing but the classical Fourier transform!

In general, the Gelfand transform need not be 1-1; however one good consequence of this fact comes from an investigation of the best possible situation. To see this, begin by noting that, in addition to being a commutative Banach algebra, the space $C(X)$ (resp., $C_0(X)$) of continuous functions on a compact Hausdorff space (resp., continuous functions vanishing at infinity on a locally compact Hausdorff space) X has the following extra structure:

There exists an involution $f \mapsto f^*$ (where $f^*(x) = \overline{f(x)}$) which satisfies the following properties:

$$\begin{aligned} (\alpha f + g)^* &= \bar{\alpha} f^* + g^* \\ (fg)^* &= g^* f^* \\ (f^*)^* &= f \\ \|f^* f\| &= \|f\|^2. \end{aligned} \tag{1}$$

These axioms are sufficiently important to warrant a definition. A Banach algebra $\mathcal{A}$ is called a C^**-algebra* if it admits an involution satisfying the conditions listed in (1). A possible justification for this definition lies in the following result.

A Banach algebra $\mathcal{A}$ is called a *C*-algebra* if it admits an involution satisfying the conditions listed in equation (1).

Theorem 6. (**Gelfand–Naimark Theorem**). *The following conditions on a Banach algebra $\mathcal{A}$ are equivalent:*

1. *The Gelfand transform is a norm-preserving isomorphism of* $\mathcal{A}$ *onto* $C(\hat{\mathcal{A}})$ *or* $C_0(\hat{\mathcal{A}})$*, according as whether* $\mathcal{A}$ *has an identity or not.*

2. $\mathcal{A}$ *has the structure of a commutative* C^**-algebra.*

Further, in that case, Γ *is automatically an isomorphism of *-algebras (i.e., also* $\Gamma(x^*) = \overline{\Gamma(x)}$*).*

Thus, not only does the Gelfand–Naimark theorem identify the precise mathematical structure possessed by the function algebra $C(X)$ (as that of a commutative unital C^*-algebra), it can also be seen to identify the important notion of a neither necessarily commutative nor unital C^*-algebra. (The astute reader might have suspected the arrival of non-commutative C^*-algebras from our listing (1) of the axioms where we had demanded that the involution reverse products.)

Classic examples of non-commutative C^*-algebras are $M_n(\mathbb{C}), n > 1$, and more generally, (their infinite-dimensional version, given by) the set $\mathcal{L}(\mathcal{H})$ of all continuous linear operators on a Hilbert space $\mathcal{H}$, where the product $AB = A \circ B$ is given by composition of maps, the norm is given by

$$\|A\| = \sup\{\|Ax\| : x \in \mathcal{H},\ \|x\| \leq 1\}\ ,$$

and the involution is given by setting A^* to be the unique element[4] of $\mathcal{L}(\mathcal{H})$ with the property that

$$\langle Ax, y\rangle = \langle x, A^*y\rangle\ \forall x, y \in \mathcal{H}.$$

[4] The existence and uniqueness of such an adjoint operator is a small proposition and an easy consequence of the so-called Riesz lemma which says that the only continuous linear functionals on are of the form $x \mapsto \langle x, z\rangle$ for some (uniquely determined) element $z \in \mathcal{H}$.

More generally, any subalgebra $\mathcal{A}_0$ of $\mathcal{L}(\mathcal{H})$, or of any C^*-algebra $\mathcal{A}$ for that matter, which is self-adjoint and norm-closed (meaning $\{x\}_n \subset \mathcal{A}_0, x \in \mathcal{A}, \|x_n - x\| \to 0 \Rightarrow x, x^* \in \mathcal{A}_0$) is a C^*-algebra in its own right. Each subset S of a C^*-algebra is contained in a smallest C^*-subalgebra $C^*(S)$, which is then said to be 'generated' by S. For instance, $C^*(\{x\})$ is the closure of the set of linear combinations of 'words' in x and x^* (such as $x^*xxx^*x^*xxx$ for instance). In particular, it is not hard to see that $C^*(\{x\})$ is commutative precisely when $xx^* = x^*x$; such elements are said to be *normal*. Thus, examples of normal elements are self-adjoint ($x = x^*$) and unitary ($uu^* = u^*u = 1$) elements. It is a pleasant consequence of the Gelfand–Naimark theorem that if x is normal, then $\widehat{C^*(\{x\})} \cong \text{sp } x$ and that $C^*(\{x\}) \cong \{f \in \{C(\text{sp } x) : f(0) = 0\}$. Thus the Gelfand–Naimark theorem gives vital information – such, for instance, as contained in part 3 of the next proposition – about the C^*-subalgebras generated by normal elements, by letting us deal with normal elements as comfortably as with functions.

By the way, one of the first facts that one proves about C^*-algebras that is not a consequence of the commutative theory (the point being that x may not commute with the z of (2) below) is this statement regarding positivity:

PROPOSITION 7.

The following conditions on an element $x \in \mathcal{A}$ are equivalent:

1. *x is self-adjoint and* $\text{sp } x \subset [0, \infty)$.
2. *There exists some $z \in \mathcal{A}$ such that $x = z^*z$.*
3. *There exists a self-adjoint $y \in \mathcal{A}$ such that $x = y^2$;.*

Such an x is said to be positive *and we write $x \geq 0$ or $x \in \mathcal{A}_+$ to indicate this fact.*

Further, the self-adjoint square-root y may be chosen to be positive, and such a positive square root of x is unique.

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Just as the Gelfand–Naimark theorem identifies commutative unital C^*-algebras as algebras of continuous functions on a compact Hausdorff space, there is a 'non-commutative Gelfand–Naimark theorem' which shows that any C^*-algebra $\mathcal{A}$ is isomorphic to a C^*-subalgebra of some $\mathcal{L}(\mathcal{H})$.

functions on a compact Hausdorff space, there is a 'non-commutative Gelfand–Naimark theorem' which shows that any C^*-algebra $\mathcal{A}$ is isomorphic to a C^*-subalgebra of some $\mathcal{L}(\mathcal{H})$ – which is another way of saying that $\mathcal{A}$ admits a faithful (= 1-1) representation on a Hilbert space $\mathcal{H}$ (= a *-homomorphism into $\mathcal{L}(\mathcal{H})$). The key is to find a way to construct one, and then enough of them, and finally a faithful one. This comes from an ingenious adaptation of integration theory to the non-commutative context.

Let us see how to construct a representation of $C[0,1]$, for instance. The first Hilbert space one can think of in connection with $[0,1]$ is the Hilbert space $L^2([0,1])$ which may be thought of as the completion of $C[0,1]$ with respect to the norm given by the inner-product

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\mathrm{d}x \ ;$$

and there is a most natural representation $\pi : C([0,1]) \to \mathcal{L}(L^2([0,1])$, given by

$$(\pi(f)\xi)(x) = f(x)\xi(x) \ ,$$

where elements $\xi \in L^2([0,1])$ are viewed as (measurable and) square-integrable functions on $[0,1]$. The fact that all this goes over perfectly, almost verbatim, to the case of any C^*-algebra, is the content of the immensely useful *Gelfand–Naimark–Segal construction.*

The bridge needed to make this transition is provided by the celebrated Riesz representation theorem which may be stated thus:

Theorem 8 (**Riesz Representation Theorem**). *The following conditions on a linear functional ϕ on $C(X)$ (X compact Hausdorff) are equivalent:*

1. *ϕ preserves positivity: $f \geq 0 \Rightarrow \phi(f) \geq 0$;*

2. *there exists a finite positive measure* μ *defined on (the Borel sets in)* X *such that*

$$\phi(f) = \int_X f \mathrm{d}\mu \ ;$$

3. ϕ *is a continuous, i.e. bounded, linear functional on* $C(X)$ *and* $\|\phi\| = \phi(1)$ *(where 1 denotes the identity of* $C(X)$*).*

Call a linear functional ϕ on a C^*-algebra *positive* if $\phi(z^*z) \geq 0 \ \forall z \in \mathcal{A}$. Given such a ϕ, the equation $\langle x, y\rangle_\phi = \phi(y^*x)$ is seen to define a 'semi-inner-product' on $\mathcal{A}$[5]; and consequently the Cauchy–Schwarz inequality is valid:

$$\begin{aligned} |\phi(y^*x)|^2 &= |\langle x, y\rangle_\phi| \\ &\leq \langle x, x\rangle_\phi \langle y, y\rangle_\phi \\ &= \phi(x^*x)\phi(y^*y) \ . \end{aligned} \tag{2}$$

[5] This means that $\langle x, y\rangle_\phi$ is linear in x and conjugate-linear in y (so $\langle \Sigma_i \alpha_i x_i, \Sigma_j \beta_j y_j\rangle_\phi = \Sigma_i \alpha_i \Sigma_j \bar{\beta}_j \langle x_i, y_j\rangle_\phi$ and satisfies $\langle x, x\rangle_\phi \geq 0 \forall x$.

Putting $y = 1$ in (2) yields

$$\begin{aligned} |\phi(x)|^2 &\leq \phi(x^*x)\phi(1) \\ &\leq \|x^*x\|\phi(1)^2 \\ &\leq \|x\|^2\phi(1)^2 \end{aligned}$$

thereby establishing that

$$\phi \geq 0 \Rightarrow \|\phi\| = \phi(1) \tag{3}$$

even for non-commutative C^*-algebras.

A second useful consequence of (2) is that for $x \in \mathcal{A}$, we have

$$\phi(x^*x) = 0 \Leftrightarrow \phi(y^*x) = 0 \ \forall y \in \mathcal{A} \ ;$$

hence the so-called radical

$$\mathrm{Rad}(\phi) = \{x \in \mathcal{A} : \phi(x^*x) = 0\}$$

of a positive linear functional is always a left-ideal in $\mathcal{A}$.

This has two consequences:

1. The quotient vector space $\mathcal{A}/\mathrm{Rad}(\phi)$ has a genuine inner product given by

$$\langle \hat{x}, \hat{y} \rangle = \phi(y^* x)$$

where we write $\hat{z}$ for the coset $z + \mathrm{Rad}(\phi)$.

2. For each $x \in \mathcal{A}$, the equation $\lambda(x)\hat{y} = \widehat{xy}$ yields an unambiguously defined linear transformation $\lambda(x)$ on $\mathcal{A}/\mathrm{Rad}(\phi)$.

We may summarise the conclusions of this Gelfand–Naimark–Segal (GNS, for short) construction thus:

Theorem 9. *Let ϕ be a positive functional on a C^*-algebra. Then there exists a representation π of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ and a vector $\xi \in \mathcal{H}$ such that*

$$\phi(x) = \langle \pi(x)\xi, \xi \rangle \ \forall x \in \mathcal{A}.$$

Proof. Let $\mathcal{H}$ be the Hilbert space completion of the inner-product space $\mathcal{A}/\mathrm{Rad}(\phi)$. We first assert that the equation in item (2) above defines a bounded operator $\lambda(x)$ on $\mathcal{A}/\mathrm{Rad}(\phi)$ and hence extends uniquely to a bounded operator $\pi(x)$ on $\mathcal{H}$. For this we begin by noting that if, for fixed $y \in \mathcal{A}$, we define $\phi_y(z) = \phi(y^* z y)$, then, $\phi_y \geq 0$ since $\phi_y(x^* x) = \phi(y^* x^* x y) = \phi((xy)^*(xy)) \geq 0$. Hence by an application of (3) to ϕ_y, we find that

$$\begin{aligned} |\phi(y^* z y)| &= |\phi_y(z)| \\ &\leq \phi_y(1)\|z\| \\ &= \|z\|\phi(y^* y) \ ; \end{aligned} \tag{4}$$

and it follows that

$$\begin{aligned} \|\lambda(x)\hat{y}\|_\phi^2 &= \|\widehat{xy}\|_\phi^2 \\ &= \phi(y^* x^* x y) \\ &\leq \|x^* x\|\phi(y^* y) \quad \textit{by (4)} \\ &= \|x\|^2 \|\hat{y}\|_\phi^2 \ . \end{aligned}$$

Hence indeed $\lambda(x)$ is a bounded operator on $A/\mathrm{Rad}(\phi)$ (of norm at most $\|x\|$), and hence extends uniquely to a bounded operator $\pi(x)$ on $\mathcal{H}$.

If we set $\xi = \hat{1}$, then, by definition $(\pi(\mathcal{A})\xi) = \{\hat{x} : x \in \mathcal{A}\}$ is dense in $\mathcal{H}$. (One says ξ is a cyclic vector for the representation π.) Also, note that for any $x, y, z \in \mathcal{A}$, we have

$$\begin{aligned} \langle \pi(x)\hat{y}, \hat{z}\rangle &= \phi(z^*xy) \\ &= \phi((x^*z)^*y) \\ &= \langle \hat{y}, \pi(x^*)\hat{z}\rangle \end{aligned} \tag{5}$$

and the density assertion in the first sentence of this paragraph permits us to conclude that $\pi(x)^* = \pi(x^*)$.

A simple reasoning using this density in a similar fashion serves to verify that π respects the algebra operations, and consequently that π is indeed a representation. Finally, setting $y = z = 1$ in (5) yields

$$\langle \pi(x)\xi, \xi\rangle = \phi(x)$$

and all parts of the theorem are proved. □

In the case of $\mathcal{A} = C([0,1])$, with $\phi(f) = \int_0^1 f(x)\mathrm{d}x$, we see that $\mathrm{Rad}(\phi) = 0$. Such a positive functional is said to be *faithful*. If ϕ were such a faithful positive functional, then the representation obtained from the associated GNS construction would also be faithful (i.e., one-to-one). (Reason: $\pi(x) = 0 \Rightarrow \phi(x^*x) = \|\pi(x)\xi\|^2 = 0 \Rightarrow x = 0$. It is a fact that any separable C^*-algebra admits a faithful state; and one can fairly easily deduce the following *non-commutative Gelfand–Naimark Theorem*, at least in the separable case.

Theorem 10. *Any C^*-algbera $\mathcal{A}$ admits a faithful representation, and is thus isomorphic to a C^*-subalgbera of some $\mathcal{L}(\mathcal{H})$. In case $\mathcal{A}$ is separable, the Hilbert space $\mathcal{H}$ can be chosen to be separable.*

Suggested Reading

[1] V S Sunder, *Functional Analysis: Spectral Theory*, TRIM Series No.13, Hindustan Book Agency, Delhi, 1997; International edition: Birkhaüser Advanced Texts, Basel, 1997.

[2] Wikipedia:http://en.wikipedia.org/wiki/Israel-Gelfand.

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