Existence and reality in mathematics and natural science*

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It must appear somewhat presumptuous of me to venture to speak on the topic mentioned in the title of this lecture. No less a figure than Albert Einstein had once said:

'Behind the tireless efforts of the investigator there lurks a stronger, more mysterious desire: it is existence and reality that one wishes to comprehend. But one shrinks from the use of such words, for one soon gets into difficulties when one has to explain what is really meant by "reality" and by "comprehend" in such a general statement . . .'.

Nevertheless, realizing full well that I am no professional in these matters, and running the risk of often being naive, let me proceed.

From ancient times to the present there have been many deep and learned discussions on this subject. Many shades of reality have been discerned. Both philosophers and scientists, particularly physicists, have contributed to the debate. It also happens that with each major phase of development in physics, new vistas and perspectives open up, which could not have been imagined previously. Our picture of reality changes constantly as we learn from nature. In recent times, to mention just a few names, thinkers like Reichenbach, Margenau and d'Espagnat, and in their general writings Schrödinger and Heisenberg, have expressed themselves on these matters. In my own limited attempt I will try to say something about physics, something about mathematics, and then of their links. Some points will be highlighted, and more questions raised than answered.

We can trace the origins of mathematics to human and social needs and experiences in antiquity. Thus numbers and arithmetic most likely grew out of trade, barter and commerce; and geometry out of the need for land survey, especially after the annual floods of the Nile. To some degree at least – and this will be elaborated later – this capacity to create and use mathematical concepts can be traced to features of the brain developed during phylogenesis – the long period of evolution of our species – subject to the pressures of natural selection while interacting with nature. The origins of the realization that mathematics is also useful in the description of natural phenomena is also very old – one thinks of the integer relationships discovered by Pythagoras among harmonics and overtones in music. In the modern sense, the crucial role of mathematics in understanding nature was first clearly stated by Galileo[1] in 1623:

'Philosophy is written in this very great book which always lies open before our eyes (I mean the universe), but one cannot understand it unless one first learns to understand the language and recognize the characters in which it is written. It is written in mathematical language . . . ; without these means it is humanly impossible to understand a word of it. . . .'

Around this time came Kepler's three laws of planetary motion and Galileo's laws for the motion of falling bodies. We must also remember that Galileo was the first one to measure time intervals of the order of a second reliably; and the person who first realized that in describing natural phenomena it is most fruitful to regard time as the basic independent variable, while other quantities varied *with respect to* time. This link between mathematics and natural phenomena has been expressed in various ways with increasing eloquence and persuasiveness over the centuries. From the present century I would like to present a few examples; first of course Einstein again:

'Our experience . . . justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas.'

Actually this seems a bit over simplified! Then in his James Scott Prize Essay of 1939 Paul Dirac says that there is *no logical reason* why mathematics should be so useful in describing nature but offers this thought[2]:

'This must be ascribed to some *mathematical quality in Nature*, a quality which the casual observer of Nature would not suspect, but which nevertheless plays an important role in Nature's scheme.'

In a famous 1959 lecture titled 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences', Eugene Wigner says[3]: '. . . the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and . . . there is no rational explanation for it'. And later after commenting on the unbelievable accuracy with which certain predictions of theory have been verified he continues: 'This shows that the mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language.'

Lastly I turn to Richard Feynman from the next generation[4]: 'Every one of our laws is a purely mathe-

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matical statement in rather complex and abstruse mathematics.... It gets more and more abstruse and more and more difficult as we go on.... It is impossible to explain honestly the beauties of the laws of nature in a way that people can feel, without their having some deep understanding of mathematics.'

This passage emphasizes that as time passes, the kind of mathematics used in fundamental physics gets continually more abstract and sophisticated.

It is illuminating to recall the ways in which mathematics and physics have advanced in the past, sometimes hand in hand, sometimes out of step. As we all know, the differential calculus and ordinary differential equations grew out of the needs of Newtonian mechanics, and also the problem of determining tangents to curves. Later general function theory and partial differential equations developed alongside continuum mechanics, heat conduction, wave phenomena and electromagnetism. But also during this period in the 19th century there were some basic advances internal to mathematics itself, whose uses in physics came much later. Here one thinks of the discovery of noneuclidean geometries, solving a long standing 'crisis' within mathematics; the whole idea of groups born out of the problem of solving higher order algebraic equations; and the theory of matrices and noncommutative algebra[5]. Noneuclidean geometry proved essential decades later in the relativistic theory of gravitation. Then in the hands initially of Felix Klein and Sophus Lie, and later of Hermann Weyl and Wigner, group theory became the perfect language for expressing symmetry in nature, especially symmetries of natural laws. This is so in both classical and quantum mechanics. Coming to matrices, we all know the role it played in various parts of classical physics, but more importantly its decisive importance in quantum mechanics. Here we should recall that when Heisenberg discovered matrix mechanics he did not know what matrices were, but essentially reinvented them and their law of multiplication guided by the Ritz Combination Law of Spectroscopy!

During this century there have been instances of independent but parallel and practically simultaneous conceptual advances in physics and in mathematics, and I mention a few. The theory of Hilbert spaces and linear operators came in just around the time quantum mechanics was developed, and this was used by von Neumann to put quantum mechanics into a definitive mathematical form already around 1932. Then the gauge idea of Weyl of 1918, through the theory of the Dirac monopole and onto abelian and nonabelian gauge theories in physics more or less paralleled the growth of the mathematical theory of principal fibre bundles and connections. In the case of the Dirac delta function, the physicist's use of this concept was well ahead of the mathematical theory of distributions; and with the Feynman path integral too, physics seems ahead of mathematics.

All this raises many deep questions. Because of these developments and advances – sometimes hand in hand, sometimes out of step, sometimes independent – we are led to ask: Is mathematics a part of nature in some sense, existing independently of us, or is it a human creation? Why is it so useful in describing nature? Is it merely a language for communication among human beings or is it much more than that? Is the human mind or brain predisposed to create and recognize mathematical ideas? In trying to cope with some of these questions we will soon see the importance and relevance of our understanding of biological evolution.

But before getting into heavy stuff some light-hearted comments are appropriate. At a conference on theoretical physics in 1967, C. N. Yang recounted this story: Some one asked a mathematician the reason for the great advances in mathematics during this century, and the reply was – it has finally freed itself from physics! Similarly, Yang said, one day people will say the tremendous advances in theoretical physics are due to the fact that it has become free of experimental physics. It is remarkable that he had anticipated so well the trend of string theory so much in advance!

Now back to our subject. First something about the nature of mathematics. From long times past, two opposing views have been held – the realist or Platonic view, and the constructivist view[6]. (For the sake of completeness one should also mention the formalist school of thought, but for the present purposes the two views mentioned will suffice.) In the Platonic view, there is a World of Ideas, of perfect forms, and things in the material world are only imperfect copies of them. Mathematics exists at this level of ideas, perfect embodiments of concepts; a world of ideal things which reason can reach and which somehow controls real objects. Ethical principles also belong – for Plato – to this world of ideas. From the constructivist point of view, in contrast, mathematics exists only in human minds; it has been created by us in response to experience and our needs. Obviously there is no clear cut resolution, and over the ages there have been distinguished mathematicians of both persuasions. Among the Platonists we count Descartes, even Newton and Leibnitz, Charles Hermite, Cantor and Godel and many others; and prominent among the constructivists are Poincaré and Brouwer. Gauss, the Prince of mathematicians, seems to have been a constructivist in relation to arithmetic, and an empirical scientist with respect to geometry. Indeed, soon after the discovery of noneuclidean geometries he declared:

'I am profoundly convinced that the theory of space occupies an entirely different position with regard to our knowledge a priori from that of arithmetic; that perfect conviction of the necessity and therefore the absolute truth which is characteristic of the latter is totally wanting in our knowledge of the former. We must confess, in all humility, that number is solely a product of our mind. Space, on the other hand, possesses also a reality outside our mind, the laws of which we cannot fully prescribe a priori.'

Many expressions of the Platonic point of view can be presented, and I give a couple. Charles Hermite, the teacher of Poincaré, says:

'I believe that the numbers and functions of analysis are not the arbitrary product of our spirits. I believe that they exist outside of us with the same character of necessity as the objects of objective reality; and we find or discover them and study them as do the physicists, chemists and zoologists.'

From across the channel, in England, G. H. Hardy put it this way[7]:

'I believe that mathematical reality lies outside us, and that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations', are simply our notes of our observations' ... '317 is a prime number, not because we think it is so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way.'

And many other such declarations can be found. For the opposite view, I quote only Poincaré:

'A reality completely independent of the spirit that conceives it, sees it or feels it, is an impossibility. A world so external as that, even if it existed, would be forever inaccessible to us.'

At this point I would like to draw attention to a recent and very stimulating book – *Conversations on Mind, Matter and Mathematics* – a dialogue between the French neurophysiologist Jean-Pierre Changeux and the French mathematician Alain Connes, dealing with just these issues[8]. Each one puts forward his position repeatedly and forcefully, and still after a few hundred pages of dialogue there is no reconciliation. The neurophysiologist says that a small number of fundamental brain processes leads to all human languages and to mathematics. Mathematical concepts exist materially in the brain, and correspond to specific brain states. Faculties of logic and reasoning are directly linked to the organization of the brain, which has been so since Homo erectus 400,000 years ago. The ability to create new mathematical objects is a capacity of the brain; later it analyses them and proves theorems. Mathematical objects do not 'exist somewhere in the universe', independent of material cerebral support. The development of mathematics in stages shows a historical progression, it is a result and part of cultural progress. Even the axiomatic method is a brain faculty, and sensitivity to mathematics is a product of the brain. Finally, he declares[9]: 'I deny the existence of a mathematical reality prior to our experience of it. The coherence of mathematics seems to me a posteriori, rather than a priori, the result quite simply of its noncontradiction....'

How does the mathematician respond? He says with equal passion that mathematical reality exists independent of the human brain, distinct from how we come to know of it and understand it. We form mental images of it as we progress, but it is always there, independent of us. Just as the reality of the material world is the result of coherence and commonality of our perceptions of it via the senses, the same holds for mathematics too. There exists, independent of us, a raw immutable mathematical reality, distinct from our tools to explore it; distinct also from what we know of it at any given time; distinct even from the mathematical regularities which we find in natural phenomena. Mathematical reality is *outside* space and time, *outside* physical reality; and our sensitivity to it is distinct from sight, touch and hearing. *It is not located in the physical world.* He sums this up in these words[10]: 'It is humility, finally, that forces me to admit that the mathematical world exists independently of the manner in which we apprehend it, that it is not localized in space and time. The evolution of our perception of mathematical reality causes a new sense to develop, which gives us access to a reality that is neither visual, nor auditory, but something else altogether.'

You can see how strongly opposed the two points of view are, and why there is no meeting ground. At the end of the dialogue each remains unconvinced of the other's position, and they agree to disagree. You can also appreciate the reason for the mathematician's position, even if it is difficult to accept at first encounter. There is a psychological aspect here too – once Hermann Weyl admitted that believing in a pre-existing mathematical world helped and motivated him to pursue more substantial problems than otherwise!

Time now for a brief detour into philosophy and biology, and then back to physics. Towards the end of the 18th century, the philosopher Immanuel Kant attempted an explanation of the great successes of Galilean–Newtonian physics along the following lines. He said that as experience of the outside world comes into our minds through the senses, there are certain innate ways in which we process this information, certain set patterns for the functioning of the brain. Thus we 'see' nature only through certain 'filters', only according to certain pre-existing patterns, and these are called the synthetic a priori truths. Here 'a priori' means 'in advance of experience'; and *synthetic*, as opposed to *analytic*, means that these truths have nontrivial content. Though the contrary could be *imagined*, it is one way and not the other. Thus 'synthetic' means: not a consequence of logic alone, not a product of pure reason. Among the synthetic a priori Kant included the Euclidean geometry of space, uniformly flowing time, the law of causality, and in later versions even the law of mass conservation and Newton's Third Law of Motion. For Kant, there was no way in which these principles could be ever violated, since they were a precondition of all interpretation of experience. Briefly, some of the empirical successes of Galilean–Newtonian physics were made into inevitable features of our understanding of nature.

Soon after Kant's time, though, things changed quite dramatically. First there was the discovery of non-euclidean geometry; and this is what led to Gauss' statement quoted earlier that the theory of space is not knowledge a priori. Later developments in physics – special and general relativity, and quantum mechanics – seemed to undermine Kant's position further. In particular, geometry of space and time became part of empirical science.

During this century, on the basis of Darwinian evolution and the efforts of Konrad Lorenz and later Max Delbruck, there has been a reinterpretation and a better understanding of Kant's ideas in ways which were not available to him in his time[11]. The key point is that in their struggle for survival according to the principles of natural selection, different species have to learn to cope with natural phenomena taking place roughly at their own scales of length, mass and time. This is the 'world of middle dimensions'. These are the phenomena directly relevant in a biological evolutionary sense; and natural selection favours the development of those faculties which are able to pick out the most important physical features of this part of nature. And what is the result of long and slow learning through evolution for a species as a whole, lasting hundreds of thousands of years, seems to the individual member of the species as a priori, as knowledge, or better as capacity for knowledge, he is born with in advance of experience. The key sentence of Delbruck is: 'What is a priori for the individual is a posteriori for his species.'

Naturally this new meaning given to Kantian ideas has a corollary – as we move away from the world of middle dimensions, into phenomena involving the very large or the very small or the very rapid, we must be prepared for departures from the intuitive notions which are our biological heritage. This is indeed what happens in physics away from the everyday world, and here mathematics is the main guide to replace ordinary intuition.

Now how does mathematics fare in this situation? The idea is that *it* comes from the analytic a priori component of human knowledge. In his 1930 Konigsberg lecture titled 'Logic and the understanding of Nature' the mathematician David Hilbert – who like Kant was a 'son of Konigsberg' – tackled precisely this problem. He says[12]:

'I admit that even for the construction of special theoretical subjects certain a priori insights are necessary. . . . I even believe that mathematical knowledge depends ultimately on some kind of such intuitive insight. . . . Thus the most general basic thought of Kant's theory of knowledge retains its importance. . . . The a priori is nothing more or less than . . . the expression for certain indispensable preliminary conditions of thinking and experiencing. But the line between that which we possess a priori and that for which experience is necessary must be drawn differently by us than by Kant – Kant has greatly overestimated the role and the extent of the a priori'. . . . 'We see now: Kant's a priori theory contains anthropomorphic dross from which it must be freed. After we remove that, only that a priori will remain which also is the foundation of pure mathematical knowledge.'

Admittedly, the insights of Lorenz and Delbruck were unavailable to Hilbert. Nevertheless one can appreciate that while the synthetic a priori is related to our understanding of natural phenomena in the world of middle dimensions, the analytic a priori is the source of mathematical knowledge. Hilbert like Poincaré was a constructivist. For him, mathematics was a human creation. Ways of logical thinking, of working out consequences of assumptions, arguments, the notion of consistency – all these are also the results of biological evolution, when faced with a world having mathematical qualities, and subject to laws governing its behaviour.

However, behind this level of understanding stands another mystery. Why does the human brain have so many faculties, so much more capacity, than seems necessary for biological survival? We can surely understand the biological advantage of the ability to construct language for communication – but why should it go so far as to create drama and poetry where elementary communication may have been enough? And why this marvellous capacity to think of mathematical objects of great depth and structure, where rudimentary arithmetic and some little geometry may have sufficed? As Wigner says[3]: '. . . it is hard to believe that our reasoning power was brought, by Darwin's process of natural selection, to the perfection which it seems to possess.'

I once asked a biologist this question and he said it was all an 'emergent phenomenon' – a brain suddenly evolved with all these capacities, which could not be understood in terms of its parts but only as a whole. But this is not a satisfying answer in the present context; it is like Churchill's description of the erstwhile Soviet Union as a mystery wrapped in an enigma! In the same vein one can ask: granted that mathematics is of some use in the formulation of natural laws, why does it then turn out so fantastically accurate, so much more so than we could have reasonably expected? This was just the question behind Wigner's essay recalled earlier[3].

At the same time, though, another view of all this is possible. Could we understand a nature which was somehow only partly mathematical and not so all the way? Within fundamental physics we are familiar with the argument that we cannot have classical physics valid over some domain and then quantum physics for the rest. We should have a unified and common pattern, or else its absence would become a problem calling for resolution!

Now let us return to the relationship between mathematics and physics. I mentioned earlier that the level of sophistication and abstractness here keeps continually increasing. However one should also realize that for his time, the creation of the concept of acceleration by Galileo was a gigantic achievement. In contrast to the first derivative or rate of change, the second derivative is highly nonintuitive and far removed from immediate experience. As Wigner put it[3], '... those of us who have tried to draw an osculating circle to a curve know that the second derivative is not a very immediate concept'. Here Wigner probably also implies that not many of us have tried to do this exercise! Be that as it may, in the early phase of classical physics the guiding principle can in retrospect be seen to be the simplicity of the mathematical concepts used in physics. But as further progress and elaboration occurred, there was a gradual transition and the quest for simplicity gave way to a quest for beauty. Dirac put it this way[2]: after relativity and even more so after quantum theory, '... we now see that we have to change the principle of simplicity into a *principle of mathematical beauty*.'

This is of course subjective, but both Dirac and the mathematician Hardy, in their respective contexts, justify it in remarkably similar words[2,13]:

'This is a quality which cannot be defined, any more than beauty in art can be defined, but which people who study mathematics usually have no difficulty in appreciating.'

'It may be very hard to *define* mathematical beauty, but that is just as true of beauty of any kind – we may not know quite what we mean by a beautiful poem, but that does not prevent us from recognizing one when we read it.'

One must only add that with some modern poetry there may be difficulties.

Parallel with this development, the notions of existence and reality in physics too have evolved and increased greatly in subtlety. During the 19th century, the success of Galilean–Newtonian physics led to a purely mechanical view of nature; and in Lord Kelvin's words: 'It seems to me that the test of "do we or do we not understand a particular point in physics?" is "Can we make a mechanical model of it?"'

As we all know, it took a lot of effort to accept electric and magnetic fields on their own terms, as primitive constituents of nature not reducible to matter. Even Maxwell on many occasions sought for explanations for them in terms of gears and wheels. Then with the coming of relativity and quantum mechanics, things became more abstract, and the constituents of the picture of nature became much more refined than before. The ideas of transformations and symmetry have gained enormous importance, so much so that Dirac once said[2]: '... both relativity and quantum theory ... show that transformations are of more fundamental importance than equations'. This is already true in classical physics, wherein both special and general relativity physically important quantities are often partly *defined* by the way they change under a given family of transformations. This becomes even more pronounced in quantum theory where physical quantities are defined by the transformations they generate and by their behaviours under them.

So we are led to ask: at what level do the ideas of transformations and symmetry exist in nature? It is already difficult to answer questions like: even though they are admittedly approximate, are Newton's equations and Maxwell's equations parts of nature? Granted that this is a naive and overly simplistic question, but they are good approximations to the workings of nature in certain domains. Subject to this qualification, do they not 'exist' one step behind the specific phenomena that obey them? Now what of their symmetries and those of other laws? Do they stand two steps behind phenomena? Natural laws unify individual phenomena or occurrences in nature, each obeying the laws concerned. In turn, transformations and symmetries bring together different laws sharing the same symmetry, and so belong to an even deeper level of functioning of nature.

After all that has happened since the 19th century, we should surely not tie ourselves down to a mechanical view of nature, or ultimately to a view based on sense impressions alone. It is true that our normal unaided senses are both quantitatively and qualitatively limited, and scientific instruments extend their range and accuracy enormously. Even so they can reach out only to space, time and their contents. Our intuitive 'middle-world' feeling is that all that exists must be within space and time. But now we must be willing to go beyond this and face up to the question: are these subtle ideas and aspects, where mathematics is the main guide, also present in nature somehow? In the general framework of quantum mechanics, we use so many mathematical objects and operations – noncommuting quantities for physical variables, spinors, group representations, Grassmann numbers, operators for electric and magnetic fields and so on. Is it not legitimate to ask: in what manner do they exist in nature? More pointedly: *is there an aspect to nature not accessible to the ordinary senses but only to a mathematical sensitivity?* Is there a level of functioning of which we cannot form any conventional mental picture except as governed by equations, transformations, symmetry and mathematical consistency?

I honestly feel a retreat into a point of view ignoring all these 'intangible' elements of the mathematical description of nature, hanging on only to the results of calculations and experimental predictions, is not available to us. That would be no way to judge the situation. Long ago in 1910–11, while Einstein was teaching a course on electromagnetic theory, he made this remarkable comment[14]:

'We set up a conceptual system the individual parts of which do not correspond directly to empirical facts. Only a certain totality of theoretical material corresponds again to a certain totality of experimental facts.'

In the same spirit, when we judge theories of physics, even though they are incomplete and evolving, we must judge them in their wholeness and taking account of their mathematical structures. Here I cannot avoid quoting from Heisenberg's[15] recollection of his feelings on the night he discovered matrix mechanics in June 1925:

'I had the feeling that, through the surface of atomic phenomena, I was looking at a strangely beautiful interior, and felt almost giddy at the thought that I now had to probe this wealth of mathematical structure nature had so generously spread out before me.' So the mathematical structure itself is a part of nature!

Mathematics then is a language but a special one because it is the language of nature, it is also a method of reasoning, and the least ambiguous mode of communication we possess. Through phylogenetic evolution we have acquired the capacity to create mathematical objects. In this way we acquire a new sense independent of and in addition to the other ordinary ones, and so we reach aspects of nature beyond space and time. This should make us sympathize with the Platonic view of mathematics as well. On the other hand, as in other respects recounted above, we have got more than we could have anticipated. Once created by us, mathematical objects take on a life of their own. The consequences of initial ideas are not all evident at the beginning but are revealed only after much effort. There is more in them than was consciously put in.

Strange as it may initially seem and hard as it may be to accept, we seem driven to this conclusion – because mathematics is essential to describe nature, we have to adopt a more open view of existence and reality going beyond space, time and the tangible. The problem of existence and reality is much subtler than our naive expectations may have been. Mathematics then, like nature, has also an intangible level of existence. This line of thinking seems to bring us close to the point of view expressed by Connes and recalled earlier. So, have we solved the problem of the existence of mathematics by saying that it is the language of natural science? No, not quite. There remains a gap, this connection between mathematics and its use in natural science goes only a certain distance and does not cover all aspects of the situation! Some differences remain since, as far as one can see, we can imagine mathematical objects and structures which may not have counterparts in nature. Thus as the mathematician Atle Selberg says[16]:

'If one looks at mathematics as a body of knowledge, I think it definitely can be characterized as a science, but if one looks at the way in which it grows and accumulates, the actual doing of mathematics seems much more to be an art.'

And for the physicist's expression of the difference, I conclude with C. N. Yang[17]:

'It would be wrong, however, to think that the disciplines of mathematics and physics overlap very much; they do not. And they have their separate aims and tastes. They have distinctly different value judgements, and they have different traditions. At the fundamental conceptual level they amazingly share some concepts, but even there, the life force of each discipline runs along its own veins.'

So at the end we just have to say: it really seems like a case of 'so near and yet so far apart!'.

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2. Dirac, P. A. M., *Proc. R. Soc. (Edinburgh)*, 1939, **59**, 122.
3. Wigner, E. P., *Commun. Pure Appl. Math.*, 1960, **13**, No. 1; reprinted in Wigner, E. P., *Symmetries and Reflections – Scientific Essays*, OX Bow Press, Woodbridge, Connecticut, USA, 1979, pp. 222.
4. Feynman, R. P., *The Character of Physical Law*, Penguin Books, 1992, pp. 39–40.
5. For a very readable account of some of these developments see Yaglom, I. M., *Felix Klein and Sophus Lie – Evolution of the Idea of Symmetry in the Nineteenth Century*, Birkhauser, Boston, 1988.
6. Many of these matters are discussed in John D. Barrow, *Pi in the Sky – Counting, Thinking, and Being*, Clarendon Press, Oxford, 1992, as well as in ref. 8 below.
7. Hardy, G. H., *A Mathematician's Apology*, Canto Books, Cambridge University Press, 1992, pp. 123–124, 130.
8. Jean-Pierre Changeux and Alain Connes, *Conversations on Mind, Matter and Mathematics*, Princeton University Press, 1995.
9. Ref. 8, p. 121.
10. Ref. 8, p. 39.
11. Konrad Lorenz, *The Natural Science of the Human Species: An Introduction to Comparative Behaviour Research – The 'Russian Manuscript, 1944–48'* (ed. Agnes von Cranach) transl. Robert D. Martin, M.I.T. Press, 1996, reviewed by Chandrashekaran, M. K., *Curr. Sci.*, 1997, **72**, 83; Max Delbruck, *Mind from Matter? An Essay on Evolutionary Epistemology*, Blackwell Scientific, 1986.
12. Constance Reid, *Hilbert*, Springer, New York, 1972, pp. 190.
13. Ref. 7, p. 85.
14. *The Collected Papers of Albert Einstein, The Swiss Years: Writings, 1909–1911*, English Translation, Princeton University Press, 1993, vol. 3, pp. 259.
15. Heisenberg, W., *Physics and Beyond – Encounters and Conversations*, Allen & Unwin, London, 1971, p. 61.
16. Selberg, A., *Collected Papers*, Springer, 1989, vol. 1, pp. 695.
17. Yang, C. N., in *To Fulfill a Vision – Jerusalem Einstein Centennial Symposium* (ed. Neeman, Y.), Addison-Wesley, 1981.