Fundamentals of the LISA Stable Flight Formation

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Abstract. The joint NASA-ESA mission LISA relies crucially on the stability of the three spacecraft constellation. Each of the spacecraft is in heliocentric orbits forming a stable triangle. The principles of such a formation flight have been formulated long ago and analysis performed, but seldom presented if ever, even to LISA scientists. We nevertheless need these details in order to carry out theoretical studies on the optical links, simulators etc. In this article, we present in brief, a model of the LISA constellation, which we believe will be useful for the LISA community.

1. Introduction

LISA is a ESA-NASA mission for observing low frequency gravitational waves in the frequency range from 10^{-5} Hz to 1 Hz [1]. In order for LISA to operate successfully, it is crucial that the three spacecraft which form the hubs of the laser interferometer in space maintain nearly constant distances between them, though their order of magnitude is 5×10^6 km. The existence of orbits having this property was firstly reported by Bender [2] as the basis of LISA. In order to thoroughly study the optical links and light propagation between these moving stations, we however need a detailed model of the LISA configuration. We therefore find it useful to recall explicitly the not so trivial principles of a stable formation flight. In this brief work, we firstly study three Keplerian orbits around the Sun with small eccentricities and adjust the orbital parameters so that the spacecraft form an equilateral triangle with nearly constant distances between them. Then we find that to the first order in the parameter $\alpha = l/2R$, where $l \sim 5 \times 10^6$ km, is the distance between two spacecraft and $R = 1$ A. U. $\sim 1.5 \times 10^8$ km, the distances between spacecraft are exactly constant; any variation in arm-lengths should result from higher orders in α or from external perturbations of Jupiter and the secular effect due to the Earth's gravitational field. (The eccentricity e is related in a simple way to α and is proportional to α to the first order in α.) In fact our analysis shows that such formations are possible with any number of spacecraft provided they lie in a magic plane making an angle of 60° with the ecliptic. We establish this general result with the help of the Hill's or Clohessy-Wiltshire (CW) equations [3].

2. The choice of orbits

2.1. The exact orbits

The exact orbits of the three spacecraft are constructed so that to the first order in the parameter α, the distances between any two spacecraft remain constant. Below we give such a choice of orbits. This choice is clearly not unique and other choices are possible which satisfy some criteria of optimality such as the distances between spacecraft vary as little as possible.

We construct the orbit of the first spacecraft and then obtain the other two orbits by rotations of 120° and 240°. The equation of an elliptical orbit in the $(X - Y)$ plane is given by [4],

$$X = R(\cos\psi + e), \quad Y = R\sqrt{1 - e^2}\sin\psi, \tag{1}$$

where R is the semi-major axis of the ellipse, e the eccentricity and ψ the eccentric anomaly. The focus is at the origin. The eccentric anomaly is related to the mean anomaly Ωt by,

$$\psi + e\sin\psi = \Omega t, \tag{2}$$

where t is the time and Ω the average angular velocity. We have chosen the zero of time when the particle is at the farthest point from the focus (this is contrary to what most books do and because of this choice of initial condition we have a positive sign instead of a negative sign on the left hand side of Eq.(2)).

We choose the barycentric frame with coordinates (X, Y, Z) as follows: The ecliptic plane is the $X - Y$ plane and we first consider a circular reference orbit of radius 1 A. U. centered at the Sun. The plane of the LISA triangle makes an angle of $60°$ with the ecliptic plane. As we shall see later, we deduce from the CW equations that this allows constant inter-spacecraft distances to the first order in α. This fact dictates the choice of orbits of the spacecraft formation. We choose spacecraft 1 to be at its highest point (maximum Z) at $t = 0$. This means that at this point, $\psi = 0$ and $Y = 0$. Thus to obtain the orbit of the first space-craft we must rotate the orbit in Eq. (1) by a small angle ϵ about the $Y-$axis so that the spacecraft 1 is lifted by an appropriate distance above the $X - Y$ plane. In order to obtain the inclination of $60°$, the spacecraft must have its Z-coordinate equal to $l/2$. The geometry of the configuration is shown in Figure 1. From

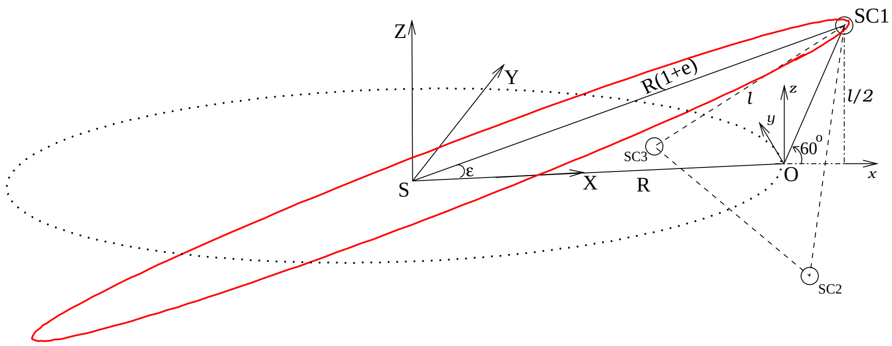

Figure 1. The figure shows the geometry of the orbits and of LISA. The barycentric frame is labelled by (X, Y, Z) while the CW frame is labelled by (x, y, z). SC1, SC2 and SC3 denote the three spacecraft. The radius of the reference orbit is taken to be $R = 1$ A. U. and S denotes the Sun.

the geometry ϵ and e are obtained as,

$$\tan\epsilon = \frac{\alpha}{1 + \alpha/\sqrt{3}}\,, \tag{3}$$

$$e = \left(1 + \frac{2}{\sqrt{3}}\alpha + \frac{4}{3}\alpha^2\right)^{1/2} - 1\,, \tag{4}$$

and the orbit equations for the spacecraft 1 are given by:

$$\begin{aligned} X_1 &= R(\cos\psi_1 + e)\cos\epsilon, \\ Y_1 &= R\sqrt{1-e^2}\sin\psi_1, \\ Z_1 &= R(\cos\psi_1 + e)\sin\epsilon. \end{aligned} \tag{5}$$

The eccentric anomaly ψ_1 is implicitly given in terms of t by,

$$\psi_1 + e\sin\psi_1 = \Omega t. \tag{6}$$

The orbits of the spacecraft 2 and 3 are obtained by rotating the orbit of spacecraft 1 by $2\pi/3$ and $4\pi/3$ about the $Z-$axis; the phases ψ_2, ψ_3 however, must be adjusted so that the spacecraft are about the distance l from each other. The orbit equations of spacecraft $k = 2, 3$ are:

$$
\begin{aligned}
X_k &= X_1 \cos\left[\frac{2\pi}{3}(k-1)\right] - Y_1 \sin\left[\frac{2\pi}{3}(k-1)\right], \\
Y_k &= X_1 \sin\left[\frac{2\pi}{3}(k-1)\right] + Y_1 \cos\left[\frac{2\pi}{3}(k-1)\right], \\
Z_k &= Z_1 ,
\end{aligned} \tag{7}
$$

with the caveat that the ψ_1 is replaced by the phases ψ_k where they are implicitly given by,

$$
\psi_k + e \sin\psi_k = \Omega t - (k-1)\frac{2\pi}{3}. \tag{8}
$$

These are the exact equations of the orbits of the three spacecraft. With these orbits the inter-spacecraft distance varies upto about 100,000 km. In Figure 2 we show how the inter-spacecraft distances vary over the course of a year. Note that there are other choices of e and ϵ close to the

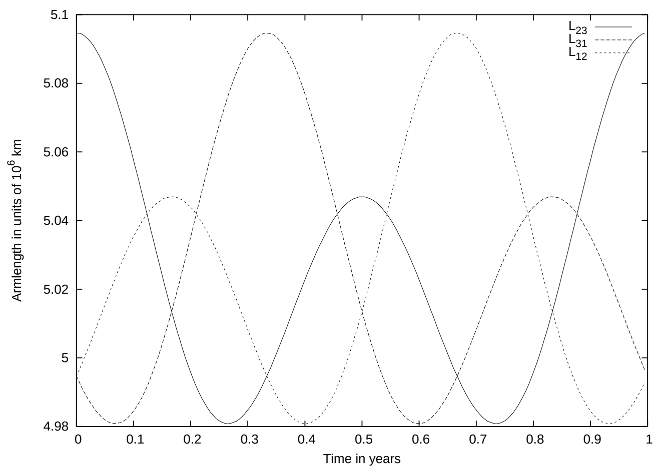

Figure 2. The variation of the lengths of the arms of LISA (the breathing modes) is shown in units of 10^6 km, when the exact orbits are computed. To the first order in α the lengths of the arms remain constant and are equal to $l = 5 \times 10^6$ km.

above values for the three orbits which give smaller variations in the armlengths.

2.2. The orbits to first order in α

In this subsection we obtain the orbits to the first order in α. The tilt ϵ and the eccentricity e are given to this order by,

$$
\tan\epsilon = \alpha, \qquad e = \frac{\alpha}{\sqrt{3}}. \tag{9}
$$

We find that e is proportional to α and, $\epsilon \sim 1.7 \times 10^{-2}$ and $e \sim 10^{-2}$. Then to this order and now writing α in terms of e, Eqs.(5) become:

$$\begin{aligned} X_1 &= R(\cos\psi_1 + e), \\ Y_1 &= R\sin\psi_1, \\ Z_1 &= \sqrt{3}eR\cos\psi_1, \end{aligned} \tag{10}$$

where the eccentric anomaly can be explicitly solved for to the first order in e in terms of the time t:

$$\psi_1 = \Omega t - e\sin\Omega t. \tag{11}$$

The approximate orbits of the spacecraft 2 and 3 can be obtained, as before, by rotating the orbit of spacecraft 1 by $2\pi/3$ and $4\pi/3$ respectively about the Z-axis as in Eq.(7). The corresponding phases ψ_2 and ψ_3 now, can be explicitly obtained in terms of t:

$$\psi_k = \Omega t - (k-1)\frac{2\pi}{3} - e\sin\left[\Omega t - (k-1)\frac{2\pi}{3}\right]. \tag{12}$$

In the next section we prove that to the first order in α, the distance between any two space-craft is l, that it is a constant and remains so *at all times*; *the LISA constellation moves rigidly as an equilateral triangle with its centroid tracing a circle with radius of 1 A. U. with the Sun as its centre.* To check this from the above equations is straightforward. We can compute the distance between spacecraft 1 and 2, which at the lowest order in α proves to be $2\alpha R$, the two other distances are equal to the preceding by symmetry. This model succeeded because we already knew the result. In the next section, we show with the help of a more sophisticated formalism, how this case is a special case of a much more general result and that stable formations with infinite number of spacecraft are possible. This result is important because we then have large number of flight formations to choose from. Depending on the required physical criteria optimal flight formations may be selected.

3. The CW frame

Clohessy and Wiltshire [3] make a transformation to a frame - the CW frame (x, y, z) which has its origin on the reference orbit and also rotates with angular velocity Ω. The x direction is normal and coplanar with the orbit, the y direction is tangential and comoving, and the z direction is chosen orthogonal to the orbital plane. They write down the linearised dynamical equations for test-particles in the neighbourhood of a reference particle (such as the Earth). Since the frame is noninertial, Coriolis and centrifugal forces appear in addition to the tidal forces. The CW equations for a free test particle of coordinates (x, y, z) are:

$$\left\{ \begin{array}{l} \ddot{x} - 2\Omega\dot{y} - 3\Omega^2 x \ = \ 0\,, \\ \ddot{y} + 2\Omega\dot{x} \ = \ 0\,, \\ \ddot{z} + \Omega^2 z \ = \ 0\,, \end{array} \right. \tag{13}$$

where $\Omega \equiv 2\pi/1$ year. The general solution, depending on six arbitrary parameters is:

$$\left\{ \begin{array}{rcl} x(t) & = & \frac{\dot{x}_0}{\Omega} \sin \Omega t - \left(3x_0 + \frac{2\dot{y}_0}{\Omega}\right) \cos \Omega t + 2\left(2x_0 + \frac{\dot{y}_0}{\Omega}\right) , \\ y(t) & = & \left(6x_0 + \frac{4\dot{y}_0}{\Omega}\right) \sin \Omega t + \frac{2\dot{x}_0}{\Omega} \cos \Omega t - 3(2\Omega x_0 + \dot{y}_0)t + \left(y_0 - \frac{2\dot{x}_0}{\Omega}\right) , \\ z(t) & = & z_0 \cos \Omega t + \frac{\dot{z}_0}{\Omega} \sin \Omega t \, . \end{array} \right. \tag{14}$$

We observe that both x and y contain constant terms and y also contains a term linear in t. The constant term in y is merely an offset and could be removed without loss of generality by a trivial translation of coordinate along the same orbit. The removal of the x offset also removes the linear term in t (the runaway solution). In contrast with the y offset, the x offset corresponds to a different orbit with a different period than that of the reference particle, namely, the origin of the CW frame. Thus the only actual and important requirement is that of vanishing of the x offset term. This term represents Coriolis acceleration in the y direction and comes from integrating the y equation in the CW equations (13). If we require a solution with no offsets, we must have:

$$\begin{aligned} &\dot{y}_0 + 2\Omega x_0 = 0, \\ &\dot{x}_0 - \frac{1}{2}\Omega y_0 = 0. \end{aligned} \tag{15}$$

With these additional constraints on the initial conditions, the bounded and centred solution is:

$$\begin{aligned} &x(t) = \frac{1}{2} y_0 \sin \Omega t + x_0 \cos \Omega t, \\ &y(t) = y_0 \cos \Omega t - 2x_0 \sin \Omega t, \\ &z(t) = z_0 \cos \Omega t + \frac{\dot{z}_0}{\Omega} \sin \Omega t. \end{aligned} \tag{16}$$

If moreover we require the distance of the particle from the origin to be constant, equal to d, say, we get the following equation:

$$\begin{aligned} &\left(\frac{1}{4} y_0^2 + 4x_0^2 + \frac{\dot{z}_0^2}{\Omega^2}\right) \sin^2 \Omega t + (x_0^2 + y_0^2 + z_0^2) \cos^2 \Omega t + \\ &\left(\frac{2 z_0 \dot{z}_0}{\Omega} - 3x_0 y_0\right) \sin \Omega t \cos \Omega t = d^2. \end{aligned} \tag{17}$$

After identifying the terms of frequencies 0 and 2Ω (sin and cos), we obtain the two equations:

$$\begin{aligned} z_0^2 - \frac{\dot{z}_0^2}{\Omega^2} &= 3\left(x_0^2 - \frac{1}{4} y_0^2\right) , \\ \frac{2 z_0 \dot{z}_0}{\Omega} &= 3 x_0 y_0 \, . \end{aligned} \tag{18}$$

Adding the first to i times the second yields the complex condition:

$$\left(z_0 + i\frac{\dot{z}_0}{\Omega}\right)^2 = 3\left(x_0 + i\frac{y_0}{2}\right)^2 , \tag{19}$$

from which we obtain,

$$z_0 = \mu\sqrt{3} x_0 \quad , \quad \frac{\dot{z}_0}{\Omega} = \frac{1}{2}\mu\sqrt{3} y_0, \tag{20}$$

where $\mu = \pm 1$. The solutions satisfying the requirements of (i) no offset and (ii) fixed distance to origin are finally of the form,

$$\left\{ \begin{array}{lcl} x(t) & = & \frac{1}{2}\,\rho_0 \cos(\Omega t - \phi_0) \\ y(t) & = & -\rho_0 \sin(\Omega t - \phi_0) \\ z(t) & = & \mu\rho_0 \frac{\sqrt{3}}{2} \cos(\Omega t - \phi_0), \end{array} \right. \tag{21}$$

where

$$\rho_0 \;=\; \sqrt{4x_0^2 + y_0^2}, \qquad \tan\phi_0 \;=\; \frac{y_0}{2x_0}\,. \tag{22}$$

The initial conditions are now expressed in terms of (ρ_0, ϕ_0) instead of (x_0, y_0). We call the solutions satisfying the above requirements as *stable*. The results that we obtained by taking Keplerian orbits to the first order in α, are the same as those obtained by using the preceding CW equations. In the CW frame the equations of the orbits simplify and it is easy to verify the result. The transformation is only in the (x, y) plane; the z coordinate is undisturbed. Since we have chosen the reference orbit to be the circle centred at the Sun and radius of $R = 1$ A. U., the CW frame (x, y, z) is related to our barycentric (X, Y, Z) frame by:

$$\begin{aligned} x &= (X - R\cos\Omega t)\cos\Omega t + (Y - R\sin\Omega t)\sin\Omega t, \\ y &= -\,(X - R\cos\Omega t)\sin\Omega t + (Y - R\sin\Omega t)\cos\Omega t, \\ z &= Z. \end{aligned} \tag{23}$$

The orbit equations for the three spacecraft derived in the last section, now simplify and can again be written in a compact form:

$$\begin{aligned} x_k &= eR\cos\left[\Omega t - (k-1)\frac{2\pi}{3}\right], \\ y_k &= -\,2eR\sin\left[\Omega t - (k-1)\frac{2\pi}{3}\right], \\ z_k &= \sqrt{3}eR\cos\left[\Omega t - (k-1)\frac{2\pi}{3}\right], \end{aligned} \tag{24}$$

where $k = 1, 2, 3$ labels the three space-craft. One immediately recognizes the form of Eqs.(21) for the special case of $\mu = 1$ with the initial conditions $\rho_0 = 2eR$ and $\phi_0 = 2\pi(k-1)/3$. The symmetry is now obvious. It is straightforward to verify that the distance between any two spacecraft is l. Thus the LISA spacecraft constellation rigidly moves as an equilateral triangle of side l in this approximation.

In fact, it is possible to establish a general result: *In the CW frame there are just two planes which make angles of $\pm\pi/3$ with the (x-y) plane, in which test particles obeying CW equations and the stability conditions (as defined above), perform rigid rotations about the origin with angular velocity $-\Omega$.*

To see this, consider a test particle at arbitrary (ρ_0, ϕ_0) whose orbit is parametrized by Eqs.(21). Consider the frame which is obtained from the CW frame (x, y, z), by first rotating about the y-axis by $\mu\pi/3$ to obtain the intermediate frame (x', y', z') and then rotating this frame about the z'-axis by $-\Omega t$. The first rotation transforms the particle trajectories to lie in the (x', y') plane. The second rotation by $-\Omega t$ about the z'-axis makes the particle in this new frame (x'', y'', z'') *stationary*! Thus we have in the double-primed coordinates:

$$x''(t) \ = \ \rho_0 \cos\phi_0 , \quad y''(t) \ = \ \rho_0 \sin\phi_0 \, , \tag{25}$$

showing that the particle is at rest in the new rotating frame. There is thus a one to one mapping from the set of all stable (as defined above) solutions of the CW equations to the two planes whose normals $\vec{n}$ are inclined at $30°$ or $150°$ with respect to the x direction and rotating at the angular velocity $-\Omega$, the rotation axis being $\vec{n}$. The LISA plane corresponds to the choice of $150°$, and it is now clear that any particle at rest in this plane, remains at rest in it, so that any number of spacecraft in this plane would remain at constant relative distances, at least in the CW approximation, equivalent to a first order calculation in the eccentricities. This further implies that so far as 'rigid' flight formations are desired, equilateral triangle is not the only choice. Arbitrary formations with any number of spacecraft are possible as long as they obey the CW equations and satisfy the stability requirements as detailed above.

4. Conclusion

We have explicitly constructed three heliocentric spacecraft orbits which to the first order in eccentricities maintain equal distances between them which is taken to be 5 million km. We have shown with the help of a more sophisticated formalism - the CW equations - that there are two planes in the CW frame, in which particles obeying the CW equations and satisfying stability requirements, namely, no offsets (and hence no runaway behaviour) and maintaining equal distance from the origin, maintain their relative distances in the CW approximation which is equivalent to a first order calculation in the eccentricities. This has the implication that formations not necessarily triangular and with any number of spacecraft are possible as long as they obey the stability constraints and lie in any one of these planes; their relative distances will be maintained within the CW approximation. This result opens up new possibilities of spacecraft constellations with various geometrical configurations and any number of spacecraft which would be useful to future space missions.

Acknowledgments

SD would like to thank IFCPAR for travel support and the Observatoire de la Côte D'Azur, Nice, France for local hospitality, where substantial part of this work was carried out. SK would like to acknowledge DST, India for the WOS-A. All the authors would like to thank E. Chassande-Mottin for detailed discussions.

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