Krishnan and the sampling theorem

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Krishnan loved mathematics and those who knew him had great respect and even awe for his skill as a mathematician. However, he chose to publish in mathematics just once, and for future generations this particular work of Krishnan remains just about the only source from which to gain a glimpse of his mathematics. This work is outlined, and it is shown that his main result is essentially equivalent to the sampling theorem of Shannon that appeared a little later. While conventional wisdom viewed the sampling theorem as a powerful engineering tool in signal processing, Krishnan saw in his main result a rich source of deep mathematical identities.

'KRISHNAN loved mathematical reasoning and his skill as a mathematician would have gained him international recognition even without his great ability as an experimental physicist' – observed Lonsdale and Bhabha[1] in their biographical notes on Krishnan. The present note is an attempt by one who has had no personal knowledge of Krishnan to understand and appreciate this curious remark on the *mathematics* of one of the greatest experimental physicists this country has ever seen[1]. Such an effort can well be a losing game, for it is likely that the content and spirit of the above remark based on personal knowledge can never be fully grasped by just reading Krishnan's published works, but one must try.

Anyone who scans through the titles of Krishnan's collected works is sure to notice one piece of work that is singularly and manifestly different from the others: every other work involves a physical phenomenon, right from the title, whereas this one makes virtually no reference to physics. This work was published in 1948, when Krishnan was Director of the National Physical Laboratory, through a short note in *Nature*[2], titled 'A simple result in quadrature', followed by a detailed one in the *Journal of the Indian Mathematical Society*[3] under the title 'On the equivalence of certain infinite series and the corresponding integrals'. The mathematical issue Krishnan pursues in this work arose during his study of a physical problem with Bhatia[4], and this if any is the only connection with physics. Thus, what Lonsdale and Bhabha were referring to when they continued their remark to say, 'Krishnan was deeply moved by a by-product of purely mathematical interest thrown up during the course of a physical investigation', is transparent. The present article, written in honour of the memory of Professor K. S. Krishnan on the occasion of his birth centenary, is confined exclusively to this particular work of his.

Krishnan's main result

Bhatia and Krishnan studied in 1947 light-scattering in homogeneous media by considering it as reflection from appropriate thermal elastic waves, and encountered the infinite series

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2},$$

where θ, α are real parameters. Under the physical conditions governing their study, α was positive and much smaller than unity. Hence the above sum can be replaced, to a good approximation, by the corresponding integral whose value is well known:

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} \approx \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\sin^2 x}{x^2} = \pi, \quad \text{for } \alpha \ll 1. \tag{1}$$

Thus π will be expected to approximate the above sum better and better as the magnitude of α becomes smaller and smaller. It may be remarked that the above argument is standard, and can be traced to at least as far back as Einstein[5]. But Bhatia and Krishnan discovered to their surprise that the above sum equals π *exactly* for *every finite* value of α in the range $-\pi \le \alpha \le \pi$, not just in the limit $\alpha \to 0$:

$$\alpha \sum_{n=-\infty}^{\infty} \frac{\sin^2(n\alpha+\theta)}{(n\alpha+\theta)^2} = \int_{-\infty}^{\infty} \mathrm{d}x\, \frac{\sin^2 x}{x^2} = \pi, \quad \text{for } -\pi \le \alpha \le \pi. \tag{2}$$

Krishnan begins his paper[3] by giving a proof of the above identity. Consider the well known series

$$\sum_{n=-\infty}^{\infty} \frac{\sin(n+\beta)z}{n+\beta} = \pi,$$

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valid for any real β, and real z in the range $0<z<2\pi$. At a fundamental level, the origin of this identity is the same as the identity in eq. (2), as will become clear in the course of this article. This series can be integrated term by term with respect to z in any closed interval $[\gamma, \delta]$, where $0<\gamma<\delta<2\pi$, since it is uniformly convergent in this interval. We thus obtain

$$\sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\gamma}{(n+\beta)^2} - \sum_{n=-\infty}^{\infty} \frac{\cos(n+\beta)\delta}{(n+\beta)^2} = (\delta-\gamma)\pi. \qquad (3)$$

Keeping δ constant and making $\gamma \to 0$, eq. (3) reduces to

$$\sum_{n=-\infty}^{\infty} \frac{1-\cos(n+\beta)\delta}{(n+\beta)^2} = \pi\delta,$$

since the first series on the left side of eq. (3) is uniformly convergent and therefore represents a continuous function of γ. Putting now $\delta = 2\alpha$, $\alpha\beta = \theta$, and dividing both sides by $2\alpha\,(\neq 0)$, we obtain eq. (2).

The above proof of Krishnan is reproduced here verbatim just to give a flavour of the rigour and style of his argument. According to the above result (2), if the graph of $\sin^2 x/x^2$ is rectangulated (see Figure 1), the area under the graph (the integral of $\sin^2 x/x^2$) and the area under the rectangulated version (the sum in eq. 2) will be exactly equal, as long as the finite spacing of the rectangulation is less than or equal to π; the rectangulation need not even be centred about $x=0$ (i.e. θ need not be zero). Clearly, this curious property is characteristic of the function $\sin^2 x/x^2$, and will not be expected to be shared by every function.

For instance, the Gaussian function $g(x) = \exp(-\pi x^2)$ does not possess this property, as can be seen from the following identity[6]:

$$\alpha \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 \alpha^2) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2/\alpha^2). \qquad (4)$$

These sums can be recognized as the theta function evaluated at specific values of its arguments, and hence the identity itself may be considered as an instance of the celebrated transformation formula for the theta function. Equivalently, this identity may be viewed as a particular case of the Poisson summation theorem. Now, the right side of eq. (4) is manifestly greater than unity for every nonzero value of α, and approaches unity only in the limit $\alpha \to 0$,. But the integral of the Gaussian function corresponding to the left side equals unity. Thus,

$$\alpha \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 \alpha^2) \neq \int_{-\infty}^{\infty} dx \exp(-\pi x^2) = 1, \qquad (5)$$

which may be compared with eq. (2).

Let us return to Krishnan who is intrigued by the two *symmetries* the sum in eq. (2) possesses, namely translation (θ) by arbitrary amount and scaling (α) by a restricted amount. He sets forth two questions for him to answer: Can the above symmetries be understood in terms of some generic property of the function under consideration? Are there other functions exhibiting these symmetries? It is clear that the two questions are related. Indeed, the generic property should necessarily be such that the second question is automatically answered.

Krishnan recognizes that his proof of eq. (2) reproduced above is too specific to a particular function, $\sin^2 x/x^2$, and hence it is unlikely to help in identifying the generic property he was after. And so he considers a second proof, with a footnote: 'We are thankful to Professor Norbert Wiener for the following elegant alternative proof'.

Consider, instead of $\sin^2 x/x^2$, a generic real even function $f(x) = f(-x)$ whose Fourier transform defined by

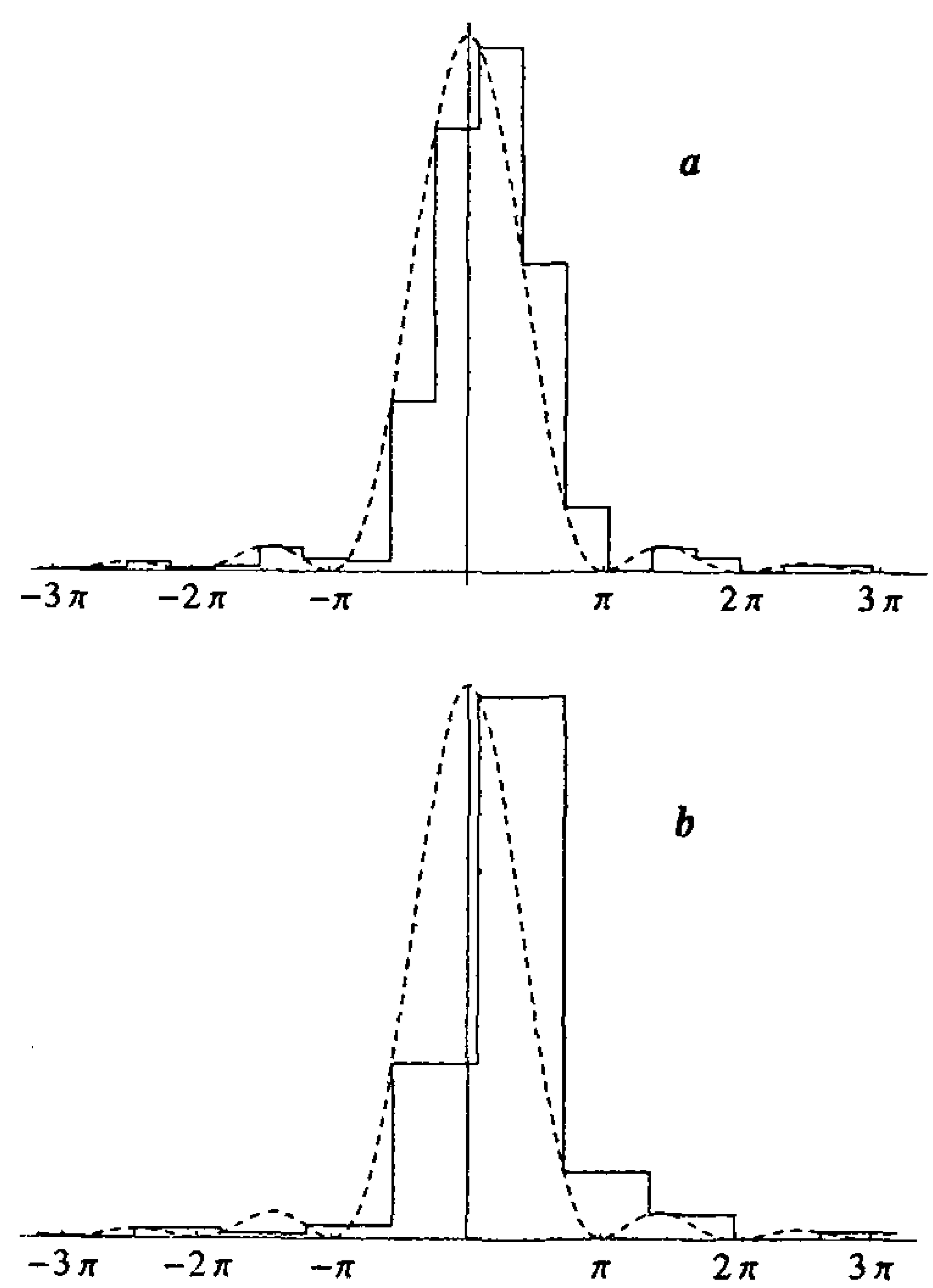

Figure 1. Illustrating the identity eq. (2): (*a*) corresponds to $\alpha = 1$, $\theta = 0.25$; and (*b*) corresponds to $\alpha = 2$, $\theta = 0.25$. In either case the area under the dotted curve representing the integral on the right side of eq. (2) equals the area under the rectangulated version representing the sum on the left.

$$g(\nu)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}x\, f(x)\exp(-i\nu x) \tag{6}$$

is assumed to exist. It follows from the definition that $g(\nu)$ too will be real and even. We see from eq. (6) that translation of a function manifests in the Fourier transform as a phase linear in the argument. That is, if $g(\nu)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(x+\theta)$ is $g(\nu)\exp(i\nu\theta)$. Applying Poisson's summation formula to $f(x+\theta)$, we have

$$\alpha\sum_{n=-\infty}^{\infty} f(n\alpha+\theta)=\sqrt{2\pi}\sum_{m=-\infty}^{+\infty} g\left(\frac{2\pi m}{\alpha}\right)\exp\left(i\frac{2\pi m\theta}{\alpha}\right). \tag{7}$$

It is seen from eq. (6) that the integral of $f(x)$ equals $\sqrt{2\pi}$ times $g(0)$, the Fourier transform of $f(x)$ evaluated at $\nu=0$. Thus, the series on the left side of eq. (7) will equal the integral of $f(x)$ if $g(\nu)$ possessed the property that $g(2\pi m/\alpha)=0$ whenever the integer variable $m\neq 0$, so that only the term corresponding to $m=0$ contributes to the sum on the right side (the phase factor makes no contribution at $m=0$). Clearly, a sufficient (but not necessary) condition for this to happen is that $g(\nu)=0$ for $|\nu|\geq 2\pi/\alpha$. This leads to the result Krishnan was deeply moved by.

Theorem: If $f(x)$ is a real even function whose Fourier transform $g(\nu)$ vanishes for $|\nu|\geq\nu_0$, then

$$\alpha\sum_{n=-\infty}^{\infty} f(n\alpha+\theta)=\int_{-\infty}^{\infty}\mathrm{d}x\, f(x),$$

$$\text{for } 0<\alpha\leq 2\pi/\nu_0. \tag{8}$$

If $g(\nu)\neq 0$ at $\nu=\pm\nu_0$, then α is restricted by $0<\alpha<2\pi/\nu_0$.

A function whose Fourier transform (spectrum) outside a finite range (band) vanishes is said to be *band-limited* and the closed interval outside of which the band-limited function vanishes is the *band-width* of the function[7]. Thus Krishnan found that band-limitedness is the *generic* property he had been after, and for any band-limited function the identity eq. (8) expressing the equivalence of the series and the corresponding integral will be valid, provided α is less than 2π/band-width.

Examples

With the generic property thus identified as band-limitedness, the answer to Krishnan's second question is immediate: scan through a source which tabulates known Fourier transform pairs and look for band-limited functions. Indeed Krishnan does list a set of band-limited functions picked from a popular monograph on Fourier integrals. Interestingly, he does not stop with this list, but produces another list of examples of band-limited functions, this time from a work[8] of the man who knew *infinity*! One is struck by this famous experimental physicist's familiarity with the works of Ramanujan. This familiarity is not a superficial one, as will become clear when we return to this aspect. If we define

$$f_k(x)=\sin^k x/x^k=(f_1(x))^k,$$

then $f_k(x)$ is band-limited and has band-width k, for every finite integer value of k. This may be seen from the convolution property of Fourier transforms. If $g(\nu)$, $g'(\nu)$, $g''(\nu)$ are the Fourier transforms of $f(x)$, $f'(x)$, $f''(x)$ respectively (primes do not refer to differentiation) and if $f''(x)=f(x)f'(x)$, then it is an immediate consequence of the definition eq. (6) that

$$g''(\nu)=g(\nu)\star g'(\nu)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}\nu\; g(\nu')\,g'(\nu-\nu')$$
$$=g'(\nu)\star g(\nu). \tag{9}$$

That is, pointwise multiplication of functions translates into the convolution operation ★ in the Fourier transform domain. Let $g_k(\nu)$ be the Fourier transform of $f_k(x)$. From the definition of Fourier transform we have

$$g_1(\nu)=\begin{cases}\sqrt{\pi/2}, & \text{for } |\nu|\leq 1,\\ 0, & \text{for } |\nu|>1,\end{cases} \tag{10}$$

which is essentially the characteristic function for the interval $[-1,1]$. Now $g_k(\nu)$ for higher values of k can be computed using the convolution property, eq. (9). For instance,

$$g_2(\nu)=g_1(\nu)\star g_1(\nu)=\begin{cases}\sqrt{\pi/2}\,(2-|\nu|)/2, & \text{for } |\nu|\leq 2,\\ 0, & \text{for } |\nu|\geq 2;\end{cases}$$

$$g_3(\nu)=g_1(\nu)\star g_2(\nu)$$

$$=\begin{cases}\sqrt{\pi/2}\,(3-\nu^2)/4, & \text{for } |\nu|\leq 1,\\ \sqrt{\pi/2}\,(3-|\nu|)^2/8, & \text{for } 1\leq|\nu|\leq 3,\\ 0, & \text{for } |\nu|\geq 3;\end{cases}$$

$$g_4(\nu)=g_2(\nu)\star g_2(\nu)=g_1(\nu)\star g_3(\nu)$$

$$=\begin{cases}\frac{1}{48}\sqrt{\pi/2}\,(32-12\nu^2+3|\nu|^3)/4, & \text{for } |\nu|\leq 2,\\ \frac{1}{48}\sqrt{\pi/2}\,(4-|\nu|)^3, & \text{for } 2\leq|\nu|\leq 4,\\ 0, & \text{for } |\nu|\geq 4;\end{cases} \tag{11}$$

These functions are illustrated in Figure 2. It is clear that $g_k(\nu)$ is piecewise polynomial, with support $[-k, k]$. Further, $g_k(\nu)$ becomes smoother with increasing value of k, consistent with the fact that $f_k(x)$ becomes sharper.

Having computed $g_k(\nu)$, we may as well use it to evaluate the integral of $\sin^{2k} x/x^{2k}$. Since Fourier transformation is a unitary operation, we have

$$\int_{-\infty}^{\infty} dx\,|f_k(x)|^2 = \int_{-\infty}^{\infty} d\nu\,|g_k(\nu)|^2, \qquad (12)$$

for any Fourier pair $f_k(x)$, $g_k(\nu)$. Using the piecewise polynomial form of $g_k(\nu)$ to evaluate the integral on the right side of eq. (12), we obtain

$$\int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} = \pi, \qquad \int_{-\infty}^{\infty} dx \frac{\sin^4 x}{x^4} = \frac{2}{3}\pi,$$

$$\int_{-\infty}^{\infty} dx \frac{\sin^6 x}{x^6} = \frac{11}{20}\pi, \qquad \int_{-\infty}^{\infty} dx \frac{\sin^8 x}{x^8} = \frac{151}{315}\pi, \qquad (13)$$

We may conclude this section with the positive note that there does exist a copious supply of band-limited functions, for these functions have a knack of cross as well as self-breeding. Clearly, a linear combination of two band-limited functions is band-limited, with band-width equal to the *larger* of the individual band-widths. From the convolution property it is transparent that the product of two band-limited functions is band-limited, with band-width equal to the *sum* of the individual band-widths. Since convolution translates under Fourier transformation into pointwise multiplication, it follows that the convolution of two band-limited functions results in a band-limited function whose band-width is the *smaller* of the individual band-widths. We can make a stronger statement, for band-limited functions have the following 'ideal' property: convolution of a band-limited function with any function is band-limited! Finally, if $f(x)$ is band-limited, then $f(x-a)$ and $f(x)\exp(icx)$, for any real a, c, are band-limited as well. Thus, for instance, $\sin^m x/x^n$ is band-limited with band-width m, for every $m \geq n$, for $\sin^{m-n} x$ and $\sin^n x/x^n$ are band-limited with band-widths $(m-n)$ and n respectively. (There is a misprint in this regard following eq. (11) of ref. 2, but ref. 2 is free of such a misprint.) Indeed, a reader who is not very familiar with Fourier transforms will find it interesting to graph the Fourier transform of $\sin^{n+2} x/x^n$ for $n = 1, 2, 3, 4$.

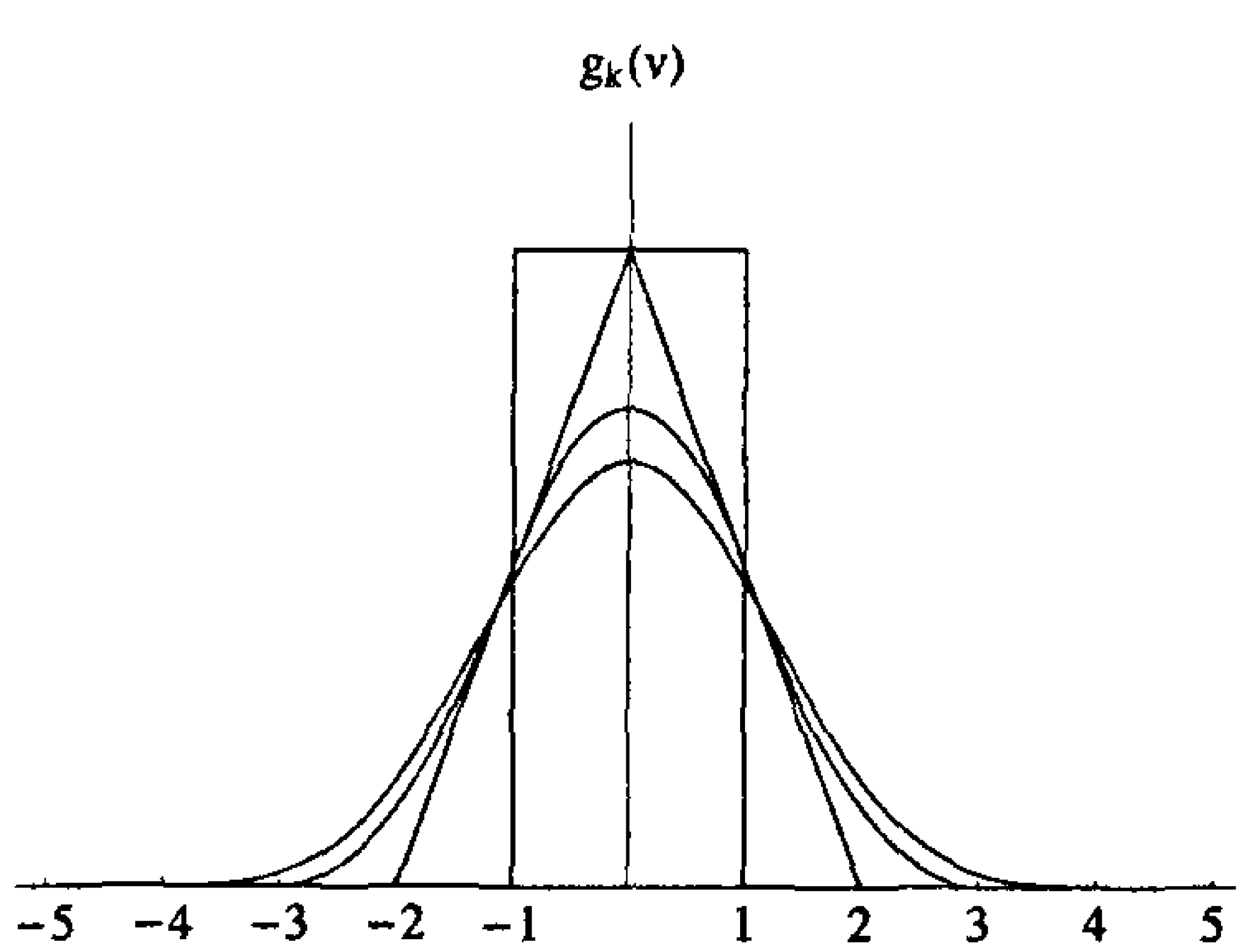

Figure 2. Showing $g_k(\nu)$, for $k = 1, 2, 3, 4$. Note that $g_k(\nu)$ becomes smoother and broader with increasing k, and vanishes outside the interval $|\nu| \leq k$.

More identities

Krishnan saw in his main result eq. (8) a rich source of mathematical identities, and he exploited eq. (8) in this regard in several different ways. While we shall now outline a few of these, it should be remarked at the outset that each identity Krishnan so derives is a 'type' identity in the sense that it applies to every function in the infinite dimensional family of band-limited functions or, in some cases, to an infinite dimensional sub-family thereof.

For instance, suppose we have a band-limited function $f(x)$ whose integral is 'standard' in the sense that it is known. For each such function eq. (8) can be viewed as an identity that sums a series and, moreover, the resulting identity will exhibit invariance under restricted scaling α and unrestricted translation θ. Indeed, the identity eq. (2) and the unnumbered one following it serve as examples of this fact, corresponding respectively to $f(x) = \sin^2 x/x^2$ and $f(x) = \sin x/x$. Since the Gaussian is not band-limited, there exists no such identity corresponding to it, and this explains the nonidentity eq. (5).

Now consider the family of band-limited functions $f(x)$ that are odd. Their Fourier transforms $g(\nu)$ will be odd as well. What is important for Krishnan is the fact that $g(\nu)$ will necessarily vanish at $\nu = 0$. Krishnan considers the example

$$f(x) = \frac{\beta \cos \beta x}{x} - \frac{\sin \beta x}{x^2},$$

whose Fourier transform turns out to be

$$g(\nu) = \begin{cases} i\sqrt{\pi/2}, & \text{for } |\nu| \leq \beta, \\ 0, & \text{for } |\nu| > \beta. \end{cases}$$

The main result, eq. (8), applied to this function reads

$$\alpha \sum_{n=-\infty}^{\infty} \left\{ \frac{\beta \cos \beta (n\alpha + \theta)}{n\alpha + \theta} - \frac{\sin \beta (n\alpha + \theta)}{(n\alpha + \theta)^2} \right\}$$

$$= \int_{-\infty}^{\infty} dx \left(\frac{\beta \cos \beta x}{x} - \frac{\sin \beta x}{x^2} \right)$$

$$= \sqrt{2\pi}\, g(0) = 0,$$

$$\text{for } 0 < \alpha < 2\pi/\beta.$$

That the integral above is zero follows from the fact that the integrand is odd. Thus Krishnan obtained the following identity:

$$\beta \sum_{n=-\infty}^{\infty} \frac{\cos \beta (n\alpha + \theta)}{n\alpha + \theta} = \sum_{n=-\infty}^{\infty} \frac{\sin \beta (n\alpha + \theta)}{(n\alpha + \theta)^2},$$

$$\text{for } 0 < \alpha < 2\pi/\beta,$$

notwithstanding the fact that the integrals corresponding to the left and right sides, namely

$$\int_{-\infty}^{\infty} dx \frac{\cos \beta x}{x} \quad \text{and} \quad \int_{-\infty}^{\infty} dx \frac{\sin \beta x}{x^2}$$

diverge! It is to be understood that α, θ are chosen such that $n + \theta$ is not equal to zero for any integer value n.

The Fourier pair $f(x)$, $g(v)$ in the above example is among the list of band-limited functions Krishnan had picked from the monograph on Fourier integrals referred to earlier. The following remark is intended to show that there is nothing mysterious regarding the band-limitedness of this $f(x)$. This remark will also help to make it transparent that the above identity is a 'type' identity, by showing that the family of odd band-limited functions is simply 'as large as' the even one.

If $g(v)$ is the Fourier transform of $f(x)$, then $ivg(v)$ is the Fourier transform of the derivative of $f(x)$. It follows that the derivative of a (odd/even) band-limited function is a (even/odd) band-limited function [In retrospect, we should have added differentiation to the operations which take band-limited functions to band-limited functions, already listed following eq. (13)]. The above Fourier pair is precisely of this type: $f(x)$ is the derivative of $f_l(x)$ and hence $g(v)$ is iv times $g_l(v)$ defined in the previous section.

Krishnan's fascination for and familiarity with the works of Ramanujan are well known. The latter had derived[9] several formulae of the Poisson summation formula type. Recall that Krishnan's main result, eq. (8), was a direct consequence of the interplay between Poisson's summation formula and band-limitedness of a function. By applying Ramanujan's more general formulae to band-limited functions, Krishnan was able to derive several interesting identities. Unfortunately, due to the constraint to keep the length of this note within reasonable limits, I am not able to detail here this pretty part of Krishnan's work. I hope this article will motivate the reader to consult the original.

Sampling theorem

I wish to conclude this note by showing that Krishnan's main result is *essentially equivalent* to the celebrated sampling theorem. Note that eq. (8) which is a direct consequence of the Poisson summation formula is valid even when $f(x)$ is neither real nor even. It says that $g(0)$, the Fourier transform of *any* band-limited function $f(x)$ evaluated at zero 'frequency', can be faithfully reconstructed from the samples of the function at regular intervals α, if the sampling interval α is bounded above by *the reciprocal of the bandwidth* of $f(x)$. The sampling theorem, on the other hand, asserts that the band-limited function itself can be faithfully reconstructed from its samples, provided the sampling interval is bounded above by *the reciprocal of twice the bandwidth.*

Now, multiplication of $f(x)$ by a phase factor linear in x goes over to translation in the Fourier transform domain:

$$f_{v'}(x) \equiv f(x)\, e^{iv'x} \underset{\text{F.T.}}{\rightarrow} g_{v'}(v) = g(v - v') \qquad (14)$$

Clearly, if $f(x)$ is band-limited with band-width v_0, then $f_{v'}(x)$, is band-limited as well with band-width $\le (v_0 + |v'|)$, and hence the result eq. (8) applies to $f_{v'}(x)$. We have (with $\theta = 0$),

$$\alpha \sum_{n=-\infty}^{+\infty} f(n\alpha)\, e^{-in\alpha v'} = \int_{-\infty}^{\infty} dx\, f(x)\, e^{-iv'x}$$

$$= \sqrt{2\pi}\, g(v'),$$

$$\text{for } 0 < \alpha < 2\pi/(v_0 + |v'|). \qquad (15)$$

Since the relevant range of v' is $|v'| \le v_0$, it is now clear from eq. (15) that if $0 < \alpha \le \pi/v_0$, then $g(v)$ over the entire range $|v| \le v_0$ can be fully reconstructed from the samples of $f(x)$. Since faithful reconstruction of $g(v)$ is the same as faithful reconstruction of $f(x)$, one indeed finds that Krishnan's result, eq. (8), is equivalent to the sampling theorem.

A pictorial representation may help to clarify why the critical sampling rate (Nyqvist rate) required for faithful reconstruction of a function is twice as tight as the one required for reconstruction of its integral alone. To this end it is useful to consider a function consisting of an array of equispaced Dirac delta functions of equal 'heights'. This function is given the descriptive name *comb* function:

$$\text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n). \qquad (16)$$

It is clear that a comb function goes to another comb function under Fourier transformation:

$$\text{comb}\left(\frac{x}{d}\right) \underset{\text{F.T.}}{\rightarrow} \frac{d}{\sqrt{2\pi}} \left(\frac{d\nu}{w\pi}\right)$$

$$= \sqrt{2\pi} \sum_{m=-\infty}^{\infty} \delta\left(\nu - \frac{2\pi}{d} m\right). \qquad (17)$$

A comb function is simply a lattice of points, and under Fourier transformation a lattice with spacing d goes to the reciprocal lattice with spacing $2\pi/d$, a result that generalizes to higher dimensions.

We may now use the comb function to conveniently express all the samples $\{f(n\alpha)\}$ of $f(x)$ in a single function

$$f_{\text{sam}}(x) \equiv \text{comb}\left(\frac{x}{\alpha}\right) f(x) = \alpha \sum_{n=-\infty}^{\infty} f(n\alpha)\, \delta(x - n\alpha). \qquad (18)$$

If $g(\nu)$ is the Fourier transform of $f(x)$ and $g_{\text{sam}}(\nu)$ that of the sampled function $f_{\text{sam}}(x)$, we have from the convolution property and eq. (17)

$$g_{\text{sam}}(\nu) = \text{F.T. of } \left\{\text{comb}\left(\frac{x}{\alpha}\right)\right\} \star g(\nu)$$

$$= \left(\sqrt{2\pi} \sum_{m=-\infty}^{\infty} \delta\left(\nu - \frac{2\pi}{\alpha} m\right)\right) \star g(\nu)$$

$$= \sum_{m=-\infty}^{\infty} g\left(\nu - \frac{2\pi}{\alpha} m\right). \qquad (19)$$

Thus, sampling at regular intervals renders the Fourier transform periodic, by producing, in addition to the original Fourier transform $g(\nu)$, perfect copies centred at $2\pi m/\alpha$, $m = \pm 1, \pm 2, \ldots$ as side bands at regular intervals on both sides (Figure 3); $g_{\text{sam}}(\nu)$ is the *superposition* of all these side bands and the original. If these are mutually nonoverlapping, as will happen if $\alpha < \pi/\nu_0$, then sampling leaves the spectrum basically undistorted, notwithstanding the multiple copies, and so multiplication of g_{sam} by a filter function $\hat{\Delta}(\nu)$ possessing the property

$$\hat{\Delta}(\nu) = \begin{cases} 1, & \text{for } |\nu| \le \nu_0, \\ 0, & \text{for } |\nu| \ge (2\pi/\alpha) - \nu_0, \\ \text{arbitrary}, & \text{for } \nu_0 < |\nu| < (2\pi/\alpha) - \nu_0, \end{cases} \qquad (20)$$

will reconstruct the original $g(\nu)$ exactly:

$$g_{\text{sam}}(\nu)\, \hat{\Delta}(\nu) = g(\nu). \qquad (21)$$

On the other hand, if our interest is restricted to $g(0)$, the integral of $f(x)$, then we will require the side bands not to overlap with the original at $\nu = 0$, and we can afford to allow overlap elsewhere. This leads to the condition $\alpha < 2\pi/\nu_0$, twice weaker than the condition for reconstruction of $g(\nu)$ in full.

Under inverse Fourier transformation the reconstruction eq. (21) takes the convolution form

$$f(x) = f_{\text{sam}}(x) \star \Delta(x)$$

$$= \left(\alpha \sum_{n=-\infty}^{\infty} f(n\alpha)\, \delta(x - n\alpha)\right) \star \Delta(x)$$

$$= \frac{\alpha}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} f(n\alpha)\, \Delta(x - n\alpha), \qquad (22)$$

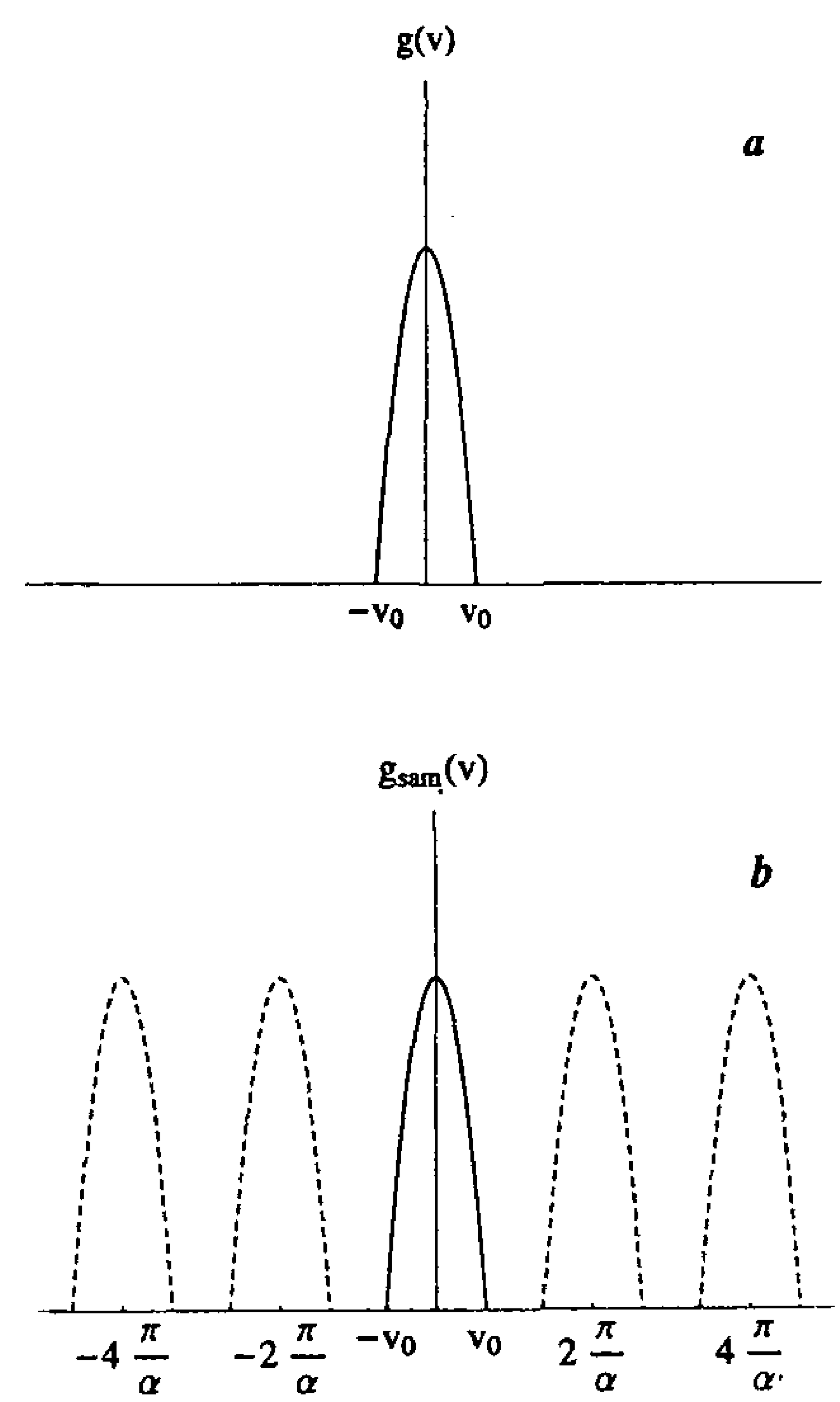

Figure 3. Effect of sampling on the Fourier spectrum of a band-limited function. (*a*) Spectrum $g(\nu)$ of the original function $f(x)$ before sampling, with band-width ν_0; (*b*) spectrum $g_{\text{sam}}(\nu)$ of the sampled function $f_{\text{sam}}(x)$. The side bands are centred at $2\pi n/\alpha$, $n = \pm 1, \pm 2, \ldots$. There is no overlap if $\alpha < \pi/\nu_0$. But no overlap at $\nu = 0$ requires the weaker condition $\alpha < 2\pi/\nu_0$.

where $\Delta(x)$ is the inverse Fourier transform of the filter function $\hat{\Delta}(\nu)$.

We see that $\Delta(x)$ plays the role of an *interpolation function*. The freedom [see eq. (20)] available in the choice of $\hat{\Delta}(\nu)$ [and hence of $\Delta(x)$] should be appreciated; each choice for the interpolation function obedient to eq. (20) will faithfully reconstruct $f(x)$ from its samples $\{f(n\alpha)\}$. This interpolation or reconstruction algorithm, which Krishnan did not consider, is often viewed as part of the statement of the sampling theorem.

An example of filter function that satisfies the requirement, eq. (20), is the *ideal low pass filter*, defined by

$$\hat{\Delta}(\nu) = \begin{cases} 1, & \text{for } |\nu| < \beta, \\ 0, & \text{for } |\nu| \geq \beta, \\ \nu_0 < \beta < (2\pi/\alpha) - \nu_0. \end{cases} \tag{23}$$

The corresponding interpolation function is the familiar *sinc* function

$$\Delta(x) = \frac{2\beta}{\sqrt{2\pi}} \frac{\sin \beta x}{\beta x}, \tag{24}$$

so that the reconstruction formula, eq. (22), reads

$$f(x) = \frac{\alpha\beta}{\pi} \sum_{n=-\infty}^{\infty} f(n\alpha) \frac{\sin[\beta(x-n\alpha)]}{\beta(x-n\alpha)}. \tag{25}$$

Clearly, $\Delta(x)$ is the output the filter $\hat{\Delta}(\nu)$ will yield for a δ-function input. In the case of electronic circuitry, where the Fourier pair (x, ν) corresponds to (time, angular frequency), the δ-function will correspond to an ideal pulse (or impulse), and so the filter Green's function $\Delta(x)$ is given the descriptive name *impulse response function* (In the case of optical systems, where the δ-function corresponds to the field amplitude of a point source, the Green's function is called the *point spread function*). If one simply sends in the time series of pulses corresponding to the samples $\{f(n\alpha)\}$ through the low pass filter, eq. (23) or eq. (20), the continuous time signal that appears at the output will be $f(x)$, exactly! This is what the reconstruction algorithm, eq. (22), asserts.

Commenting on a draft of this article, N. Kumar rephrased in an interesting manner Krishnan's main result and the sampling theorem which reconstruct, under certain physically meaningful conditions, the *whole* from its *parts*: A band-limited function is 'implicated' by any lattice whose lattice spacing does not exceed a characteristic value. He found this reminding of the *implicate* nature of the true works of the human mind on the one hand, and a theorem of F. Carlson which characterizes the holomorphic functions which are implicated by the set of integer points 0, 1, 2, ... (bounded on one side) on the other[10].

Shannon's sampling theorem[11], whose influence in the field of signal processing cannot be over-emphasized, appeared in 1949. It turns out that the sampling theorem was known to Whittaker[12]. But it is only after the work of Shannon that the sampling theorem became popular as a powerful tool in signal processing. For this reason, the theorem is often referred to as the Whittaker–Shannon sampling theorem. We may note in passing that with the appearance in 1968 of Goodman's book[0], whose system approach revolutionized the teaching of classical wave optics, the sampling theorem became part of the folklore in optics as well.

It is interesting that the mathematical result Krishnan was deeply moved by is equivalent to the sampling theorem, and independent of Whittaker and Shannon. While conventional wisdom viewed the sampling theorem as an engineering tool in signal processing, Krishnan saw it as a rich source of deep mathematical identities. I have not been able to trace the extent of the influence this particular work of Krishnan had on the work of others. It seems Krishnan would not have cared. In any case, he chose not to publish more in mathematics. His interest in mathematics appears to have been a life long affair of the heart. Only once did he choose not to conceal it, or so it will seem to the future generations.

A colleague remarked, on seeing an earlier draft of this note, that he was reminded of a celebrated world class batsman, who would have deserved a place in any international cricket team simply for his skills as a bowler, but chose to bowl in public only once. And he found this imagery irresistible.

1. Kathleen Lonsdale and Bhabha, H. J., in *Biographical Memoirs of Fellows of the Royal Society*, 1967, vol. 13. Krishnan's published works are listed. It has the following footnote of interest: 'The first draft of this notice was prepared by the late Dr Bhabha, H. J., F.R.S., and was found amongst his papers late in 1966'. I am grateful to Professor G. Rajasekaran for drawing my attention to this remark.
2. Krishnan, K. S., *Nature*, London, 1948, **162**, 215.
3. Krishnan, K. S., *J. Indian Math. Soc.*, 1948, **12**, 79.
4. Bhatia, A. B. and Krishnan, K. S., *Proc. R. Soc.*, 1948, **A192**, 181. Professor N. Mukunda has been kind enough to remind me of the following statement that occurs in the preface of M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Oxford, 1959) [some authors rightly prefer to refer to this treatise as *the bible of non-quantum optics*]: 'This approach is of considerable physical significance and its power is illustrated in a later chapter (Chapter XII) in connection with the diffraction of light by ultrasonic waves, first treated in this way by A. B. Bhatia and W. J. Noble; Chapter XII was contributed by Prof. Bhatia himself'.
5. Einstein, A., *Ann. Phys.*, 1910, **33**, 1294.
6. Bellman, R., *A Brief Introduction to Theta Functions*, Holt, Rinehart and Winston, New York, 1961, p. 11.
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10. Hille, E., *Analytic Function Theory*, Chelsea Publishing Co., New York, 1962, vol. II; Dodson, M., *Curr. Sci.*, 1992, **63**, 253.
11. Shannon, C. E., *Proc. IRE*, 1949, **37**, 10.
12. Whittaker, E. T., *Proc. R. Soc. Edinburgh*, 1915, **A35**, 181.

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