# General Theory of Vibrations of Cylindrical Tubes†

## PART—III: UNCOUPLED FLEXURAL VIBRATIONS OF CLOSED TUBES

*by*

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**Summary**

*Flexural vibration characteristics of unstiffened doubly symmetric cylindrical tubes are studied using the theory proposed by the authors in Part I. An exact solution of a simply supported tube with the cross-section given by* $x = \frac{s^2}{4\pi^2} \sin \frac{2\pi s}{S}$ *is presented. Natural frequencies of a simply supported tube of rectangular cross-section are obtained by using first and second order approximation equations. Numerical results indicate that it is essential to consider, at least, second order approximation equations to study the flexural vibration characteristics of doubly symmetric tubes.*

### ADDITIONAL NOTATION**

| | | |
|---|---|---|
| a | — | Half the width of the tube |
| b | — | Half the depth of the tube |
| $K_t^4$ | = | $\frac{\omega^2 \rho L^4 A}{E B_{xx}}$ |
| $k_w^2$ | = | $\omega^2 \rho L^2/E$ |
| P | = | a/b |
| Q | = | L/a |
| $\alpha_1, \beta_1$ | — | Defined by Eq. (3.37) |
| $\delta_{ij}$ | — | Kronecker delta |
| $\zeta_1, \zeta_2$ | — | Defined by Eq. (3.63) |
| $\lambda_t$ | — | Defined by Eq. (3.63) |
| $\lambda_1^2, \lambda_2^2, \lambda_3^2$ | — | Roots of the polynomial (Eq. 3.39 and Eq. 3.65) |

†This forms part of a Thesis entitled "Vibration studies of some basic aircraft structural components" by A.V. Krishna Murty approved for the award of the Degree of Doctor of Philosophy in the Faculty of Engineering, Indian Institute of Science, Bangalore

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**In addition to the notation used in Part I.

$\mu_t^2 = \frac{\bar{S}_{xx}}{k^2 B_{xx}}$

$\mu_1^2, \mu_2^2$ — Defined by Eq. (3.61)

$\upsilon_1^2$ — Defined by Eq. (3.61)

$\upsilon_t^2$ — Defined by Eq. (3.44)

$\Gamma_1^2, \Gamma_2^2, \Gamma_3^2, \Gamma_4^2$ — Defined by Eq. (3.61)

## 3.0 Introduction

Flexural mode of vibration of doubly symmetric tubes, although uncoupled with torsion mode, generally involves warping motion also. This is because of the fact that the assumption of CSRMD allows complete freedom to warp. This is the typical difference between tube and classical beam theories. Warping displacement is a function of two variables; and so the analysis gets considerably complicated when compared to beam theory.

Over the years, this is one of the basic problems of intense research activity. Owing to the similarities in the formulation of beam and tube theories, several attempts were made to adopt or to modify the beam equations suitably. The necessary modifications are the incorporation of transverse shear, longitudinal inertia and shear lag effects. The differential equations are improved by including the rotary inertia[2*] and transverse shear effects[4]. Investigations by subsequent workers[5–10] confirmed that these led to more accurate determination of natural frequencies and mode shapes of tubular structures. Nevertheless, shear lag effects are not included into the differential equation in literature in an elegant way so far.

Modified beam theories cannot be as good for short tubes, since the shear lag effects have substantial influence on the natural frequencies and mode shapes. Budiansky and Kruszewski[13] used Rayleigh-Ritz method to study the natural frequency characteristics of closed tubes. Mansfield[23] obtained a stress function solution for the stiffness of a rectangular box-section, utilising which the natural frequencies can be calculated. These investigations divulge the importance of shear lag effects in short tubes. However, it is difficult to adopt them to tubes of complex cross-sections.

The present theory involves the development of governing equations for the natural vibrations of thin cylindrical tubes of arbitrary cross-section, with the assumption of CSRMD. Using the Kantorovich form of Rayleigh-Ritz method, these are reduced to governing equations to various orders of approximation; an elegant splitting of warp results in the incorporation of shear lag effects into the governing differential equations. This work is reported in Ref. 32.

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*References are given in Part I.

In this report, we utilise the equations deduced in Part I[32] to study the flexural vibrational characteristics of two tubes.

(1) A simply-supported tube with the cross section given by $x = \frac{S^2}{4\pi^2} \sin \frac{2\pi s}{S}$.

(2) A simply- supported rectangular tube.

The first has an exact solution and brings out the nature of frequency spectrum whereas the second does not have an exact solution, but shows that, at least, second order approximation equations need to be used in order to get reasonable estimates of natural frequencies.

### 3.1 Governing equations - rigorous formulation.

The governing equations of equilibrium are given by Eqs. (1.31)*

$$\frac{d^2u}{dz^2} + k_u^2 u = -\frac{1}{S_{xx}} \frac{d}{dz} \oint \frac{\partial w}{\partial s} \frac{dx}{ds} t\, ds \quad \ldots (3.1)$$

$$k^2 \frac{\partial^2 w}{\partial z^2} + \frac{\partial^2 w}{\partial s^2} + k_s^2 w = -\frac{d^2x}{ds^2} \frac{du}{dz} \quad \ldots (3.2)$$

and the boundary conditions at each end are ( Eqs. (1.30) and (1.35) )

$$\text{either } u = 0 \text{ or } \frac{du}{dz} + \frac{1}{S_{xx}} \oint \frac{\partial w}{\partial s} \frac{dx}{ds} t\, ds = 0 \quad \ldots (3.3)$$

$$\text{either } w = 0 \text{ or } \frac{\partial w}{\partial z} = 0$$

$$\text{and} \quad \left(\frac{\partial w}{\partial s}\right)_{s=0} = \left(\frac{\partial w}{\partial s}\right)_{s=S} \quad \ldots (3.4)$$

### 3.2 A simply supported tube with the boundary of the cross section given by $x = \frac{S^2}{4\pi^2} \sin \frac{2\pi s}{S}$ – exact solution

The boundary conditions in this case are

$$u(0) = u(1) = 0 \quad \ldots (3.5)$$

$$\frac{\partial w(0, s)}{\partial z} = \frac{\partial w(1, s)}{\partial z} = 0 \quad \ldots (3.6)$$

$$\frac{\partial w(z, 0)}{\partial s} = \frac{\partial w(z, S)}{\partial s} = 0 \quad \ldots (3.7)$$

Examination of Eqs. (3.6) and (3.7) suggests the expression for w in the form

$$w = \sum_{m=1, 2, 3, \ldots}^{\infty} \sum_{n=1, 2, 3, \ldots}^{\infty} A_{mn} \cos m\pi z \sin \frac{2n\pi s}{S} \quad \ldots (3.8)$$

* Equations referred to as (1.. ) can be seen in Part I.

Substituting Eq. (3.8) in Eq. (3.1)

$$\frac{d^2u}{dz^2} + k_u^2 u = \sum_{m=1,2,3,\dots}^{\infty} \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \frac{2mn\pi^2}{SS_{xx}} \sin m\pi z \oint \frac{dx}{ds} t \cos \frac{2n\pi s}{S} ds \quad \dots (3.9)$$

The boundary of the cross section is given by

$$x = \frac{S^2}{4\pi^2} \sin \frac{2\pi s}{S} \quad \dots (3.10)$$

Substituting Eq. (3.10) in (3.9), we have

$$\frac{d^2u}{dz^2} + k_u^2 u = \sum_{m=1,2,3,\dots}^{\infty} \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \frac{mn\pi St}{2S_{xx}} \delta_{1n} \sin m\pi z \quad \dots (3.11)$$

The value of $S_{xx}$ is given by ( Eq. (1.19) )

$$S_{xx} = \oint \left(\frac{dx}{ds}\right)^2 t \, ds = \frac{S^3 t}{8\pi^2} \quad \dots (3.12)$$

Substituting Eq. (3.12) in Eq. (3.11)

$$\frac{d^2u}{dz^2} + k_u^2 u = \sum_{m=1,2,3,\dots}^{\infty} \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \frac{4mn\pi^3}{S^2} \delta_{1n} \sin m\pi z \quad \dots (3.13)$$

The solution of Eq. (3.13) is

$$u = A_1 \sin k_u z + A_2 \cos k_u z + \sum_{m=1,2,3,\dots}^{\infty} \sum_{n=1,2,3,\dots}^{\infty} \left\{ A_{mn} \frac{4mn\pi^3 \delta_{1n}}{S^2(-m^2\pi^2 + k_u^2)} \right\} \sin m\pi z \quad \dots (3.14)$$

Satisfaction of the boundary conditions (3.5) yields $A_1 = A_2 = 0$ ... (3.15)

Hence

$$u = \sum_{m=1,2,3,\dots}^{\infty} \sum_{n=1,2,3,\dots}^{\infty} A_{mn} \frac{4mn\pi^3 \delta_{1n}}{S^2(-m^2\pi^2 + k_u^2)} \sin m\pi z \quad \dots (3.16)$$

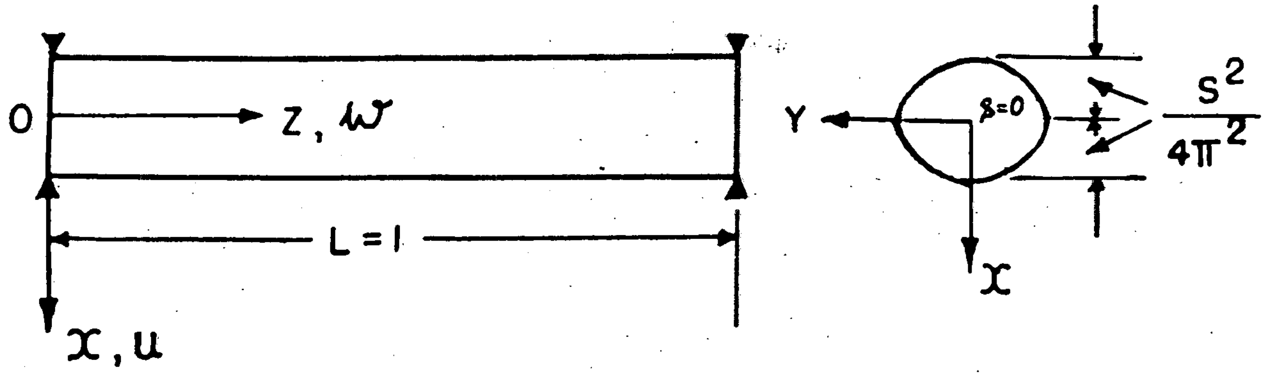


Fig. 3.1 : A simply supported Tube with the Boundary of the Cross Section given by $x = \frac{S^2}{4\pi^2} \sin \frac{2\pi s}{S}$

$$\sum_{m=1,2,3,\ldots}^{\infty} \sum_{n=1,2,3,\ldots}^{\infty} A_{mn}\left\{- k^2 m^2 \pi^2 - \frac{4n^2\pi^2}{S^2} + k_s^2\right\} \cos m\pi z \sin \frac{2n\pi S}{S}$$

$$= \sin \frac{2\pi S}{S} \sum_{m=1,2,3,\ldots}^{\infty} \sum_{n=1,2,3,\ldots}^{\infty} \left\{A_{mn} \frac{4m^2 n \pi^4 \delta_{1n}}{S^2(-m^2\pi^2 + k_u^2)}\right\} \cos m\pi z \quad \ldots (3.17)$$

from which it follows

$$A_{mn}\left(-k^2 m^2 \pi^2 - \frac{4n^2\pi^2}{S^2} + k_s^2\right) - \frac{4m^2 n \pi^4 \delta_{1n}}{S^2(-m^2\pi^2 + k_u^2)} = 0 \quad \ldots (3.18)$$

For the nontrivial solution of $A_{mn}$'s, we have the characteristic equation as

$$\left(-k^2 m^2 \pi^2 - \frac{4n^2\pi^2}{S^2} + k_s^2\right) - \frac{4m^2 n \pi^4 \delta_{1n}}{S^2\left(-m^2\pi^2 + k_u^2\right)} = 0 \quad \ldots (3.19)$$

$n, m = 1, 2, 3, \ldots$

and the corresponding mode shapes are given by

$$w = A_{mn} \cos m\pi z \sin \frac{2n\pi s}{S} \quad \ldots (3.20)$$

$$u = A_{mn} \frac{4mn\pi^3 \delta_{1n}}{S^2\left(-m^2\pi^2 + k_u^2\right)} \sin m\pi z$$

The case of $n \neq 1$ represents pure warping modes. Here natural frequencies are given by

$$k_s^2 = k^2 m^2 \pi^2 + \frac{4n^2\pi^2}{S^2} \qquad n = 2, 3, 4, \ldots \quad m = 1, 2, 3, \ldots \quad \ldots (3.21)$$

and the mode shapes are

$$w = A_{mn} \cos m\pi z \sin \frac{2n\pi s}{S}$$

and

$$u = 0 \qquad n = 2, 3, \ldots \quad m = 1, 2, 3, \ldots \qquad \ldots (3.22)$$

when n = 1, the characteristic equation is

$$\left(-k^2 m^2 \pi^2 - \frac{4\pi^2}{S^2} + k_s^2\right) - \frac{4m^2\pi^4}{S^2\left(-m^2\pi^2 + k_u^2\right)} = 0 \qquad \ldots (3.23)$$

$$m = 1, 2, 3, \ldots$$

and the mode shapes are

$$w = A_{m1} \cos m\pi z \sin \frac{2\pi S}{S} \qquad \ldots (3.24)$$

$$u = A_{m1} \frac{4m\pi^3}{S^2\left(-m^2\pi^2 + k_u^2\right)} \sin m\pi z \qquad m = 1, 2, 3, \ldots$$

and

$$\frac{u_{max}}{w_{max}} = \frac{4m\pi^3}{S^2\left(-m^2\pi^2 + k_u^2\right)} \qquad \ldots (3.25)$$

Eqs. (3.23) and (3.25) are rewritten for convenience as

$$\left(-1 + k_u^{*2}\right)\left(-1 - \frac{\mu_t^2}{m^2\pi^2} + k_w^{*2}\right) - \frac{\mu_t^2}{m^2\pi^2} = 0 \qquad \ldots (3.26)$$

and

$$\frac{u_{max}}{w_{max}} = \frac{4\pi}{mS^2\left(-1 + k_u^{*2}\right)} \qquad \ldots (3.27)$$

where

$$k_u^{*2} = k_u^2 / m^2\pi^2$$

$$k_w^{*2} = k_s^2 / k^2 m^2 \pi^2 \qquad \ldots (3\ 28)$$

$$\mu_t^2 = 4\pi^2 / k^2 S^2$$

Neglecting longitudinal inertia Eq. (3.26) reduces to a very simple form as

$$k_u^{*2} = \frac{m^2\pi^2}{\mu_t^2 + m^2\pi^2} \qquad \ldots (3.29)$$

**Numerical results**

The values of the frequency parameter $k_u^{*2}$ associated with primarily transverse motion and the mode shapes, obtained by including longitudinal inertia (Eqs. 3.26

and 3.27) and by neglecting logitudinal inertia (Eqs. 3.29 and 3.27) are compared below. In these calculations S = 1.

Table 3.1 shows that the influence of longitudinal inertia, in this case is extremely small. As mentioned earlier inclusion of longitudinal inertia gives two infinite sets of frequencies, one of which has primarily transverse motion and the other involves primarily warping motion. The second set involving primarily warping motion does not appear if longitudinal inertia is neglected. As such primarily flexural modes only are compared in Table 3.1. Table 3.2 shows both the sets of frequencies and amplitude ratios obtained by using Eqs. (3 26) and (3.27) respectively, for the value S = 1.

**Table 3.1 — Influence of longitudinal inertia**

| m | Frequency parameter $k_u^{*2}$ longitudinal inertia | | Modal ratio $-u_{max}/w_{max}$ longitudinal inertia | |
|---|---|---|---|---|
| | Included | Ignored | Included | Ignored |
| 1 | 0.3980 | 0.3985 | 20.8744 | 20.9082 |
| 2 | 0.7254 | 0.7260 | 22.8812 | 22.9337 |
| 3 | 0.8559 | 0.8564 | 29.0686 | 29.1646 |
| 4 | 0.9135 | 0.9138 | 36.3190 | 36.4426 |
| 5 | 0.9428 | 0.9431 | 43.9384 | 44.1395 |

**Table 3.2 — Natural frequencies and mode shapes (n = 1)**

| m | First set Primarily transverse motion | | Second set Primarily warping motion | |
|---|---|---|---|---|
| | $k_u^{*2}$ | $-u_m/w_m$ | $k_u^{*2}$ | $u_m/w_m$ |
| 1 | 0.3980 | 20.8744 | 525.6650 | 0.0239 |
| 2 | 0.7254 | 22.8812 | 288.4672 | 0.0275 |
| 3 | 0.8559 | 29.0686 | 244.4717 | 0.0171 |
| 4 | 0.9135 | 36.3190 | 229.0614 | 0.0137 |
| 5 | 0.9428 | 43.9385 | 221.9259 | 0.0113 |

The influence of shear lag on the natural frequencies involving primarily transverse motion is shown in Fig. 3.2. These values are computed after ignoring longitudinal inertia since its effect is small. The frequency parameter used in this graph is $K_t^{*4}$ and is given by

$$K_t^{*4} = \frac{k_u^{*2} \mu_t^2}{m^2 \pi^2} \qquad \ldots (3.30)$$

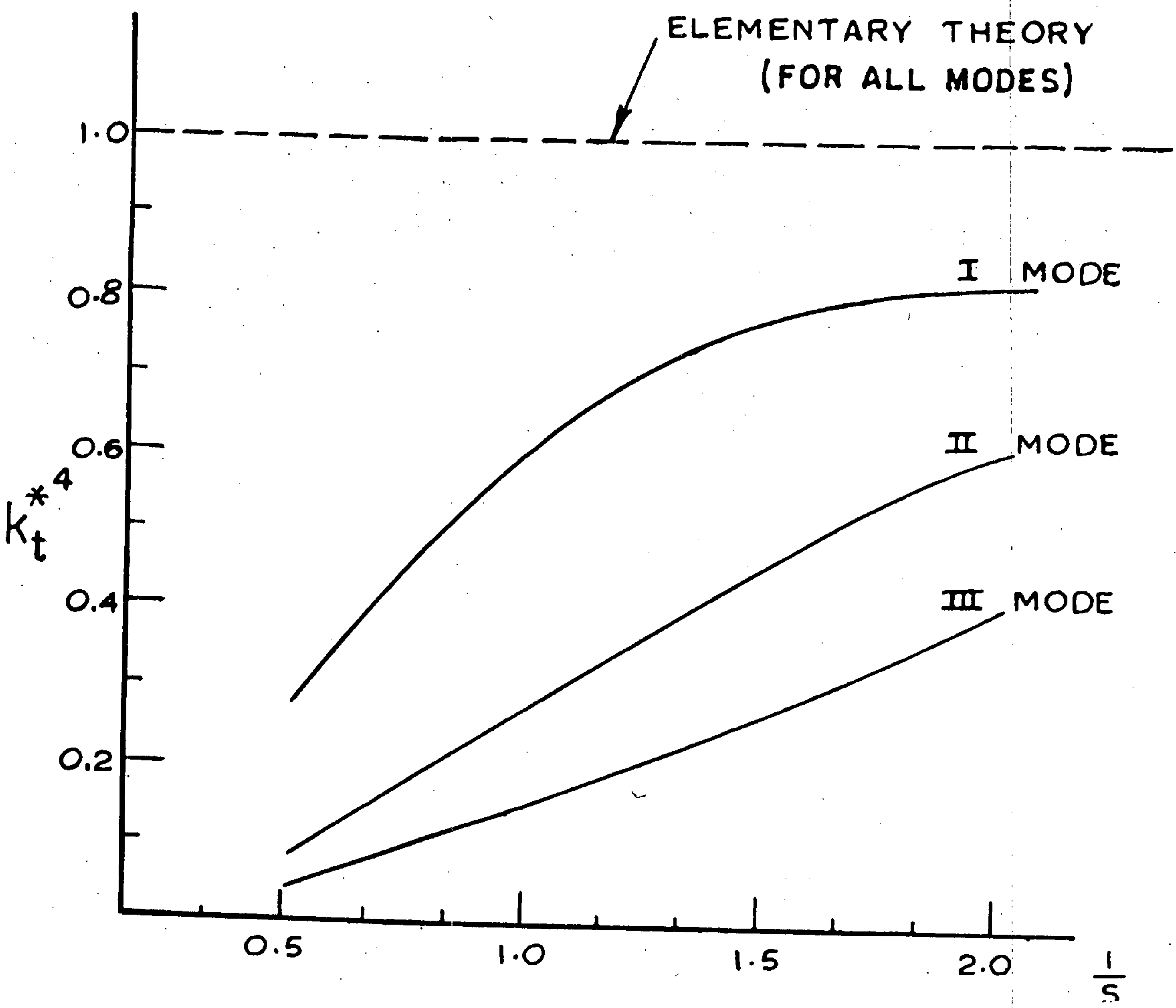


Fig. 3.2 : Frequency parameter for a simply supported tube with the boundary of the Cross Section given by

$$x = \frac{S^2}{4\pi^2} \sin \frac{2\pi s}{S}$$

Longitudinal Inertia Ignored

Alternately

$$K_t^{*4} = \left( \frac{\omega^2 \rho L^4 A}{EB_{xx}} \right) / m^4 \pi^4$$

and the value is unity in elementary theory.

### 3.3 First order approximation equations

The governing equations, in this case, are obtained by putting $t_d = t_s = t$ in Eqs. (1.58 and 1.59). The equations of equilibrium are

$$S_{xx} \frac{d^2 u}{dz^2} - b_{xx} \frac{d\phi_x}{dx} + k_s^2 A u = 0 \qquad \ldots (3.31)$$

$$k^2 B_{xx} \frac{d^2 \phi_x}{dz^2} + \left( S_{xx} \frac{du}{dz} - b_{xx} \phi_x \right) + k_S^2 B_{xx} \phi_x = 0$$

The boundary conditions at each end are (Eqs. 1.59)

$$\text{either } u = 0 \text{ or } S_{xx} \frac{du}{dz} - b_{xx} \phi_x = 0 \qquad \ldots (3.32)$$

$$\text{either } \phi_x = 0 \text{ or } \frac{d\phi_x}{dz} = 0$$

where, as defined in Eqs. (1.19), (1.55) and (1.56)

$$S_{xx} = b_{xx} = \oint \left( \frac{dx}{ds} \right)^2 t \, ds \qquad \ldots (3.33)$$

$$B_{xx} = \oint x^2 t \, ds$$

Eqs. (3.31) may be written as

$$\frac{d^2 u}{dz^2} + k_u^2 u - \frac{d\phi_x}{dz} = 0 \qquad \ldots (3.34)$$

and

$$\frac{d^2 \phi_x}{dz^2} + k_w^2 \phi_x - \mu_t^2 \phi_x + \mu_t^2 \frac{du}{dz} = 0 \qquad \ldots (3.35)$$

where $\mu_t^2 = \frac{S_{xx}}{k^2 B_{xx}}$ ... (3.36)

Using the following notation

$$k_u^2 = \alpha_1 K_t^4 \text{ where } \alpha_1 = 1/\mu_t^2 \qquad \ldots (3.37)$$

$$k_w^2 = \beta_1 K_t^4 \text{ where } \beta_1 = B_{xx}/A$$

and simplifying Eqs. (3.34) and (3.35) one gets

$$\left\{ \frac{d^4}{dz^4} + K_t^4 \left( \alpha_1 + \beta_1 \right) \frac{d^2}{dz^2} - K_t^4 \left( 1 - K_t^4 \alpha_1 \beta_1 \right) \right\} \left\{ u \text{ or } \phi_x \right\} = 0 \qquad \ldots (3.38)$$

This is the same equation used by Traill-Nash and Collar[7]; the solution of this equation is discussed by them in great detail and hence will not be discussed here in detail.

The solution of the simply supported tube with the boundary of the cross section given by $x = \frac{S^2}{4\pi^2} \sin \frac{2\pi s}{S}$ by first order approximation equations is found to be the same as the exact solution for n = 1. The solution of a rectangular simply supported tube by first order approximations is summarized in section (3.4) in order to facilitate comparison with the solution by second order approximation equations.

### 3.4 Simply supported rectangular tube — first order approximation

Let $-\lambda_1^2$ and $\lambda_2^2$ be the roots of the quadratic equation (Eq. (3.38))

$$\xi^2 + K_t^4 (\alpha_1 + \beta_1)\, \xi - K_t^4 \left(1 - K_t^4 \alpha_1 \beta_1\right) = 0 \qquad \ldots (3.39)$$

The following results are obtained by using Eqs. (3.34) and (3.35) along with the boundary conditions of the simply supported tube (Eq. (1.34)) namely,

$$\begin{aligned} u(0) &= u(1) = 0 \\ \phi'_x(0) &= \phi'_x(1) = 0 \end{aligned} \qquad \ldots (3.40)$$

Appropriate expressions for u and $\phi_x$ can be obtained using the procedure adopted in section 2.3 ; satisfying the boundary conditions (3.40) the following results can be obtained

(a) $\lambda_2^2 > 0$ ; the secular equation is

$$\frac{\lambda_1 \lambda_2 \left(\lambda_1^2 + \lambda_2^2\right)}{\left(-\lambda_1^2 + k_u^2\right)\left(\lambda_2^2 + k_u^2\right)} \sin \lambda_1 \sinh \lambda_2 = 0 \qquad \ldots (3.41)$$

or the natural frequencies are given by

$$\lambda_1^2 = m^2 \pi^2 \qquad m = 1, 2, 3, \ldots \qquad \ldots (3.42)$$

the corresponding mode shapes are given by

$$u = -\frac{m \pi \upsilon_t^2}{-\lambda_1^2 + k_u^2} A'_2 \sin m \pi z$$

and $\qquad m = 1, 2, 3, \ldots \qquad \ldots (3.43)$

$$w = -x B'_2 \cos m \pi z$$

where $\upsilon_t^2 = b_{xx}/S_{xx}$ ... (3.44)

(b) $\lambda_2^2 < 0$; besides the above set of frequencies, we have another set in this case given by

$$\lambda_2^2 = -m^2 \pi^2 \qquad m = 1, 2, 3, \ldots$$

and the mode shapes are as given in Eqs. (3.43). It may be mentioned here that the values of $k_u^2$ in cases (a) and (b) are different. Besides those presented above, there is a possibility of getting an extra frequency corresponding to either $\lambda_1 = 0$ or $\lambda_2 = 0$ (Eq. 3.41) and is given by

$$k_w^2 = \mu_t^2 .$$

The natural frequencies associated with primarily transverse motion may be obtained by substituting

$$\xi = -\lambda_1^2 = -m^2 \pi^2$$

in Eq. (3.39) and ignoring longitudinal inertia, as

$$k_u^2 = m^4 \pi^4 / \left( m^2 \pi^2 + \mu_t^2 \right)$$

which can be written as

$$K_t^{*4} = \frac{k_u^2 \mu_t^2}{m^4 \pi^4} = \mu_t^2 / \left( m^2 \pi^2 + \mu_t^2 \right) \qquad \ldots (3.45)$$

Some numerical results of Eq. (3.45) are discussed in section (3.8)

### 3.5 Second order approximation equations

The governing equations in this case are obtained by putting $t_d = t_s = t$ in Eq. (1.84) The equations of equilibrium are

$$S_{xx} \frac{d^2u}{dz^2} - b_{xx} \frac{d\phi_x}{dz} - \text{Ł}_{xx} \frac{d\psi_x}{dz} + k_s^2 Au = 0$$

$$k^2 \left( B_{xx} \frac{d^2\phi_x}{dz^2} + \bar{\text{Ł}}_{xx} \frac{d^2\Psi_x}{dz^2} \right) + \left( S_{xx} \frac{du}{dz} - b_{xx}\phi_x - L_{xx}\Psi_x \right)$$

$$+ k_s^2 \left( B_{xx}\phi_x + \bar{\text{Ł}}_{xx}\Psi_x \right) = 0$$

$$k^2 \left( \bar{\text{Ł}}_{xx} \frac{d^2\phi_x}{dz^2} + \bar{L}_{xx} \frac{d^2\psi_x}{dz^2} \right) + \left( \text{Ł}_{xx} \frac{du}{dz} - L_{xx}\phi_x - \mathfrak{L}_{xx}\psi_x \right)$$

$$+ k_s^2 \left( \bar{\text{Ł}}_{xx}\phi_x + \bar{L}_{xx}\psi_x \right) = 0 \qquad \ldots (3.46)$$

and the boundary conditions at each end are ( Eqs. (1.85) )

either $u = 0$ or $S_{xx} \frac{du}{dz} - b_{xx} \phi_x - ⅃_{xx} \psi_x = 0$

either $\phi_x = 0$ or $B_{xx} \frac{d\phi_x}{dz} + \bar{⅃}_{xx} \frac{d\psi_x}{dz} = 0$

either $\psi_x = 0$ or $\bar{⅃}_{xx} \frac{d0_x}{dz} + \bar{L}_{xx} \frac{d\psi_x}{dz} = 0$ ... (3.47)

In obtaining above, the following relation has been used

$$w = - x \phi_x - \bar{w}_{2,x} \psi_x \qquad \dots (3.48)$$

where

$$\bar{w}_{2,x} = -\iint x \, ds \, ds \qquad \dots (3.49)$$

The two constants of integration in the above integral have to be evaluated using the condition of continuity of $\frac{d\bar{w}_{2,x}}{ds}$ and the condition of zero net axial force. Various cross-sectional constants involved in Eqs. (3.46) and (3.47) are given below (Eqs. (1.19). (1.55), (1.56), (1.82) and (1 83))

$$S_{xx} = b_{xx} = \oint \left(\frac{dx}{ds}\right)^2 t \, ds$$

$$⅃_{xx} = L_{xx} = \oint \frac{dx}{ds} \frac{d\bar{w}_{2,x}}{ds} t \, ds$$

$$\mathfrak{L}_{xx} = \oint \left(\frac{d\bar{w}_{2,x}}{ds}\right) t \, ds \qquad \dots (3.50)$$

$$B_{xx} = \oint x^2 t \, ds$$

$$\bar{⅃}_{xx} = \oint x \bar{w}_{2,x} t \, ds$$

$$\bar{L}_{xx} = \oint (\bar{w}_{2,x})^2 t \, ds$$

**3 6 Cross-sectional constants of a rectangular tube—second approximation equations.**

Since the tube under consideration is doubly symmetric as shown in Fig. (3.3) and since we are considering vibration in the plane of XOZ. warp is symmetric about FB (Fig 3.3) and antisymmetric about OD. Therefore, it is sufficient to consider the region OAB for evaluation of $\bar{w}_{2,x}$ and the distribution of $\bar{w}_{2,x}$ in other regions may be obtained considering the above features.

x in the region OAB is given by

$$x = s \quad 0 \le s \le b$$

$$= b \quad b \le s \le a + b \qquad \dots (3.51)$$

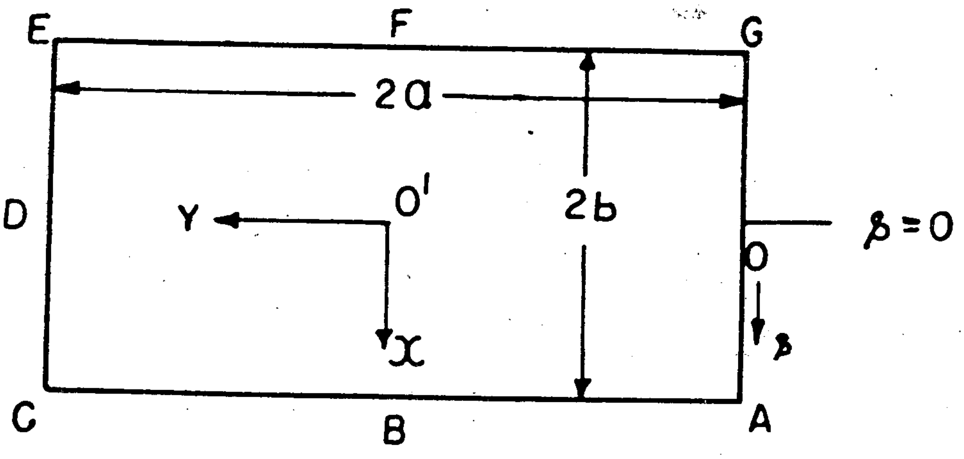


Fig. 3,3 : A Rectangular Tube.

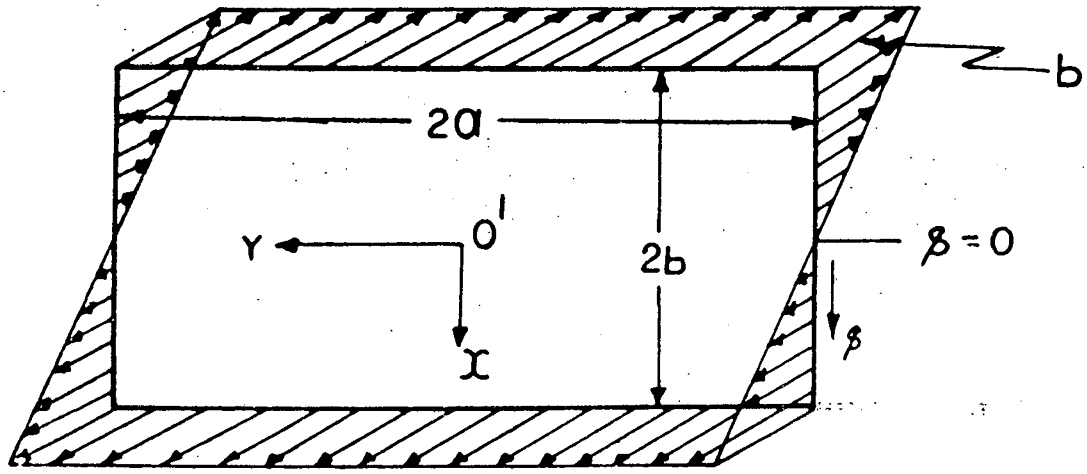


Fig. 3.4 : x = x (s) in a Rectangular Tube.

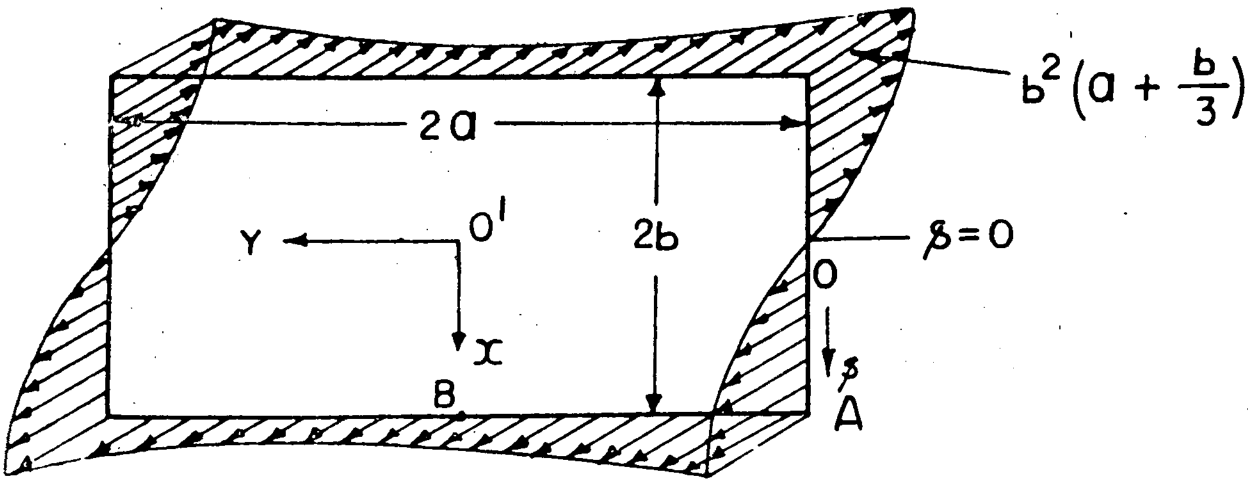


Fig. 3,5 : $\bar{w}_{2x}$ in a Rectangular Tube.

$\bar{w}_{2,x}$ is obtained from the relation given in Eq. (3.49). From symmetry and antisymmetry, we have

$$(\bar{w}_{2,x})_{x=o} = 0$$

$$\left(\frac{d\bar{w}_{2,x}}{ds}\right)_{s=a+b} = 0 \qquad \dots (3.52)$$

and

$\bar{w}_{2,x}$ and $\frac{d\bar{w}_{2,x}}{ds}$ have to be continuous at A.

Using Eqs. (3.51) in (3.49) and satisfying conditions (3.52), we obtain

$$\bar{w}_{2,x} = -\frac{s^3}{6} + (b^2/2 + ab)\, s \text{ in } 0 \leq s \leq b$$

$$= -\frac{b(s-b)^2}{2} + ab\,(s-a) + b^2\,(a+b/3) \text{ in } b \leq s \leq a+b \qquad \dots (3.53)$$

The distributions of x and $\bar{w}_{2,x}$ are shown in Figs. (3.4) and (3.5).

Using Eqs. (3.51) and (3.53) in Eqs. (3.50), the cross-sectional constants are evaluated as

$$S_{xx} = b_{xx} = 4bt$$

$$\mathit{l}_{xx} = L_{xx} = B_{xx} = 4\, b^2\, t\, (a+b/3)$$

$$\mathfrak{L}_{xx} = \bar{\mathit{l}}_{xx} = \frac{4b^2t}{15}(5a^3 + 15a^2b + 25ab^2 + 2b^3)$$

$$\bar{L}_{xx} = \frac{4b^2t}{630}(84a^5 + 420a^4b + 770a^3b^2 + 630\, a^2b^3 + 238\, ab^4 + 34\, b^5) \qquad \dots (3.54)$$

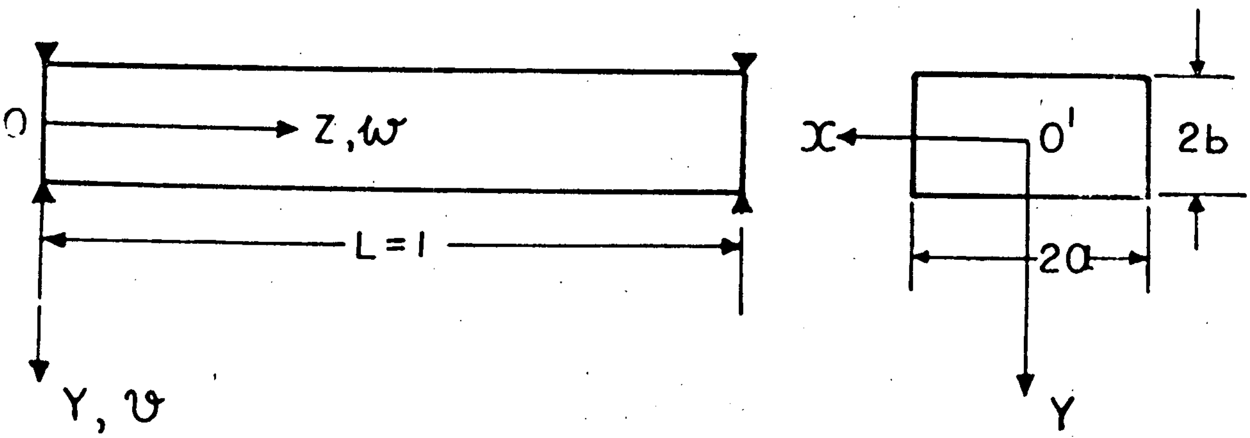


Fig. 3.6 : A simply supported Rectangular Tube.

### 3.7 Simply supported rectangular tube—second order approximation equations

The equations of equilibrium are given by Eqs. (3.46); while the boundary conditions are Eqs. (3.47).

$$u(0) = u(1) = 0 \qquad \dots (3.55)$$

$$\phi'_x(0) = \phi'_x(1) = 0 \qquad \dots (3.56)$$

$$\Psi'_x(0) = \Psi'_x(1) = 0 \qquad \dots (3.57)$$

Eqs. (3.46) may be written as

$$u'' - \phi'_x - \upsilon_1^2 \Psi'_x + k_u^2 u = 0 \quad \dots (3.58)$$

$$\phi''_x + \left( k_w^2 - \Gamma_1^2 \right) \phi_x + \mu_1^2 \left\{ \Psi''_x + \left( k_w^2 - \Gamma_3^2 \right) \Psi_x \right\} + \Gamma_1^2 u' = 0 \quad \dots (3.59)$$

$$\phi''_x + \left( k_w^2 - \Gamma_2^2 \right) \phi_x + \mu_2^2 \left\{ \Psi''_x + \left( k_w^2 - \Gamma_4^2 \right) \Psi_x \right\} + \Gamma_2^2 u' = 0 \quad \dots (3.60)$$

where

$$\upsilon_1^2 = \frac{B_{xx}}{S_{xx}} \quad \Gamma_1^2 = \frac{S_{xx}}{k^2 B_{xx}} = \mu^2$$

$$\mu_1^2 = \frac{\bar{L}_{xx}}{B_{xx}} \quad \Gamma_3^2 = \Gamma_2^2 = \frac{B_{xx}}{k^2 \bar{L}_{xx}} \quad \dots (3.61)$$

$$\mu_2^2 = \frac{\bar{\bar{L}}_{xx}}{\bar{L}_{xx}} \quad \Gamma_4^2 = \frac{\bar{L}_{xx}}{k^2 \bar{\bar{L}}_{xx}}$$

Combining Eqs. (3.58), (3.59) and (3.60), we have

$$\left\{ \left( D^2 + k_w^2 \right)^2 \left( D^2 + k_w^2 \right) - \lambda_t^2 \left( D^2 + k_w^2 \right) \left( D^2 + \zeta_1^2 k_u^2 \right) + \zeta_2^2 k_u^2 \right\} \left\{ u \text{ or } \phi_x \text{ or } \psi_x \right\} = 0 \quad \dots (3.62)$$

where

D denotes d/dz

$$\text{and } \lambda_t^2 = \frac{\mu_1^2 \Gamma_3^2 - \mu_2^2 \Gamma_4^2 + \upsilon_1^2 \left( \Gamma_2^2 - \Gamma_1^2 \right)}{\mu_2^2 - \mu_1^2}$$

$$\zeta_1^2 = \frac{\mu_1^2 \left( \Gamma_2^2 + \Gamma_3^2 \right) - \mu_2^2 \left( \Gamma_1^2 + \Gamma_4^2 \right)}{\mu_1^2 \Gamma_3^2 - \mu_2^2 \Gamma_4^2 + \upsilon_1^2 \left( \Gamma_2^2 - \Gamma_1^2 \right)} \quad \dots (3.63)$$

$$\zeta_2^2 = \frac{\mu_2^2 \Gamma_1^2 \Gamma_4^2 - \mu_1^2 \Gamma_2^2 \Gamma_3^2}{\mu_2^2 - \mu_1^2}$$

From Eq. (3.62), the expressions for u, $\phi_x$ and $\Psi_x$ are,

$$u = A_1 \sin \lambda_1 z + A_2 \cos \lambda_1 z + A_3 \sinh \lambda_2 z + A_4 \cosh \lambda_2 z + A_5 \sinh \lambda_3 z + A_6 \cosh \lambda_3 z$$

$$\phi_x = A'_1 \sin \lambda_1 z + A'_2 \cos \lambda_1 z + A'_3 \sinh \lambda_2 z + A'_4 \cosh \lambda_2 z + A'_5 \sinh \lambda_3 z + A'_6 \cosh \lambda_3 z$$

$$\psi_x = A''_1 \sin \lambda_1 z + A''_2 \cos \lambda_1 z + A''_3 \sinh \lambda_2 z + A''_4 \cosh \lambda_2 z + A''_5 \sinh \lambda_3 z + A''_6 \cosh \lambda_3 z \quad \dots (3.64)$$

—

where

$-\lambda_1^2$, $\lambda_2^2$ and $\lambda_3^2$ are roots of the cubic equation

$$\xi^3 + C_1 \xi^2 + C_2\xi + C_3 = 0 \quad \ldots (3.65)$$

and

$$C_1 = k_u^2 + 2k_w^2 - \lambda_t^2$$
$$C_2 = k_w^2 \left(2k_u^2 + k_w^2\right) - \lambda_t^2 \left( k_w^2 + \zeta_1^2 k_u^2\right) \quad \ldots (3.66)$$
$$C_3 = k_u^2 \left\{ k_w^2 \left( k_w^2 - \lambda_t^2 \xi_1^2 \right) + \zeta_2^2 \right\}$$

In writing Eqs (3.64), it is assumed that $\lambda_1^2$, $\lambda_2^2$ and $\lambda_3^2$ are positive. However, situations may exist in which either of $\lambda_2^2$ and $\lambda_3^2$ or both may be negative, but such cases are expected to represent primarily warping motion. As the present interest lies in assessing the influence of shear lag on the natural frequencies associated with primarily transverse motion, the case of negative $\lambda_2^2$ and $\lambda_3^2$ is excluded from the present discussion.

It is obvious that all the arbitrary constants in Eqs. (3.64) are not independent for they have to satisfy any two of Eqs. (3.58), (3.59) and (3.60). Substitution of Eqs (3.64) in Eqs. (3.58) and (3.59) yields the following sets of relationships respectively.

$$\left(-\lambda_1^2 + k_u^2\right) A_1 + \lambda_1 A_2^1 + \lambda_1 \upsilon_1^2 A_2'' = 0$$
$$\left(-\lambda_1^2 + k_u^2\right) A_2 - \lambda_1 A_1^1 + \lambda_1 \upsilon_1^2 A_1'' = 0$$
$$\left(\lambda_2^2 + k_u^2\right) A_3 - \lambda_2 A_4^1 - \lambda_2 \upsilon_1^2 A_4'' = 0$$
$$\left(\lambda_2^2 + k_u^2\right) A_4 - \lambda_2 A_3^1 - \lambda_2 \upsilon_1^2 A_3'' = 0 \quad \ldots (3.67)$$
$$\left(\lambda_3^2 + k_u^2\right) A_5 - \lambda_3 A_6^1 - \lambda_3 \upsilon_1^2 A_6'' = 0$$
$$\left(\lambda_3^2 + k_u^2\right) A_6 - \lambda_3 A_5^1 - \lambda_3 \upsilon_1^2 A_5'' = 0$$

and

$$\Gamma_1^2 \lambda_1 A_1 + \left(-\lambda_1^2 + k_w^2 - \Gamma_1^2\right) A_2^1 + \mu_1^2 \left(-\lambda_1^2 + k_w^2 - \Gamma_3^2\right) A_2'' = 0$$
$$-\Gamma_1^2 \lambda_1 A_2 + \left(-\lambda_1^2 + k_w^2 - \Gamma_1^2\right) A_1^1 + \mu_1^2 \left(-\lambda_1^2 + k_w^2 - \Gamma_3^2\right) A_1'' = 0$$
$$\Gamma_1^2 \lambda_2 A_3 + \left(\lambda_2^2 + k_w^2 - \Gamma_1^2\right) A_4^1 + \mu_1^2 \left(\lambda_2^2 + k_w^2 - \Gamma_3^2\right) A_4'' = 0$$
$$\Gamma_1^2 \lambda_2 A_4 + \left(\lambda_2^2 + k_w^2 - \Gamma_1^2\right) A_3^1 + \mu_1^2 \left(\lambda_2^2 + k_w^2 - \Gamma_3^2\right) A_3'' = 0$$

$$\Gamma_1^2 \lambda_3 A_5 + ( \lambda_3^2 + k_w^2 - \Gamma_1^2) A_6^1 + \mu_1^2 ( \lambda_3^2 + k_w^2 - \Gamma_3^2 ) A_6'' = 0$$

$$\Gamma_1^2 \lambda_3 A_6 + ( \lambda_3^2 + k_w^2 - \Gamma_1^2) A_5^1 + \mu_1^2 ( \lambda_3^2 + k_w^2 - \Gamma_3^2 ) A_5'' = 0 \qquad \ldots (3.68)$$

Satisfaction of the boundary conditions at $z = 0$, namely $u(0)$ $\phi_x'(0)$

and $\Psi_x'(0) = 0$ yield

$$A_2 + A_4 + A_6 = 0$$

$$\lambda_1 A_1^1 + \lambda_2 A_3^1 + \lambda_3 A_5^1 = 0 \qquad \ldots (3.69)$$

$$\lambda_1 A_1'' + \lambda_2 A_3'' + \lambda_3 A_5'' = 0$$

It can be shown from Eqs. (3.69) with the help of Eqs. (3.67) and (3.68) that

$$A_2 = A_4 = A_6 = A_1^1 = A_3^1 = A_5^1 = A_1'' = A_3'' = A_5'' = 0 \qquad \ldots (3.70)$$

Satisfying the boundary conditions at $z = 1$, namely

$u(1) = 0$, $\phi_x^1(1) = 0$ and $\Psi_x^1 = 0$, and using Eqs. (3.70)

one obtains

$$A_1 \sin \lambda_1 + A_3 \sinh \lambda_2 + A_5 \sinh \lambda_3 = 0$$

$$- \lambda_1 A_2^1 \sin \lambda_1 + \lambda_2 A_4^1 \sinh \lambda_2 + \lambda_3 A_6^1 \sinh \lambda_3 = 0$$

$$- \lambda_1 A_2'' \sin \lambda_1 + \lambda_2 A_4'' \sinh \lambda_2 + \lambda_3 A_6'' \sinh \lambda_3 = 0 \qquad \ldots (3.71)$$

Using Eqs. (3.67), (3.68) and (3.71) it can be shown that for nontrivial solution

$$\sin \lambda_1 \sinh \lambda_2 \sinh \lambda_3 = 0 \qquad \ldots (3.72)$$

or

$$\lambda_1 = m \pi \quad m = 1, 2, 3 \ldots\ldots 00 \qquad \ldots (3.73)$$

Since our interest at this stage is confined only to primarily transverse vibrations, the corresponding mode shapes are

$$u = A_1 \sin m \pi z \qquad \ldots (3.74)$$

$$w = - x\phi_x - \bar{w}_{2,x} \Psi_x = - x A_2^1 + \bar{w}_{2,x} A_2'' \cos m \pi z$$

where $A_2^1$ and $A_2''$ are obtained in terms of $A_1$ by solving the two first equationsi n Eqs. (3.67) and (3.68).

Neglecting longitudinal inertia a simple expression for natural frequency can be obtained; substitutiug

$$\xi = - \lambda_1^2 = - m^2 \pi^2 \quad \text{... (3.75)}$$

in Eq. (3.65); ignoring longitudinal inertia ($k_w^2 = 0$), and simplifying

$$k_u^2 = \frac{m^6 \pi^6 + \lambda_t^2 m^4 \pi^4}{m^4 \pi^4 + m^2 \pi^2 \lambda_t^2 \zeta_1^2 + \zeta_2^2}$$

which can also be written as

$$K_t^{*4} = \frac{k_u^2 \mu_t^2}{m^4 \pi^4} = \frac{\mu_t^2 \left( m^2 \pi^2 + \lambda_t^2 \right)}{m^4 \pi^4 + m^2 \pi^2 \lambda_t^2 \zeta_1^2 + \zeta_2^2} \quad \text{... (3.76)}$$

The values of $K_t^{*4}$, obtained by using Eqs. (3.45) and (3.76) and presented in Table 3.3, denote the results obtained by using first and second order approximations respectively.

**Table 3.3—Influence of shear lag on $K_t^{*4}$**

| P | Q | Order of approximation | Fundamental | Second mode | Third mode |
|---|---|---|---|---|---|
| 3.6 | 2 | First | 0.819 | 0.531 | 0.335 |
| | | Second | 0.730 | 0.367 | 0.115 |
| | 6 | First | 0.926 | 0.759 | 0.583 |
| | | Second | 0.883 | 0.638 | 0.424 |
| | 10 | First | 0.966 | 0.876 | 0.759 |
| | | Second | 0.946 | 0.810 | 0.648 |
| | 14 | First | 0.981 | 0.926 | 0.848 |
| | | Second | 0.969 | 0.885 | 0.770 |

The results of the first approximation can be obtained from the equations presented in Ref. (7) also. The results of Table (3.3) are presented in a graphical form in Fig. 3.7. This reveals that the effect of shear lag is considerable particulary at short aspect ratios and this effect is more significant for higher modes. Although there

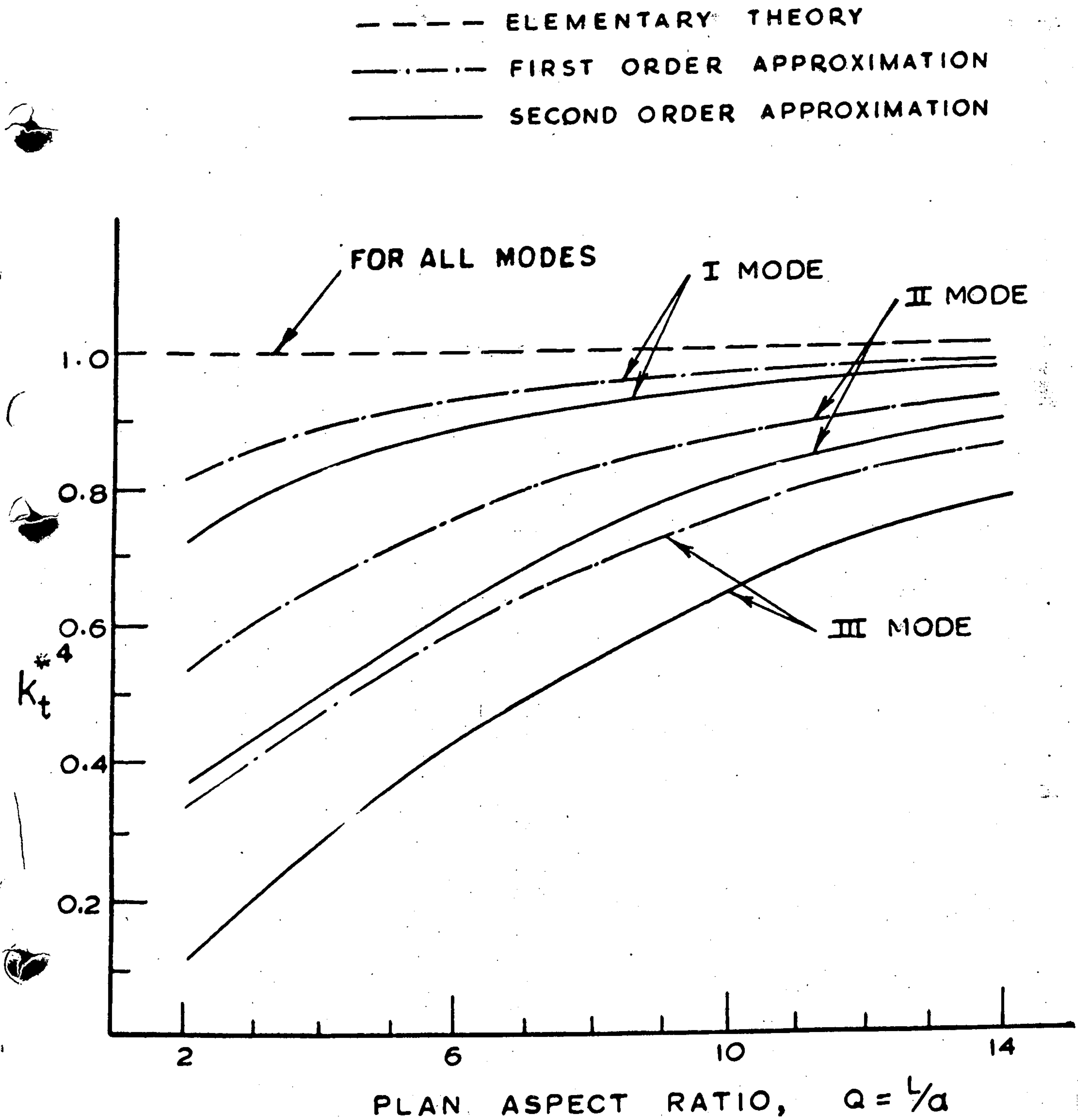


Fig. 3.7 : Frequency Parameter for a simply supported Rectangular Tube, P = 3.6 Longitudinal Inertia Ignored.

are no exact results to provide comparison for the results presented in Table 3.3, the trends are in agreement with the results of Ref. (13) which considers a free-free tube in transverse vibration.

## 3 9 Conclusions

In this chapter problems of flexural vibrations of doubly symmetric unstiffened closed tubes have been considered. The exact solution of a simply supported tube with the cross section given by $x = \frac{S^2}{4\pi^2} \sin \frac{2\pi s}{S}$ has been presented. This reveals that the frequency spectrum in this case includes a doubly infinite set of frequencies besides the well known infinite set of frequencies involving primarily transverse motion. One infinite set of this additional doubly infinite set involves large warping motion associated with small transverse motion while the remaining pertain to pure warping motions.

The numerical results reveal that the influence of shear lag is considerable for short aspect ratios and increases for higher modes. From this it can be inferred that first order approximation equations are not adequate ; it is essential to consider at least second order approximation equations to get reasonably good results.

## Acknowledgements

The authors are grateful to Professor A. Kameswara Rao for many valuable discussions.