From Natural Numbers to Numbers and Curves in Nature – I

A K Mallik

The interconnection between number theory, algebra, geometry and calculus is shown through Fibonacci sequence, golden section and logarithmic spiral. In this two-part article, we discuss how simple growth models based on these entities may be used to explain numbers and curves abundantly found in nature.

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Introduction

According to Kronecker, a famous European mathematician, only natural numbers, i.e., positive integers like 1, 2, 3, … are given by God or belong to the nature. All other numbers like negative numbers, fractional numbers, irrational numbers, transcendental numbers, complex numbers, etc., are a creation of the human mind. Of course, all these other numbers are created using the natural numbers. We are so used to natural numbers that we may fail to notice some interesting patterns in them. For example, let us observe the simple yet beautiful regularity of appearance of all the consecutive natural numbers in the following equations:

$$1+2 = 3$$
$$4+5+6 = 7+8$$
$$9+10+11+12 = 13+14+15,$$

and it continues in this fashion *ad infinitum*.

In this article, we shall see, how by playing around with natural numbers some other curious numbers and curves are generated, and discuss their appearances in nature. Of late, such an approach is becoming popular to propose mathematical models for the growth of living organisms and other patterns commonly found in nature.

Keywords
Fibonacci sequence, golden section, logarithmic spiral, mathematical models.

Fibonacci Sequence

In 1202, Italian mathematician Leonardo of Pisa, nicknamed Filius Bonacci or Fibonacci, used the natural numbers to construct the following sequence

1, 1, 2, 3, 5, 8, 13, 21, 34, 55,.... (A)

In this sequence, called the Fibonacci sequence, each number (from the third onwards) is the sum of its immediate two predecessors. Incidentally, Fibonacci is also credited with the exposition of Hindu-Arabic numerals to Europe through his book *Liber Abaci*.

The ratios of two consecutive numbers in the Fibonacci sequence are seen to generate the following sequence:

1, 0.5, 0.666…, 0.6, 0.625, 0.615…, 0.619…, 0.61818.. (B)

This sequence is seen to be alternating in magnitude but converging. We may write the Fibonacci sequence as

$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3 .$$

Thus for $n \geq 3$, $\frac{F_n}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}}$. (1)

As $n \to \infty$, let $\lim \frac{F_n}{F_{n+1}} \to j$. So from (1)

$$\frac{1}{j} = 1 + j \quad \text{or,}$$

$$j^2 + j - 1 = 0. \qquad (2)$$

Taking only the physically meaningful root,

$$j = \frac{\sqrt{5} - 1}{2} = 0.618034\ldots . \qquad (3)$$

Note that j is independent of F_1 and F_2. Some easy to prove but

Box 1.

$$F_1 + F_2 + F_3 + \dots\dots\dots + F_n = F_{n+2} - 1.$$
$$F_1 + F_3 + F_5 + \dots\dots\dots + F_{2n-1} = F_{2n}.$$
$$F_1^2 + F_2^2 + F_3^2 + \dots\dots\dots + F_n^2 = F_n F_{n+1}.$$
$$\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \frac{1}{F_3F_5} + \dots\dots\dots = 1 - \frac{1}{F_nF_{n+1}}.$$

useful identities involving the Fibonacci sequence are given in *Box* 1.

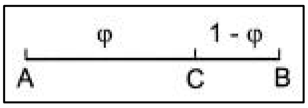


Figure 1a. Geometric interpretation of Golden Section.

Golden Section

The sequence of the ratio of two successive numbers in the Fibonacci sequence converges to a very special (as we shall see) irrational, fractional number which was well known to the early Greek mathematicians as 'Golden Section'. To the Greeks, who were predominantly geometers, this Golden Section was a harmonious, almost mystical, constant of nature. To interpret the Golden Section geometrically, we consider a line AB of unit length (*Figure* 1a). Let C be a point on and inside AB, such that

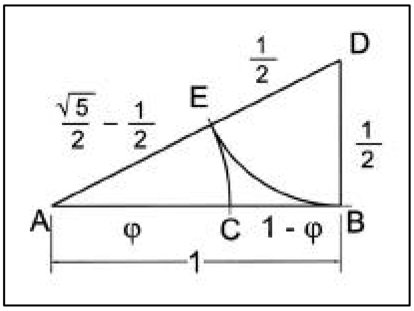
(AB/AC) = (AC/BC). If AC = j , then

$$\frac{1}{j} = \frac{j}{1-j},$$

or $j^2 + j - 1 = 0$.

This is the same as equation (2). *Figure* 1b explains how the point C is obtained geometrically. In this figure AB = 1, BD (perpendicular to AB) = ½, DE = DB and AC = AE.

Figure 1b. Geometric determination of Golden Section.



The reciprocal of Golden Section is $y = 1/j = \frac{\sqrt{5}+1}{2}$. It may be noted that

$$y = 1/j = 1 + j = j/(1-j). \qquad (4)$$

Box 2.

Let the integer part be n and the fractional part be m, then

$$\frac{n+m}{n} = \frac{n}{m}.\quad \text{This implies}$$

$$m^2+mn-n^2=0,\quad i.e.$$

$$m = \left(\sqrt{5}-1\right)\frac{n}{2}.$$

Since m is a fraction, $n = 1$, therefore the number $m + n = \dfrac{\sqrt{5}+1}{2}$.

It is easy to show that y is the only number for which the number, its integer part and its fractional part are in geometrical progression. (See *Box* 2).

Using the values of

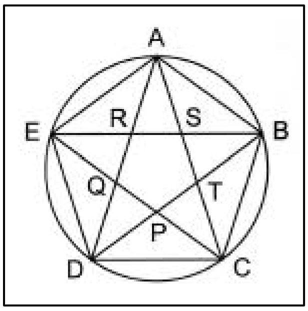
$\sin 36^{o}\left(=\sqrt{\dfrac{10-2\sqrt{5}}{4}}\right)$ and $\sin 72^{o}\left(=\sqrt{\dfrac{10+2\sqrt{5}}{4}}\right)$, it is easy to

show that the ratio of the side to the diagonal of a regular pentagon is also equal to the Golden Section (see *Figure* 2). Furthermore, in this figure it can be shown that, (AQ/AD) = (QD/AQ) = j .

Golden Section, being an irrational number, can be expressed as an infinitely continued fraction as shown below:

$$j = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\dots}}}}. \qquad (5)$$

Figure 2. Golden Section in regular pentagon.



Terminating this infinitely continued fraction at successive steps we generate exactly the sequence (B) which converges ultimately to the Golden Section. Also remembering that a continued fraction never ceases, (5) can be rewritten as $j = (1/1+j)$, which defines the Golden Section as seen earlier. It is interesting to note that the reciprocal of Golden Section can also be expressed as an infinite series using only the digit 1 as

$$y = 1/j = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}\ .$$

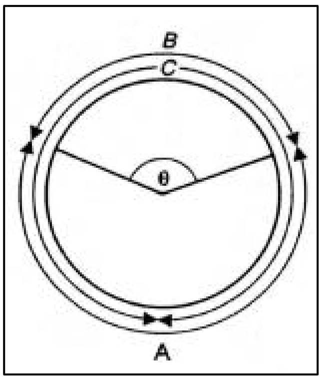
At this stage it is worthwhile to note the continued fraction representations of some other fractional irrational numbers as shown below.

$$\sqrt{2}-1=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ldots}}}} \qquad \sqrt{3}-1=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\ldots}}}}$$

$$e-2=\cfrac{1}{1+\cfrac{1}{2+\cfrac{2}{3+\cfrac{3}{4+\cfrac{4}{5+\ldots}}}}}\ .$$

Looking at these continued fraction expressions, one suspects that the Golden Section enjoys a special status, employing only the digit 1, even among the irrational fractional numbers. Number theorists call it the 'worst' irrational fraction, implying that it is most badly 'approximable' by a rational number. This approximation is carried out by truncating the continued fraction. One can easily verify that at every step this approximation is worst for the Golden Section amongst the four irrational fractions mentioned above.

Figure 3. Golden Angle.



Golden Angle

One can easily extend the concept of Golden Section to a unit circle instead of a straight line of unit length. This is shown in *Figure* 3. The Golden Angle is defined as the smaller angle q, where,

$$\frac{360^{\circ}}{360^{\circ}-q}=\frac{360^{\circ}-q}{q}$$

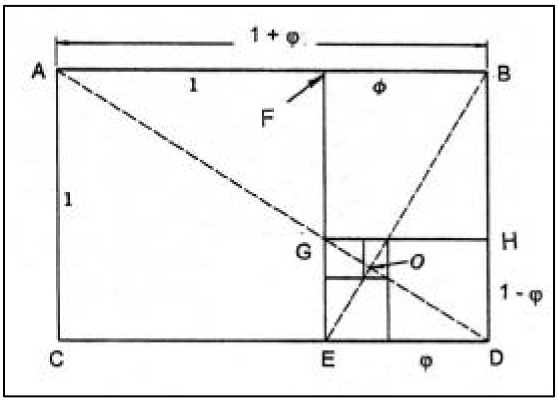


Figure 4. Golden Rectangle and its nesting.

or, $q \approx 137.5^0 = \lfloor (1-j)360^o \rfloor$. (6)

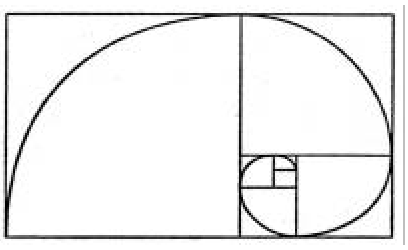
Golden Rectangle

Let us now define a Golden Rectangle ABDC of sides of length having the ratio 1:1+j (*Figure* 4). If this rectangle is subdivided into a square of unit side (AFEC), then the remaining rectangle FBDE is also a Golden Rectangle (reduced in scale and rotated through a right angle). This process can be continued indefinitely to generate similar figures with a reduction in the scale at every step by a factor j. The continued similarity implies that diagonals AD and BE serve also as diagonals of all the nested rectangles of the same orientation (i.e., rotated through 180°) as seen in *Figure* 4. The point (denoted by O in *Figure* 4) of intersection of AD and BE is the limit point or pole around which all the figures are nested.

Golden Spiral

A very remarkable curve can be drawn using these nested Golden Rectangles as shown in *Figure* 5. This curve passes through the points C, F, H, … of *Figure* 4. These points divide the longer sides of the Golden Rectangles in the Golden Section. Of course, as will be seen later, a similar curve could have also been drawn through three successive corners of each Golden Rectangle. From the similarity of the nested rectangles, it is not hard to see that the curve remains similar everywhere but for its size.

Figure 5. Golden Spiral (left-handed) in a set of nested Golden Rectangles.



With the limit point O (*Figure* 4) as the origin, the equation of the curve in polar coordinates is of the form

$$r = e^{aq}. \qquad (7)$$

Since the curve intersects any radial reference line infinite times, one can define r=1 at $q = 0$ arbitrarily, with q going from $-\infty$ (at the origin) to ∞. As the value of q (measured in the counterclockwise direction) increases

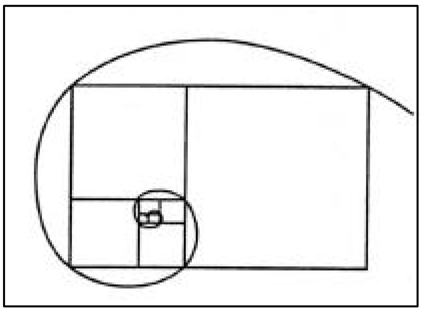
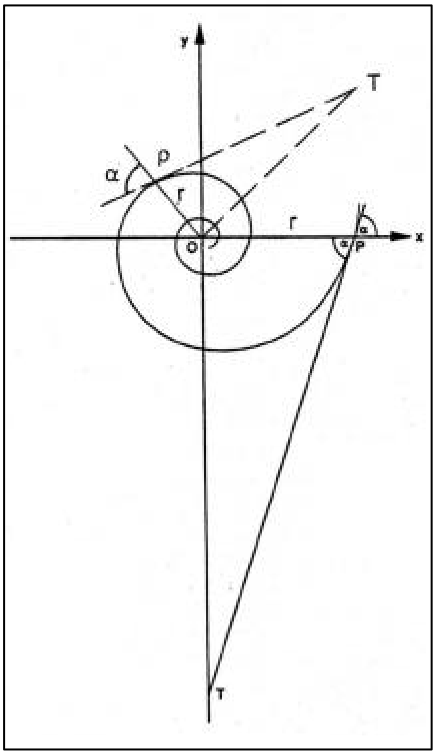


Figure 6. Golden Spiral (right-handed) circumscribing a set of nested Golden Rectangles.

by *p*/2, the curve (also the value of *r*) expands by a factor $1/j\left(=\frac{\sqrt{5}+1}{2}\right)$. So the value of $a=\frac{2}{p}\ln\frac{\sqrt{5}+1}{2}$ defines the value of the exponential growth rate with rotation. The curve spiralling out in the counterclockwise direction, as shown in *Figure* 5 with the positive value of *a* mentioned above, is called a left-hand spiral. *Figure* 6 shows a right-hand spiral with $a=-\frac{2}{p}\ln\frac{\sqrt{5}+1}{2}$. Notice that the nesting of the Golden rectangles in *Figures* 5 and 6 are carried out in opposite directions. In *Figure* 6, the spiral is drawn through three successive corners of each Golden Rectangle (as mentioned earlier) and circumscribes the set of nested rectangles.

Figure 7. Rectification of a logarithmic or equiangular spiral.



Logarithmic Spiral

The spiral of the form $r = e^{gq}$, studied extensively by mathematician Jakob Bernoulli, is called an equiangular or logarithmic spiral, $g > 0$ indicates a left hand spiral $g < 0$ a right hand spiral and $g = 0$ refers to a unit circle. Some remarkable properties of this logarithmic spiral are mentioned here.

(i) At all points on the curve, the radius vector OP makes an equal angle a to the curve (i.e., to the tangent to the curve at P) and hence the name *equiangular* spiral. It can be readily

Box 3.

(i) The property of equiangularity can be proved by the conformal (angle preserving) property of the mapping by the analytic function $w=e^z$ for the complex variable z. If $z = x + iy$, then $w = u + iv = \mathrm{R}(\cos \Phi + i \sin \Phi)$ with $R = e^x$ and $\Phi = y$. So y = constant, i.e., lines parallel to the x-axis are mapped into Φ = constant , i.e., radial lines in the u–v plane. The line, passing through the origin and making an angle a to the x-axis in the x–y plane, i.e., with the equation $y = (\tan a)x$, is mapped on to the logarithmic spiral $R = e^{(\Phi/\tan a)}$. Thus by the conformal property of the transformation the radial lines makes equal angle a to the logarithmic spiral $R = e^{(\Phi/\tan a)}$ at every point. For the curve $R = e^{\gamma q}$, $\cot a = g$.

(i) In polar coordiantes (r, q) , the radius of curvature

$$r = \frac{\left[r^2+\left(\frac{dr}{dq}\right)^2\right]^{3/2}}{r^2+2\left(\frac{dr}{dq}\right)^2-r\frac{d^2r}{dq^2}}.$$

For $R = e^{\gamma q}$, $r = \sqrt{1+g^2}\,e^{gq}$. So the radius of curvature subtends a right angle at the pole and the equation of the evolute comes out as $r = g\,e^{-(gp)/2}e^{gq}$.

(iii) $$s = \int ds = \int\sqrt{(dr)^2+(rdq)^2} = \int_{-\infty}^{q}\sqrt{\left(\frac{dr}{dq}\right)^2+r^2}\,dq\ .\quad \text{For } r=e^{gq}$$

$$s = \frac{\sqrt{1+g^2}}{g}e^{gq} = r/(\cos a), \ \text{since}\ \ g = \cot a.$$

In 1645, Evangelista Torricelli obtained this result without using integral calculus, which was yet to arrive!

seen that $\tan a = r\frac{dq}{dr} = 1/g$, i.e., $a = \cot^{-1} g$ is a constant (independent of the point P). For another approach to obtain the same result see *Box* 3.

(ii) The evolute of a logarithmic spiral is also a logarithmic spiral. This implies that the locus of the centre of curvature at all points of a logarithmic spiral forms another logarithmic spiral. For proof, see *Box* 3.

(iii) The pedal curve, i.e., the locus of the foot of the perpendicular from the pole to the tangents of a logarithmic spiral is another logarithmic spiral.

(iv) The caustic, i.e., the envelope formed by rays of light emanating from the pole and reflected by the curve, is also a logarithmic spiral.

(v) The length of the spiral from a point P (at a distance r from the pole) to the pole is $s = r/\cos a$. In *Figure* 7, s = PT. For proof, see *Box* 3.

(vi) The process of inversion by $r \to (1/r)$ changes a logarithmic spiral into its mirror image, i.e., into the same spiral but of opposite hand.

Jakob Bernoulli was so enchanted with this 'Spira Mirabilis' (marvellous spiral), which retains its shape under so many operations, that he wished to have this spiral engraved on his tomb with the inscription '*Eadem mututa resurgo*' ('Though changed, I shall arise the same'). However, the engraver (a non-mathematician, of course) made a mistake by engraving the Archimedean spiral $r = gq$ [1]!

Suggested Reading

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[2] **http://www.goldenmeangauge.co.uk/golden.htm**

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