Partial decoherence in a qubit coupled to a control register subject to telegraph noise

Amnon Aharony,[1,*] Ora Entin-Wohlman,[1,†] and Sushanta Dattagupta[2]

[1] *Department of Physics and the Ilse Katz Center for Meso- and Nano-Scale Science and Technology, Ben-Gurion University, Beer Sheva 84105, Israel*

[2] *Indian Institute of Science Education and Research-Kolkata, Mohanpur 741252, India*

(Dated: August 30, 2009)

A qubit (containing two quantum states, 1 and 2), is coupled to a control register (state 3), which is subject to telegraph noise. We study the time evolution of the density matrix ρ of an electron which starts in some coherent state on the qubit. At infinite time, ρ usually approaches the fully decoherent state, with $\rho_{nm} = \delta_{nm}/3$. However, when the Hamiltonian is symmetric under $1 \leftrightarrow 2$, the element ρ_{12} approaches a non-zero real value, implying a partial coherence of the asymptotic state. The asymptotic density matrix depends only on $\mathrm{Re}[\rho_{12}(t=0)]$. In several cases, the information stored on the qubit is protected from the noise.

PACS numbers: 03.67.-a, 05.40.-a, 03.65.Yz, 03.67.Lx

Quantum computation operates on information stored in "qubits", which are superpositions of two basic quantum states [1]. Clearly, quantum computation requires the stability of the quantum state stored on each qubit, and therefore it can be used only while this state remains coherent [2]. Interactions between qubits and their environment, including input-output measurement devices, can cause decoherence which destroys the information stored in the qubits. Attempts to avoid decoherence have led to studies of decoherence-free subspaces, within which the quantum state remains protected [3]. The states in such subspaces are practically decoupled from the sources of the decoherence, due to symmetries of the system.

Several proposals [4, 5] to reduce the decoherence of the single qubit states use *three* basic states: two degenerate states ($|1\rangle$ and $|2\rangle$) forming the qubit, and the third state ($|3\rangle$) representing a control register. Models include an electron in a three-state atom [4], or in three single-level quantum dots [6]. Below we refer to these states as sites in a tight binding model on a ring. The general Hamiltonian for such a system contains "site" energies ϵ_n and "hopping" matrix elements J_{nm},

$$\mathcal{H}_0 = \sum_n \big(\epsilon_n |n\rangle\langle n| - (J_{nn+1}|n\rangle\langle n+1| + \text{h.c.})\big) \ , \qquad (1)$$

where we identify $n = 4$ with $n = 1$. In the non-stochastic case, one observes Rabi oscillations of the electron wave function, with frequencies equal to differences between the three eigen-energies of $\mathcal{H}_0$. These oscillations are expected to decay when the system is coupled to the environment, due to decoherence.

We represent the latter coupling by a stochastic telegraph noise [7], which affects only the control register: ϵ_3 follows a Markov process or a two level jump process, and is given by $\epsilon_3 = \epsilon - f(t)\zeta$, where $f(t)$ jumps randomly from 1 to -1 (or from -1 to 1) with rate w_{-+} (or w_{+-}). These jumps in $f(t)$ result from a contact with some noise source, so that the probabilities $p_\pm$ to find $f = \pm 1$ obey a Boltzmann distribution with energy gap ΔE at temperature T, $p_+ = 1 - p_- = 1/[1 + e^{-\Delta E/k_B T}]$. Detailed balance implies $p_- w_{+-} = p_+ w_{-+}$, and therefore the jump rates can be written as

$$w_{\pm\mp} = \lambda p_\pm \ , \qquad (2)$$

where $\lambda = w_{+-} + w_{-+}$. Such a fluctuating site energy can be envisaged to occur because of noisy gate voltages or because of the coupling to a device (e.g. a point contact) which measures the occupation on that site [8]. Being based on symmetry considerations, our results may be expected to hold in other models as well.

Writing the quantum state of the system as $|\psi\rangle = \sum_n c_n |n\rangle$, the density matrix $\rho = |\psi\rangle\langle\psi|$ has elements $\rho_{nm} = c_n c_m^*$. Decoherence is usually associated with *dephasing*: the noise generates random fluctuations in the relative phases of c_n and c_m, causing the decay of the average off-diagonal matrix elements. Full decoherence corresponds to a fully "mixed" state, where all the off-diagonal elements of ρ vanish and all the three basis states (in any representation) have equal occupation probabilities (in our case $\rho_{nn} = 1/3$). In contrast, *relaxation* describes the decay of the system towards a "pure" state, which is a coherent eigenstate (the ground state at zero temperature) of the non-stochastic Hamiltonian [9]. Possible measures of decoherence include

$$\mathcal{D} \equiv 1 - \mathrm{Tr}(\rho^2) \ , \qquad (3)$$

(one always has $\mathrm{Tr}\rho = 1$) or the von Neumann entropy,

$$\mathcal{S} = -\mathrm{Tr}(\rho \log \rho) \ . \qquad (4)$$

In these expressions, ρ represents a certain average over the stochastic noise (see below). Both measures vanish for a "pure" state, where the eigenvalues of ρ are equal to 1 and two zeroes, and have maxima (2/3 and $\log 3$, respectively) for the fully decoherent state. Intermediate values of these measures require that off-diagonal matrix elements of ρ differ from zero, representing *partial decoherence*. Such deviations from full decoherence, as found below, can probably be interpreted as a mixture of decoherence and relaxation [9].

Below we find the stochastic evolution in time of ρ, beginning with an initial coherent state in which the electron is on the qubit [i.e. $R^+ = 1$, where $R^\pm \equiv \rho_{11}(t=0) \pm \rho_{22}(t=0)$]. For arbitrary parameters of the non-stochastic Hamiltonian ($\epsilon_n,\ J_{nm}$) and of the noise ($\zeta,\ w_{\mp\pm}$), our system develops full decoherence. However, when the Hamiltonian is symmetric under the interchange of the two qubit states, $1 \leftrightarrow 2$, then the system maintains *partial* decoherence: although ρ_{13} and ρ_{23} (i.e., the matrix elements which mix the qubit and the register states) approach zero, the matrix element ρ_{12} can approach a non-zero real value, implying an entanglement of the final qubit states $|1\rangle$ and $|2\rangle$. Furthermore, these states attain asymptotically equal populations $\rho_{11} = \rho_{22} = (1-\rho_{33})/2$. Surprisingly, these asymptotic values of ρ depend *only* on the initial value of the parameter $y = \mathrm{Re}[\rho_{12}(t=0)]$, which contains information on the initial entanglement of the two qubit states, and *not* on any of the other parameters (of the Hamiltonian, the noise or the initial state)! A fully coherent initial state would have $|\rho_{12}|^2 = \rho_{11}\rho_{22}$, and therefore $y^2 \le [1-(R^-)^2]/4$. In particular, the limiting values $y = \pm 1/2$ (and $R^- = 0$) correspond to the "bonding" and "anti-bonding" states $(|1\rangle \pm |2\rangle)/\sqrt{2}$, which are shown below to be protected from the noise, forming an example for a decoherence-free subspace.

Here we discuss the simplest symmetric case, which contains only hopping between $|1\rangle$ or $|2\rangle$ to $|3\rangle$: we replace $\mathcal{H}_0$ by $\mathcal{H} = \mathcal{H}_0 - f(t)\mathcal{V}$, with $\mathcal{V} = \zeta|3\rangle\langle 3|$, and set $\epsilon_1 = \epsilon_2 = 0,\ \epsilon_3 = \epsilon,\ J_{12} = 0,\ J_{13} = J_{23} = J$. The 3×3 density matrix $\rho(t)$ obeys the von-Neumann-Liouville equation, $i\partial_t\rho(t) = [\mathcal{H}, \rho(t)]$. It is convenient to rewrite this equation using the Liouville super-operators, $i\partial_t\rho(t) = \mathcal{H}^\times\rho(t) = \big[\mathcal{H}_0^\times - f(t)\mathcal{V}^\times(t)\big]\rho(t)$, with the obvious definition $\mathcal{A}^\times\rho \equiv [\mathcal{A}, \rho(t)]$.

A treatment of the equation of motion with the underlying stochasticity in $f(t)$ can be found in the literature on the stochastic theory of lineshapes [10]. Here we follow Blume [11], and replace each dynamic operator $\mathcal{O}(t)$ by a 2×2 matrix $\widetilde{\mathcal{O}}(t)$, such that $(b|\widetilde{\mathcal{O}}(t)|a)$ represents the *conditional average* of $\mathcal{O}(t)$, given that the stochastic variable $f(t)$ had the value $a (=\pm 1)$ at time $t = 0$ and the value b $(=\pm 1)$ at time t. At the end one may average over the stochastic process. The choice of the averaging procedure depends on the particular experiment. If one performs only one experiment, with a single initial state of the noise, then the initial value of $f(t)$ is fixed (say at a). The final value of $f(t)$ can then be either $b = +1$ or $b = -1$, and thus an appropriate average is

$$[\mathcal{O}]_{av,a} = \sum_{b=\pm 1} (b|\widetilde{\mathcal{O}}|a)\ . \tag{5}$$

In many quantum measurements it is more appropriate to repeat the measurement many times, on systems which are prepared in the same way. In this case, it is necessary also to average over the initial value of $f(t)$ (which equals $a = \pm 1$ with probability p_a), namely

$$[\mathcal{O}]_{av} = \sum_{a,b=\pm 1} p_a (b|\widetilde{\mathcal{O}}|a) \equiv \sum_a p_a [\mathcal{O}]_{av,a}\ . \tag{6}$$

For finite temperature T, i.e. $|\Delta p| < 1$, where $\Delta p = p_+ - p_-$, our calculations yield similar qualitative results for all these averages. In particular, all the averages coincide in the asymptotic limit $t \to \infty$.

We next construct a Markov chain of events in the time interval 0 to t, where the back and forth jumps of $f(t)$ are random and Poisson distributed [12]. The time evolution of $(b|\widetilde{\rho}(t)|a)$ is now divided into two parts:

$$\begin{aligned}\partial_t(b|\widetilde{\rho}|a) = &-i\big[\mathcal{H}_0^\times - b\mathcal{V}^\times\big](b|\widetilde{\rho}|a)\\ &+ w_{b,-b}(-b|\widetilde{\rho}|a) - w_{-b,b}(b|\widetilde{\rho}|a)\ .\end{aligned} \tag{7}$$

The first term on the RHS applies if $f(t)$ remains unchanged at time t (i.e. equal to b). In this case, the time evolution proceeds with the Liouville operator which corresponds to the original Hamiltonian, with $\epsilon_3 = \epsilon - b\zeta$. The last two terms arise if $f(t)$ flips exactly at time t, either from $-b$ to b (first term in the second line) or from b to $-b$ (last term). Equation (7) can be put into a matrix form (in the stochastic 2−dimensional space),

$$\partial_t\widetilde{\rho}(t) = \big[-i\big(\mathcal{H}_0^\times\widetilde{1} - \mathcal{V}^\times\widetilde{F}\big) + \widetilde{W}\big]\widetilde{\rho}(t)\ , \tag{8}$$

where $\widetilde{1}$ is the 2×2 unit matrix, while

$$\widetilde{W} = \begin{pmatrix} -w_{-+} & w_{+-} \\ w_{-+} & -w_{+-} \end{pmatrix} \equiv \lambda(\widetilde{\mathcal{T}} - \widetilde{1}) \tag{9}$$

is the relaxation matrix [see Eq. (2)]. In the above we have used the notations

$$\widetilde{F} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ ,\quad \widetilde{\mathcal{T}} \equiv \begin{pmatrix} p_+ & p_+ \\ p_- & p_- \end{pmatrix}\ . \tag{10}$$

Note that Eq. (8) contains two types of operators: $\mathcal{H}_0^\times$ and $\mathcal{V}^\times$ are Liouville super-operators (represented by 9×9 matrices), while $\widetilde{F}$ and $\widetilde{W}$ are 2×2 matrices acting in the 2−dimensional space of the stochastic variable. The matrix $\widetilde{\rho}$ is of order 18×18.

Using a Laplace transform, $\widetilde{\rho}(s) = \int_0^\infty dt e^{-st}\widetilde{\rho}(t)$ (unless specifically stated, all $\widetilde{\rho}$'s below depend on s), Eq. (8) becomes

$$\begin{aligned}&(s\widetilde{1} - \widetilde{W})(\widetilde{\rho}_{11} \pm \widetilde{\rho}_{22}) = R^\pm\widetilde{1} + 2J\mathrm{Im}[\widetilde{\rho}_{13} \pm \widetilde{\rho}_{23}]\ ,\\ &(s\widetilde{1} - \widetilde{W})\widetilde{\rho}_{12} = \rho_{12}(0) - iJ(\widetilde{\rho}_{13} - \widetilde{\rho}_{32})\ ,\\ &[s\widetilde{1} - \widetilde{W} - i\widetilde{e}]\widetilde{\rho}_{13} = iJ(\widetilde{\rho}_{33} - \widetilde{\rho}_{11} - \widetilde{\rho}_{12})\ ,\\ &[s\widetilde{1} - \widetilde{W} - i\widetilde{e}]\widetilde{\rho}_{23} = iJ(\widetilde{\rho}_{33} - \widetilde{\rho}_{22} - \widetilde{\rho}_{21})\ ,\end{aligned} \tag{11}$$

where we have set $\rho_{13}(0) = \rho_{23}(0) = 0$ and defined $\widetilde{e} \equiv \epsilon\widetilde{1} - \zeta\widetilde{F}$. Using the last two equations, and defining

$$\widetilde{A} = [s\widetilde{1} - \widetilde{W} - i\widetilde{e}]^{-1} \equiv \widetilde{A}_R + i\widetilde{A}_I\ , \tag{12}$$

we find $\mathrm{Im}[\widetilde{\rho}_{13}+\widetilde{\rho}_{23}] = J\widetilde{A}_R\widetilde{B}$, where $\widetilde{B} \equiv 2\widetilde{\rho}_{33} - \widetilde{\rho}_{11} - \widetilde{\rho}_{22} - 2\mathrm{Re}[\widetilde{\rho}_{12}]$. Substitution into Eq. (11) then yields

$$(s\widetilde{1}-\widetilde{W})(\widetilde{\rho}_{11}+\widetilde{\rho}_{22}) = R^+\widetilde{1} + 2J^2\widetilde{A}_R\widetilde{B}\ ,$$
$$(s\widetilde{1}-\widetilde{W})\mathrm{Re}[\widetilde{\rho}_{12}] = y\widetilde{1} + J^2\widetilde{A}_R\widetilde{B}\ . \tag{13}$$

It is easy to check that $\sum_n \rho_{nn}(t) \equiv 1$ implies that $(s\widetilde{1}-\widetilde{W})\sum_n\widetilde{\rho}_{nn} = 1$, i.e. $\sum_n\widetilde{\rho}_{nn} = \widetilde{P}$, where $\widetilde{P} \equiv (s\widetilde{1}-\widetilde{W})^{-1}$ is the Laplace transform of the probabilities to start at a and end up at b at time t. Combining this identity with the above equations, one has

$$\widetilde{B} = \big(2-3R^+-2y\big)[s\widetilde{1}-\widetilde{W}+8J^2\widetilde{A}_R]^{-1}. \tag{14}$$

To complete the solution of the Liouville equations we also find two coupled equations for $\widetilde{\rho}_{11}-\widetilde{\rho}_{22}$ and $\mathrm{Im}[\widetilde{\rho}_{12}]$:

$$\widetilde{\rho}_{11}-\widetilde{\rho}_{22} = \widetilde{C}\big(R^-\widetilde{1} + 4J^2\widetilde{A}_I\widetilde{Z}\mathrm{Im}[\rho_{12}(0)]\big)\ ,$$
$$\mathrm{Im}[\widetilde{\rho}_{12}] = \widetilde{Z}\big(\mathrm{Im}[\rho_{12}(0)] - J^2\widetilde{A}_I(\widetilde{\rho}_{11}-\widetilde{\rho}_{22})\big)\ ,$$
$$\widetilde{Z} = (s\widetilde{1}-\widetilde{W}+2J^2\widetilde{A}_R)^{-1}\ ,$$
$$\widetilde{C} = (\widetilde{Z}^{-1}+4J^4\widetilde{A}_I\widetilde{Z}\widetilde{A}_I)^{-1}\ . \tag{15}$$

The time evolution of the density matrix follows inverse Laplace transforms, which involve decaying oscillations, approaching asymptotic time independent limits as $t\to\infty$. Before presenting an example of this time evolution, we consider these latter limits, using the identity $\widetilde{\rho}(t\to\infty) = \lim_{s\to 0}[s\widetilde{\rho}(s)]$. Noting that $\widetilde{A}_I$, $\widetilde{Z}$ and $\widetilde{C}$ all remain finite as $s\to 0$, we conclude that $\widetilde{\rho}_{11}(\infty)-\widetilde{\rho}_{22}(\infty) = \Im[\widetilde{\rho}_{12}(\infty)] = 0$, $\widetilde{\rho}_{13}(\infty) = \widetilde{\rho}_{23}(\infty) = 0$. Tedious but straightforward algebra is needed to show that

$$\lim_{s\to 0} J^2\widetilde{A}_R[s\widetilde{1}-\widetilde{W}+8J^2\widetilde{A}_R]^{-1}$$
$$= \frac{1}{16}\begin{pmatrix} 1+\epsilon/\zeta & 1+\epsilon/\zeta \\ 1-\epsilon/\zeta & 1-\epsilon/\zeta \end{pmatrix}\ . \tag{16}$$

Using also $\widetilde{P}(t\to\infty) = \lim_{s\to 0}[s\widetilde{P}] = \widetilde{\mathcal{T}}$, we find that the matrix $\widetilde{\rho}$ splits as

$$(b|\rho_{mn}|a) = \rho^\infty_{mn}(b|\widetilde{\mathcal{T}}|a) = \rho^\infty_{mn}p_b\ , \tag{17}$$

with

$$\rho^\infty = \frac{1}{8}\begin{bmatrix} R^++2-2y & 6y+2-3R^+ & 0 \\ 6y+2-3R^+ & R^++2-2y & 0 \\ 0 & 0 & 4-2R^++4y \end{bmatrix}. \tag{18}$$

Since the factor p_b in (17) gives the probability to find the stochastic variable in state b, it does not affect the actually measured density matrix, which will always approach ρ^∞. In particular, $[\widetilde{\mathcal{T}}]_{av,a} = 1$ and therefore all the averages will yield the same results. When the electron is initially placed on the qubit, i.e. $R^+ = 1$, then ρ^∞ depends only on y, and *not* on the other details of the initial state. The result is also independent of the non-stochastic Hamiltonian (ϵ, J) and of the stochastic noise (Δp, ζ, λ), as long as $\lambda > 0$ and $|\Delta p| < 1$. In contrast, when $\Delta p = \pm 1$, i.e. at zero temperature, the system jumps to the state with $\epsilon_3 = \epsilon \mp \zeta$ and then it stays there forever, behaving like the non-stochastic system with that energy.

Substituting Eq. (18) with $R^+ = 1$ into the decoherence measures (3) and (4) yields maximal decoherence, $\rho_{nm}(\infty) = \delta_{nm}/3$, when $y = 1/6$. When the electron starts in state $|1\rangle$, then $y = 0$, and one has asymptotic partial decoherence, $\rho_{11} = \rho_{22} = 3/8$, $\rho_{12} = -1/8$. When the initial state of the qubit is fully entangled, $(|1\rangle + e^{i\alpha}|2\rangle)/\sqrt{2}$, then

$$\rho(0) = \frac{1}{2}\begin{bmatrix} 1 & e^{-i\alpha} & 0 \\ e^{i\alpha} & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\ , \tag{19}$$

so that $y = \frac{1}{2}\cos\alpha$ and $\rho^\infty_{33} = \frac{1}{2}\cos^2(\alpha/2)$. Measuring this occupation of the control register then yields a measure of the phase α, in spite of the decoherence!

Both measures of decoherence, Eqs. (3) and (4), vanish at all times if the electron starts with $y = -1/2$ ($\alpha = \pi$), which corresponds to the "anti-bonding" state $|AB\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$. Being fully antisymmetric under $1 \leftrightarrow 2$, $|AB\rangle$ does not couple to $|3\rangle$, and is not affected by the symmetric Hamiltonian. Therefore, this state forms the *decoherent free subspace*, which remains constant and fully protected at all times. As y increases from $-1/2$, one encounters a mixture of this "anti-bonding" state and the other two orthogonal eigenstates of $\mathcal{H}_0$, and thus the system develops various degrees of partial decoherence. The value $y = 1/2$ ($\alpha = 0$) corresponds to the initial "bonding" state, $|B\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$. In this case, $\rho^\infty_{11} = \rho^\infty_{22} = \rho^\infty_{12} = 1/4$, and the projection of the quantum state onto the qubit subspace remains in the fully coherent state $|B\rangle$.

To demonstrate typical results, we start with an arbitrary choice of parameters, $y = -0.3$, $R^+ = 1$, $R^- = \sqrt{1-4y^2}$ and set $\epsilon = J$. Figure 1 shows the three occupations $\rho_{nn}(t)$. The non-stochastic limit (left panel) exhibits Rabi oscillations with period $2\pi/J$, corresponding to the difference between eigen-energies of $\mathcal{H}_0$; the coupling of the qubit to the control register causes periodic partial periodic occupation of the latter. In the stochastic cases, small values of $\lambda(\ll J)$ and/or $\zeta(\ll J)$ yield small corrections to this non-stochastic picture at short times, with an eventual "flow" to the asymptotic behavior described above. The remaining panels of Fig. 1 show $[\rho]_{av}$ for a relatively large stochasticity, $\zeta = \lambda = J/2$ and $\Delta p = p_+ - p_- = 0.1$ (high T) or 0.9 (low T). The other averages $[\rho]_{av,\pm}$ exhibit similar qualitative behavior, and all the averages approach the asymptotic values given by Eq. (18). Clearly, the decoherence time is longer for the lower temperature (larger $|\Delta p|$). Figure 2 shows the decoherence measure $\mathcal{D}$, calculated using the three averages $[\rho]_{av,\pm}$ (dashed lines) and $[\rho]_{av}$ (full lines). Note also that the asymptotic limit $\mathcal{D} = 0.34$ is much smaller than the fully decoherent value $2/3$, demonstrating the partial decoherence. Also, the approach to this asymptotic value is much slower for the lower temperature.

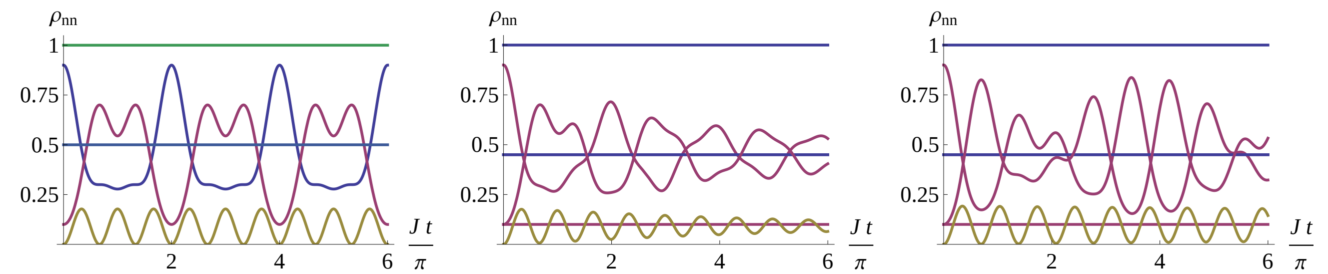

FIG. 1: The site occupations for an initial coherent state with $y = -0.3$, and $\epsilon = J$. Left: non-stochastic limit. Center and right: average occupations for stochastic cases, with $\zeta = \lambda = 0.5J$, $\Delta p = .1$ and $.9$. ρ_{11} and ρ_{22} approach $(3-2y)/8 = 0.45$ and ρ_{33} approaches $(1+2y)/4 = 0.1$ (shown by horizontal lines).

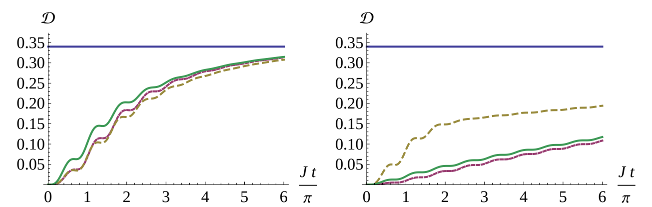

FIG. 2: Time dependence of the decoherence measure $\mathcal{D}$, for the same parameters as in Fig. 1. Left (right): $\Delta p = 0.1$ (0.9) and with $\zeta = \lambda = 0.5J$. Small (large) dashes correspond to $[\rho]_{av,+}$ ($[\rho]_{av,-}$). Full lines correspond to $[\rho]_{av}$. The horizontal line shows the asymptotic value, $\mathcal{D} = 0.34$.

The equations of motion become more complicated when one adds direct hopping between the two qubit states, i.e. $J_{12} \neq 0$. We have solved the equations of motion for this generalized case, by inverting the full 9×9 matrices for the super-operators [13]. Surprisingly, the asymptotic result (18) remains the same, independent of J_{12}. This robust result depends only on the symmetry $1 \leftrightarrow 2$. We have also checked that any deviation from this symmetry (e.g. by introducing a Aharonov-Bohm flux inside the ring formed by the three sites, or by having different energies on sites 1 and 2) immediately changes the asymptotic limit back into the fully decoherent one.

It would be very interesting to check if Eq. (18) also holds for other sources of noise, e.g. a capacitive coupling to a point contact [8]. It would also be interesting to consider generalizations of these results to systems containing several qubits. We hope that our results will also be tested experimentally.

SD is grateful to BGU for the warm hospitality he received during several visits, when this project was carried out. AA and OEW acknowledge discussions with S. Gurvitz and with Y. Imry and support from the DIP.

* Electronic address: aaharony@bgu.ac.il; Also at Tel Aviv University, Tel Aviv 69978, Israel
† Also at Tel Aviv University, Tel Aviv 69978, Israel
[1] C. Bennett, Phys. Today **48**, 24 (1995); D. DiVincenzo, Science **270**, 255 (1995); D. Aharonov, in Ann. Rev. Comp. Phys., edited by D. Stauffer (World Scientific, Singapore, 1998); A. M. Steane, Rep. Prog. Phys. **61**, 117 (1998).
[2] W. H. Zurek, Phys. Today **44** (no. 10), 36 (1991).
[3] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
[4] A. Beige, D. Baun, B. Tregenna, and P. L. Knight, Phys. Rev. Lett. **85**, 1762 (2000).
[5] I. O. Kulik, T. Hakioğlu, and A. Barone, Eur. Phys. J. B **30**, 219 (2002).
[6] D. Loss and D. P. Di Vincenzo, Phys. Rev. A **57**, 120 (1998); T. Tanamoto, Phys. Rev. A **61**, 22305 (2000).
[7] See for instance S. Dattagupta, *Relaxation Phenomena in Condensed Matter Physics* (Academic Press Orlando, 1987).
[8] S. A. Gurvitz and G. P. Berman, Phys. Rev. B **72**, 073303 (2005).
[9] S. A. Gurvitz and D. Mozyrsky, Phys. Rev. B **77**, 075325 (2008).
[10] R. Kubo, in *Fluctuation, Relaxation and Resonance in Magnetic systems* (D. ter Haar, editor), (Oliver and Boyd, Edinburg, 1962); also in Adv. Chem. Phys. **15**, 101 (1969).
[11] M. Blume, Phys. Rev. **174**, 351 (1968).
[12] See Chapter VIII of Ref. 7.
[13] Details will be published separately.