STUDIES IN TURBINE GEOMETRY—II. ON THE SUB-GEOMETRIES OF LIE WHICH BELONG TO THE MOBIUS-LAGUERRE PENCIL.

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1. WE have seen in the first paper of this series[1] that the totality of turbines in a metrical S_n may be represented by the points of a projective S_{n+2}, the oriented spheres (which are particular turbines whose Mobius- and Laguerre-spheres coincide) corresponding to points on the Lie quadric

$$\Omega\,(X, X) = -x_0^2 + x_1^2 + \cdots\cdots + x_{n+1}^2 - x_{n+2}^2 = 0 \qquad (1\cdot1)$$

The Projective Group in S_{n+2} which keeps Ω invariant (the Lie Group) is associated in the sense of Klein's Erlanger programme with Lie Geometry which is the invariant theory of transformations which carry turbines into turbines and oriented spheres into oriented spheres and conserve proper contact. Among the sub-geometries of Lie's geometry only two have received detailed attention, namely, those associated with the sub-groups of the Lie Group which have the prime $x_0 + x_1 = 0$ corresponding to spheres of infinite radius, and the prime $x_{n+2} = 0$ corresponding to spheres of zero radius as invariant primes. The former of these is Laguerre's geometry of oriented lines and spheres, and the latter is the one based on the inversion group and may be called Inversion Geometry or Mobius Geometry.

In this paper it is proposed to discuss the main features of the sub-geometries of Lie obtained by taking any one of the pencil of primes $k\,(x_0 + x_1) + x_{n+2} = 0$ as an invariant variety, and to obtain a principle of transference by which any result of Mobius Geometry may be associated with a corresponding result in any other Geometry of the Mobius-Laguerre pencil. While the existence of these geometries is hinted[2] in Prof. Blaschke's monumental work *Vorlesungen uber Differentialgeometrie* of which the third volume is devoted to circle and sphere geometries, the introduction of the turbine concept is a powerful help in giving concreteness and reality to these geometries.

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[1] "The concepts of Turbine Geometry," *Journal of the Indian Mathematical Society*, 1938, Vol. III, No. 3. This paper will be referred to as T.G.I. The reference here is to T.G.I., para 4.

[2] Blaschke, *Differentialgeometrie*, III, 1929, p. 208.

Invariant Concepts of G_k.

2. We define the geometry G_k as the invariant theory of the $(n+1)(n+2)/2$ parameter group of projective transformations which carry Ω into itself and have the prime

$$\pi_k \equiv \Omega\,(X, K) \equiv k\,(x_0 + x_1) - x_{n+2} = 0 \qquad (2\cdot 1)$$

as an invariant prime and hence also its pole $K \equiv (-k, k, 0, 0, \cdots 0, 1)$ as an invariant point. The geometries G_0 and G_∞ corresponding to $k = 0$ and $k \to \infty$ are the geometries of Mobius and Laguerre respectively.

On the algebraic side, the determination of a complete system of invariants of a system of n oriented spheres or turbines is equivalent to the determination of the invariants of one quadratic form and a set of $n+1$ linear forms in cogredient variables, one associated with each turbine (by polarisation w.r.t. Ω), and the invariant form $(2\cdot 1)$.

In this geometry, oriented spheres of radius k play a peculiar rôle so that it is convenient to have a special name for them. We call them k-spheres. We shall also use a prefix k attached to any concept of Mobius or Laguerre geometry to denote the corresponding concept of G_k, *e.g.*, k-inversion.

Corresponding to the angle and the tangential segment which are invariant respectively in G_0 and G_∞, we have here a basic invariant which can be expressed either in terms of an angle or of a length which we proceed to define. Consider two surface elements of a k-sphere. These determine two points P, Q and also two oriented primes p, q, one through each point. The angle $\pm\,\theta$ between the oriented primes is determinate mod. $2\,\pi$ and will be called the " angle between the two surface elements". The arc length $\pm\, s$ between the points P, Q measured along the great circle through PQ is also determinate mod. $2\,\pi k$ and will be called " the arc segment " between the oriented surface elements. They are connected by the relation that the latter is k times the former.

3. Firstly we notice that $x_{n+2}/(x_0 + x_1) = r$ the oriented radius of the inner sphere of the turbine X. From $(2\cdot 1)$ we see that

the transformations of G_k carry every turbine of inner radius k into another turbine of inner radius k. In particular they carry oriented spheres of radius k into oriented spheres of radius k. $(3\cdot 1)$

We have seen that in Lie's geometry and hence also in its sub-geometries proper contact is conserved and hence that these are essentially geometries of surface elements. It is thus important to know how the surface elements of a k-sphere are carried over into the surface elements of the transformed k-sphere. Taking n neighbouring k-spheres, these are carried over into n

neighbouring k-spheres. In the limiting position n such spheres have in common a pair of diametrically opposite surface elements. Thus a pair of diametrically opposite elements of a k-sphere correspond to a pair of diametrically opposite surface elements of the transformed k-sphere. From this follows easily the following basic theorem of G_k geometry:

The angle and the arc segment between two surface elements of a k-sphere are invariants in the geometry G_k. (3·2)

If we take two oriented spheres we can in general find a k-sphere S having proper contact with both. The transformations of G_k carry the two spheres into two others which touch S′ the k-sphere which is the transform of S. Applying (3·2) to the two contact surface elements and their transforms, we see that

the angle and the arc segment between the surface elements in which a k-sphere touches two oriented spheres are invariants in G_k. (3·3)

In G_0 (Mobius Geometry) the arc segment is of zero length and the angle (angle of intersection) is the only invariant. On the other hand, in G_∞ (Laguerre Geometry) the angle is zero and the arc segment becomes the common tangential segment.

4. The Fundamental Involution I : Associated Spheres and Turbines.

The involution I (projective transformation of period two) which has K for a fixed point and its polar prime π_k for fixed prime carries Ω into itself and is called the fundamental involution I of G_k. It associates oriented spheres in pairs in such a manner that the tangent primes at the corresponding points on Ω meet on π_k. Interpreting this we have the result

Two "associated spheres" S, S′ which correspond to points on Ω collinear with K have the property that each is the envelope of k-spheres touching the other. Hence the spheres S, S′ are concentric and the sum of their radii is $2k$. (4·1)

In G_0 (Mobius Geometry) two associated spheres differ only in orientation, while in Laguerre Geometry the associate of every sphere is the point sphere l at infinity (*vide* T.G.I. 4·5).

We have seen in T.G.I. para 2, that a turbine may be defined by means of its inner sphere c and the tangential segment t or through its outer sphere C and the angle α, these two specifications using respectively the invariant concepts of G_0 and G_∞ respectively. Exactly in the same way, we may define a turbine in terms of the concepts of G_k as follows. All the k-spheres which touch a turbine have for envelope two "associated spheres" S S′ whose radii are $k+\rho$ and $k-\rho$ where

$$\rho^2 = \{t^2 + (r-k)^2\}. \qquad (4·2)$$

To obtain the turbine (c, t), we take every oriented surface element of S and slide it along the unique k-sphere through it in all directions so that it makes a constant angle β with its old position; in other words move it through a constant arc segment $k\,\beta$—where

$$\tan\ \beta = t/(r - k). \tag{4·3}$$

The same turbine may also be obtained by treating the surface elements of S′ in a similar manner the angle and the arc segment being, however, now $(\pi - \beta)$ and $k\,(\pi - \beta)$. If the angles (or the arc segments) to be taken with S and S′ be interchanged, we shall get another turbine which is the transform of the original turbine in the involution I. Two such turbines correspond to points in S_{n+2} collinear with K and separated harmonically by K and π_k. When β is a right angle, the two turbines coincide and the point lies on π_k so that the inner radius of the turbine is k. From the equality of the arc segments $k\,(\pi - \beta)$ and $k\,\beta$ we deduce that

When a k-sphere touches a k-turbine (a turbine whose inner radius is k), its section with the outer sphere of the latter is a diametral section of the k-sphere. (4·4)

In such a case we say that *the k-sphere is bisected by the outer sphere of the turbine.* In the correspondence which we shall be later setting up between G_0 and G_k, non-oriented spheres of radius k correspond to points and their bisection by other spheres to the incidence relation of points and spheres (*vide* para 7).

Since by (3·2) the angles β or $\pi - \beta$ either of which may be used to specify the turbine are invariants in G_k, it may be expected that $\tan^2 \beta$ will be an absolute invariant of the turbine under the transformations of G_k.

5. Invariants of a System of Turbines.

For real transformations of the Lie group a single turbine X has but one invariant—the sign of Ω (X X). If this is positive $R^2 - r^2 = t^2$ is positive and t is real; if it is negative $R^2 < r^2$ and t is imaginary.

In G_k a turbine X has a single absolute invariant, namely

$$\Omega\,(K, X)^2/\Omega\,(X\,X)\ \Omega\,(K\,K) = -(r - k)^2/t^2 \tag{5·1}$$

which represents the square of the cosine of the non-Euclidian separation of the points K and X with Ω as the absolute quadric. Its value is $-\cot^2\beta$ where β is the angle mentioned in the previous para. When the invariant (5·1) vanishes $\beta = \pi/2$ and the turbine has an inner radius k. When it becomes infinite $\beta = 0$ and the turbine is a sphere.

For two turbines $X_1 = (c_1, t_1)$ and $X_2 = (c_2, t_2)$ we have, besides their separate invariants $\cot^2\beta_1$ and $\cot^2\beta_2$ a joint invariant in Lie's Geometry and hence also in its sub-geometries, namely

$$\Omega\,(X_1\,X_2)^2/\Omega\,(X_1\,X_1)\;\Omega\,(X_2\,X_2) = (t_{12}^2 - t_1^2 - t_2^2)/4t_1^2t_2^2, \qquad (5\cdot2)$$

where t_{12} is the length of the common tangential segment of c_1 and c_2. If both the turbines are of inner radius k, we have a simple geometrical interpretation for the joint invariant (5·2). We take a k-sphere which has proper contact with both the k-turbines X_1, X_2 and which is therefore cut by their outer spheres in diametral prime sections which are respectively perpendicular to the lines joining its centre with those of the turbines. The triangle formed by the three centres has sides t_1, t_2 and t_{12} and hence the invariant (5·2) is the square of the cosine of the angle between the two diametral primes. Since the first member of (5·2) represents the square of the cosine of the non-Euclidean separation of X_1 and X_2 with respect to the absolute Ω, we see that

> *we may identify the non-Euclidian separation of the points representing two k-turbines with the angle between the two diametral prime sections of a k-sphere touching both the turbines, along which it is cut by the outer spheres of the two turbines.* (5·3)

If the given turbines have an inner radius other than k, we have under G_k an absolute invariant

$$\begin{aligned}&\Omega\,(X_1\,X_2)\;\Omega\,(K\,K) \div \Omega\,(X_1\,K)\;\Omega\,(X_2\,K)\\ &\quad= (t_{12}^2 - t_1^2 - t_2^2) \div (k - r_1)\,(k - r_2)\\ &\quad= (d^2 - r_1^2 - r_2^2 + 2r_1r_2 - t_1^2 - t_2^2) \div (k - r_1)\,(k - r_2)\\ &\quad= 2 + [d^2 - (r_1 - k)^2 - (r_2 - k)^2 - t_1^2 - t_2^2] \div (r_1 - k)\,(r_2 - k)\end{aligned}$$

where d is the distance between the centres of X_1 and X_2. (5·4)

When the two turbines are spheres, t_1 and t_2 vanish and $r_1 - k$, $r_2 - k$, d are the sides of a triangle formed by the centres of the spheres and of a k-sphere which has proper contact with both of them. Hence the invariant (5·4) is $4\sin^2\lambda/2$ where λ is the angle subtended at the centre of the k-sphere by the centres of the given spheres, *i.e.*, it is the angle between the surface elements which the k-sphere has in common with the two given spheres.

6. The non-oriented Sphere in the Geometry G_k.

If we restrict ourselves to turbines of inner radius k, the representative point lies on π_k and is subject to transformations induced on this prime by the projective group of G_k in the ambient space S_{n+2}. The resulting geometry may be considered as a geometry of non-oriented spheres in S_n, since every k-turbine has a non-oriented Mobius sphere, and conversely from any non-oriented sphere we may obtain a k-turbine by associating with it a concentric inner sphere of radius k. The section Ω_k of Ω by π_k is an invariant variety, the points on which correspond to spheres of radius k and such spheres are carried over into other spheres of radius k. They play the same part here as points do in ordinary Inversion Geometry. Unless otherwise stated, every sphere mentioned in this para is to be considered non-oriented.

The fundamental operation here is " k-inversion with respect to a given sphere S". This is an involutoric transformation of k-spheres into k-spheres which interchanges the two k-spheres in any coaxal system of spheres of which S is a member. If a k-sphere is bisected by S, then it is its own transform in the k-inversion determined by S. If we take the k-sphere as the basic element of the geometry G_k, any sphere not of radius k may be regarded as the totality of k-spheres bisected by it. In this sense S is its own inverse in the k-inversion defined by it. On π_k, S is represented by a point s which is not on Ω_k since we suppose that S is not a k-sphere. The k-inversion I_k (S) defined by S corresponds to the involutoric projective transformation of π_k which has s for fixed point and its polar variety with respect to Ω_k as fixed prime. The points on the section of this polar prime with Ω_k correspond to k-spheres bisected by S as may be seen from (4·4). More generally, two spheres of radii R_1, R_2 for which $d^2 = t_1^2 + t_2^2 = R_1^2 + R_2^2 - 2k^2$ correspond to points conjugate with respect to Ω_k since when they are converted into k-turbines we have $d = t_{12}$, $R_1^2 = k^2 + t_1^2$ and $R_2^2 = k^2 + t_2^2$. If p, q be two corresponding points in I_k (S), the sections of Ω_k by their polar primes are carried over into each other. Thus

The inversion I_k (S) replaces every k-sphere by the other k-sphere in the pencil of spheres determined by it and S. All the k-spheres which are bisected by a sphere P are carried over into k-spheres which are bisected by another sphere Q. P and Q are then transforms of each other in the k-inversion I_k (S).

It is known that all projective transformations which carry a quadric into itself may be obtained by successive involutoric transformations of the type discussed above. Thus the k-inversion is the basic operator which generates the group G_k.

7. Principle of Transference from G_0 to G_k.

Given any theorem in Mobius geometry G_0, we may express it as a projective relation between a set of points on the prime $\pi_0 = x_{n+2} = 0$ and the section Ω_0 of Ω by π_0. Any projective transformation between π_0 and π_k which carries Ω_0 into Ω_k will give a configuration with the same projective relationship to Ω_k as the original system had to Ω_0. Interpreting these relations in terms of the concepts of G_k we have a dual theorem in G_k for every theorem in G_0. All that is necessary is a scheme of interpretation like the one given below of projective relations in π_k in terms of the concepts of G_k.

(All the spheres referred to in this table are non-oriented.)

Situation in a Projective S_{n+1} with an invariant quadric ω	Interpretation in G_0 $S_{n+1} \equiv \pi_0$; $\omega \equiv \Omega_0$	Interpretation in G_k $S_{n+1} \equiv \pi_k$; $\omega \equiv \Omega_k$
Point	Sphere	Sphere
Point on the quadric.	Sphere of radius zero.	Sphere of radius k.
Projective separation of two points p, q with respect to the quadric ω as absolute.	Angle of intersection of the two spheres P and Q.	The angle between the two diametral sections of a k-sphere bisected by both P and Q.
The tangent prime at p passes through q.	The point P lies on the sphere Q.	The k-sphere P is bisected by Q.
Two points p, q conjugate w.r.t. the quadric.	Two spheres P, Q which are mutually orthogonal.	Two spheres of radii R_1, R_2 with their centres d apart where $d^2 = R_1{}^2 + R_2{}^2 - 2k^2$.
Two points p, q whose join touches the quadric.	The spheres P and Q touch.	The common section of P and Q is a sphere (in a lower dimension) of radius k.
Projective transformation of period 2 with p for invariant point and its polar prime w.r.t. the quadric for fixed prime.	Mobius Inversion.	k-inversion.
The automorphic group of ω in S_{n+1}.	Inversion group of sphere geometry in S_n.	The group of the geometry G_k

As an example of such generalisation let us take the configuration formed by two Hart-tetrads of circles. We have the result[3] :—

It is possible to find two tetrads of circles in a plane such that each circle of one system has with each circle of the other system a chord of the same length k.

In the same manner if the Miquel-Clifford configuration be considered as one in inversion Geometry, and generalised into a result in G_k, we have the following chain of theorems[4]

Take two circles C_1, C_2 which bisect a k-circle (circle of radius k*) named C. There is then another k-circle C_{12} (or C_{21}) which is also bisected by them.*

[3] *Vide* my paper, " On a principle of duality in circle sphere and line Geometries," *Mathematische Zeitschrift*, Bd. **44**, Heft 2, p. 194.

[4] A different generalisation of the Miquel-Clifford configuration is given in the paper referred to in the previous footnote (*vide* page 187).

If we take three circles C_1, C_2, C_3 *which bisect a k-circle* C, *the three k-circles* C_{12}, C_{23}, C_{31} *are all bisected by a circle* C_{123}.

Taking four circles C_1, C_2, C_3, C_4 *all of which bisect the* k-*circle* C *the four circles* C_{123}, C_{234}, C_{341}, C_{412} *obtained by the process mentioned above all bisect a k-circle* C_{1234} *and so on.*

a k-circle being associated with an even number, and a circle of different radius with an odd number of circles C_r.

When $k = 0$ we have the classical Miquel-Clifford configuration.

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