Correlation-induced spectral changes and energy conservation

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An energy conservation law is derived for fields generated by random, statistically stationary, scalar sources of any state of coherence. It is shown that correlation-induced spectral changes are in strict agreement with this law and that, basic to the understanding of such changes, is a distinction that must be made between the spectrum of a source and the spectrum of the field that the source generates. This distinction, which is obviously relevant for spectroscopy, does not appear to have been previously recognized. [S1050-2947(96)06610-3]

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I. INTRODUCTION

It has been predicted theoretically some years ago [1,2] that the spectrum of light and of other radiation may change on propagation, even in free space. This phenomenon, which was soon verified experimentally [3–5], has attracted a good deal of attention and has resulted in the publication of about 100 papers on this subject [6]. The spectral changes, which have their origin in spatial correlation properties of sources, may be of very different kinds. They may, for example, consist of redshifts or blueshifts of spectral lines, narrowing or broadening of the lines, and generation of new lines. Moreover, different spectral changes may occur in different directions of observation.

In spite of the considerable interest that has been shown in this effect, an explicit demonstration that such spectral changes do not violate energy conservation has, up to now, been demonstrated only under somewhat restricted circumstances [12–14]. In this paper we derive an energy conservation law for fields produced by random, statistically stationary sources of any state of coherence and we demonstrate that correlation-induced spectral changes do not violate this law. Our analysis, which is valid for both classical and quantum sources, shows also that basic for the understanding of this phenomenon is a distinction that must be made between the spectrum of a source and the spectrum of the field that the source generates. This distinction, which is obviously very relevant to the interpretation of spectroscopic data, does not appear to have been previously appreciated.

II. THE SOURCE SPECTRUM AND THE SPECTRUM OF THE RADIATED FIELD

We consider radiation from a scalar source, localized in a finite region D of space (Fig. 1). Let $p(\mathbf{r},t)$ denote the source density distribution and $E(\mathbf{r},t)$ the field generated by the source, at a point $\mathbf{r}$, at time t. We take both $p(\mathbf{r},t)$ and $E(\mathbf{r},t)$ to be the complex analytic signal representations [15] of a real source variable and a real field variable. In order to illustrate the main aspects of the theory we will ignore the vectorial nature of the problem and take $p(\mathbf{r},t)$ and $E(\mathbf{r},t)$ to be scalars. We may think of p and E as representing Cartesian components of the polarization vector and of one of the electromagnetic field vectors, respectively, although this, of course, is only a rough analogy. We will show elsewhere that the main conclusions of our analysis hold when the full electromagnetic nature of the source and of the field are taken into account.

Since we are interested in spectral properties it is convenient to deal with the Fourier transforms

$$\tilde{p}(\mathbf{r},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}p(\mathbf{r},t)e^{i\omega t}dt \tag{2.1}$$

and

$$\tilde{E}(\mathbf{r},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}E(\mathbf{r},t)e^{i\omega t}dt \tag{2.2}$$

of $p(\mathbf{r},t)$ and $E(\mathbf{r},t)$, respectively. They are related by the inhomogeneous Helmholtz equation

$$(\nabla^2+k^2)\tilde{E}(\mathbf{r},\omega)=-4\pi k^2\tilde{p}(\mathbf{r};\omega), \tag{2.3}$$

where

$$k=\omega/c \tag{2.4}$$

is the free-space wave number associated with the frequency ω, c being the speed of light *in vacuo*. The outgoing solution of Eq. (2.3) is

$$\tilde{E}(\mathbf{r},\omega)=k^2\int_D\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\tilde{p}(\mathbf{r}',\omega)d^3r'. \tag{2.5}$$

The field $\tilde{E}^{(\infty)}(r\mathbf{u},\omega)$ at a point $\mathbf{r}=r\mathbf{u}$ ($\mathbf{u}^2=1$) in the far zone (the radiation field) is obtained at once from Eq. (2.5) by making use of the asymptotic approximation (see Fig. 1)

$$\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\sim e^{-ik\mathbf{u}\cdot\mathbf{r}'}\frac{e^{ikr}}{r} \tag{2.6}$$

($kr\to\infty$ with $\mathbf{u}$ fixed) and hence

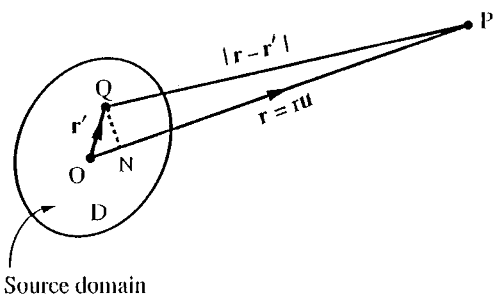

FIG. 1. Illustrating notation relating to Eq. (2.6). P is a point in the far zone of the source, $\mathbf{u}$ is a unit vector, and $\mathbf{u}\cdot\mathbf{r}'=\overline{ON}$.

$$\tilde{E}^{(\infty)}(r\mathbf{u},\omega)=k^2\frac{e^{ikr}}{r}\int_D\tilde{p}(\mathbf{r}',\omega)e^{-ik\mathbf{u}\cdot\mathbf{r}'}d^3r'. \quad (2.7)$$

In any realistic situation the source distribution and, consequently, the field distribution are not deterministic but are random functions of time. We assume that the ensembles that characterize the temporal fluctuations of $p(\mathbf{r},t)$ and $E(\mathbf{r},t)$ are statistically stationary, at least in the wide sense [16]. The spectrum $S_p(\mathbf{r},\omega)$ of the source distribution and the spectrum $S_E(\mathbf{r},\omega)$ of the field distribution are then given by the formulas

$$\langle\tilde{p}^*(\mathbf{r},\omega)\tilde{p}(\mathbf{r},\omega')\rangle=S_p(\mathbf{r},\omega)\,\delta(\omega-\omega'), \quad (2.8a)$$

$$\langle\tilde{E}^*(\mathbf{r},\omega)\tilde{E}(\mathbf{r},\omega')\rangle=S_E(\mathbf{r},\omega)\,\delta(\omega-\omega'), \quad (2.8b)$$

where the angular brackets denote the ensemble average and δ is the Dirac delta function.

On substituting from Eq. (2.7) into Eq. (2.8b) we obtain, for the spectrum of the radiation field, the expression

$$S_E^{(\infty)}(r\mathbf{u},\omega)=\frac{k^4}{r^2}\int_D\int_D W_p(\mathbf{r}',\mathbf{r}'',\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}''-\mathbf{r}')}d^3r'\,d^3r'', \quad (2.9)$$

where $W_p(\mathbf{r}',\mathbf{r}'',\omega)$ is the cross-spectral density of the polarization density defined by the formula [18]

$$\langle p^*(\mathbf{r}',\omega)p(\mathbf{r}'',\omega')\rangle=W_p(\mathbf{r}',\mathbf{r}'',\omega)\,\delta(\omega-\omega'). \quad (2.10)$$

It is convenient to express the right-hand side of Eq. (2.9) in a somewhat different form. For this purpose we introduce the spectral degree of coherence [19] of the polarization density

$$\mu_p(\mathbf{r}',\mathbf{r}'',\omega)=\frac{W_p(\mathbf{r}',\mathbf{r}'',\omega)}{\sqrt{S_p(\mathbf{r}',\omega)}\sqrt{S_p(\mathbf{r}'',\omega)}}. \quad (2.11)$$

On substituting for W_p from Eq. (2.11) into the integral in Eq. (2.9) we see at once that

$$S_E^{(\infty)}(r\mathbf{u},\omega)=\frac{k^4}{r^2}\int_D\int_D\sqrt{S_p(\mathbf{r}',\omega)}\sqrt{S_p(\mathbf{r}'',\omega)}\mu_p(\mathbf{r}',\mathbf{r}'',\omega)\times e^{-ik\mathbf{u}\cdot(\mathbf{r}''-\mathbf{r}')}d^3r'\,d^3r''. \quad (2.12)$$

If, as is often the case, the source is homogeneous in the sense that the source spectrum $S_p(\mathbf{r},\omega)$ is the same at every source point, i.e., if

$$S_p(r\mathbf{u},\omega)\equiv S_p(\omega)\quad\text{for all}\quad \mathbf{r}\in D, \quad (2.13)$$

formula (2.12) reduces to

$$S_E^{(\infty)}(\mathbf{r},\omega)=M(\omega,\mathbf{u},r)S_p(\omega), \quad (2.14)$$

where

$$M(\omega,\mathbf{u},r)=\frac{k^4}{r^2}\int_D\int_D\mu_p(\mathbf{r}',\mathbf{r}'',\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}''-\mathbf{r}')}d^3r'\,d^3r''. \quad (2.15)$$

Formula (2.14), together with Eq. (2.15), confirms again the result established in several previous publications, that the spectrum of the radiation field depends, in general, not only on the spectrum of the source but also on its correlation properties, represented by the spectral degree of coherence $\mu_p(\mathbf{r}',\mathbf{r}'',\omega)$. Consequently, the two spectra will, in general, differ from each other. It is not difficult to show that this is so even when the source is spherically symmetric, as has already been previously demonstrated [12–14]. Because the proportionality factor M in formula (2.14) depends not only on the frequency ω but also on the unit vector $\mathbf{u}$, the spectrum of the radiation will be different, in general, in different directions of observation. These conclusions have also been confirmed by quantum-mechanical calculations relating to simple atomic systems [20,21].

III. ENERGY CONSERVATION IN PARTIALLY COHERENT FIELDS

One of several misconceptions surrounding the subject of correlation-induced spectral changes concerns the question of energy conservation. In order to show that energy is indeed conserved in such situations, we will first derive an energy conservation law that holds for statistically stationary fields of any state of coherence.

With a suitable choice of units, the average energy flux vector $\mathbf{F}_\omega(\mathbf{r})$ at frequency ω, in a stationary optical field, is given by the formula [cf. Eq. (5.7-13), Ref. [11]]

$$\mathbf{F}_\omega(\mathbf{r})\,\delta(\omega-\omega')=-\frac{i}{2k}[\langle\tilde{E}^*(\mathbf{r},\omega)\boldsymbol{\nabla}\tilde{E}(\mathbf{r},\omega')-\tilde{E}(\mathbf{r},\omega')\boldsymbol{\nabla}\tilde{E}^*(\mathbf{r},\omega)\rangle]. \quad (3.1)$$

If we use an elementary vector identity we readily find that

$$\boldsymbol{\nabla}\cdot\mathbf{F}_\omega(\mathbf{r})\,\delta(\omega-\omega')=-\frac{i}{2k}[\langle\tilde{E}^*(\mathbf{r},\omega)\nabla^2\tilde{E}(\mathbf{r},\omega')-\tilde{E}(\mathbf{r},\omega')\nabla^2\tilde{E}^*(\mathbf{r},\omega)\rangle]. \quad (3.2)$$

Let us eliminate $\nabla^2\tilde{E}$ and $\nabla^2\tilde{E}^*$ on the right-hand side by the use of Eq. (2.3). We then obtain, for $\boldsymbol{\nabla}\cdot\mathbf{F}_\omega(\mathbf{r})$, the expression

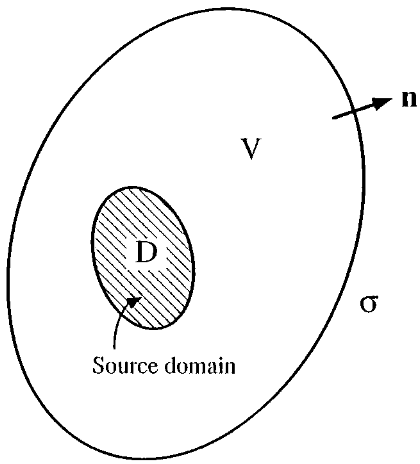

FIG. 2. Illustrating notation relating to Eq. (3.5).

$$\nabla \cdot \mathbf{F}_\omega(\mathbf{r})\,\delta(\omega-\omega') = 2\pi i k[\langle \tilde{p}(\mathbf{r},\omega)\tilde{E}^*(\mathbf{r},\omega') - \tilde{p}^*(\mathbf{r},\omega)\tilde{E}(\mathbf{r},\omega')\rangle]. \tag{3.3}$$

Next we eliminate $\tilde{E}$ and $\tilde{E}^*$ by the use of Eq. (2.5) and find that

$$\nabla \cdot \mathbf{F}_\omega(\mathbf{r}) = 4\pi k^3 \ \mathrm{Im} \int_D W_p^*(\mathbf{r},\mathbf{r}',\omega)\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\,d^3r', \tag{3.4}$$

where $W_p(\mathbf{r},\mathbf{r}',\omega)$ is the cross-spectral density of the polarization defined by Eq. (2.10) and Im denotes the imaginary part.

Formula (3.4) expresses an *energy conservation law* for stationary fields of any state of coherence. When the point $\mathbf{r}$ is outside the source domain D, $W_p(\mathbf{r},\mathbf{r}',\omega)=0$ and Eq. (3.4) reduces to

$$\nabla \cdot \mathbf{F}_\omega(\mathbf{r}) = 0. \tag{3.4$'$}$$

The physical significance of formula (3.4) becomes more apparent if we convert it into integral form. Let us integrate both sides of that equation throughout a domain V, bounded by a closed surface σ, which contains the source domain D in its interior (see Fig. 2). On using the Gauss theorem and the fact that $W_p(\mathbf{r},\mathbf{r}',\omega)=0$ for all points $\mathbf{r}$ located outside D, we find that

$$\int_\sigma \mathbf{F}_\omega(\mathbf{r})\cdot\mathbf{n}\,d\sigma = 4\pi k^3\ \mathrm{Im}\int_D\int_D W_p^*(\mathbf{r},\mathbf{r}',\omega)\,\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} \times d^3r\ d^3r', \tag{3.5}$$

where $\mathbf{n}$ is the unit outward normal to σ. Finally, if we use the fact that the cross-spectral density is Hermitian $[W^*(\mathbf{r},\mathbf{r}',\omega)=W(\mathbf{r}',\mathbf{r},\omega)]$ and also that the free-space Green's function $\exp[ik|\mathbf{r}-\mathbf{r}'|]/|\mathbf{r}-\mathbf{r}'|$ is symmetric with respect to $\mathbf{r}$ and $\mathbf{r}'$, formula (17) may be expressed in the form [22]

$$\int_\sigma \mathbf{F}_\omega(\mathbf{r})\cdot\mathbf{n}\,d\sigma = 4\pi k^4\int_D\int_D W_p(\mathbf{r},\mathbf{r}',\omega)\,\frac{\sin(k|\mathbf{r}-\mathbf{r}'|)}{k|\mathbf{r}-\mathbf{r}'|} \times d^3r\ d^3r'. \tag{3.6}$$

Since the left-hand side of Eq. (3.6) represents the rate at which energy emerges from the volume V, the right-hand side evidently represents the rate at which the source radiates energy. This term is seen to depend on the second-order correlation properties of the polarization, represented by the cross-spectral density $W_p(\mathbf{r},\mathbf{r}',\omega)$.

Formula (3.6) is *the integral form of the energy conservation law for statistically stationary fields of any state of coherence*. It is to be noted that it *holds at each frequency*. This fact is a consequence of the assumed stationarity of the source because different frequency components of members of a statistically stationary ensemble are uncorrelated [23].

IV. CONSISTENCY OF THE PHENOMENON OF SPECTRAL CHANGES IN FREE PROPAGATION WITH ENERGY CONSERVATION

We will now show that neither the difference between the spectra of the source polarization and of the radiated field nor the dependence of the field spectrum on the direction of observation noted in Sec. II [cf. Eq. (2.14)] violates the law of energy conservation.

The energy flux vector $\mathbf{F}_\omega^{(\infty)}(r\mathbf{u})$ and the spectral density $S_E^{(\infty)}(r\mathbf{u},\omega)$ of the radiation field are, for a suitable choice of units, simply related [24]:

$$\mathbf{F}_\omega^{(\infty)}(r\mathbf{u}) = S_E^{(\infty)}(r\mathbf{u},\omega)\mathbf{u}. \tag{4.1}$$

If we substitute for $S_E^{(\infty)}$ expression (2.9), Eq. (4.1) gives

$$\mathbf{F}_\omega^{(\infty)}(r\mathbf{u}) = \mathbf{u}\,\frac{k^4}{r^2}\int_D\int_D W_p(\mathbf{r}',\mathbf{r}'',\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}''-\mathbf{r}')}d^3r'\,d^3r''. \tag{4.2}$$

Hence the total energy flux radiated by the source is given by the expression

$$\int_{4\pi}\mathbf{F}_\omega^{(\infty)}(r\mathbf{u})\cdot\mathbf{u}r^2d\Omega = k^4\int_{4\pi}d\Omega\int_D\int_D W_p(\mathbf{r}',\mathbf{r}'',\omega) \times e^{-ik\mathbf{u}\cdot(\mathbf{r}''-\mathbf{r}')}d^3r'\,d^3r'' \tag{4.3}$$

$(kr\to\infty)$, where $d\Omega$ is the element of solid angle generated by the real unit vector $\mathbf{u}$ and the Ω integration extends over the whole 4π solid angle. Let us interchange, on the right-hand side of Eq. (4.3), the order of the angular and the spatial integrations and use the identity [25]

$$\int_{4\pi}d\Omega\ e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r})} = 4\pi\,\frac{\sin(k|\mathbf{r}'-\mathbf{r}''|)}{k|\mathbf{r}'-\mathbf{r}''|}. \tag{4.4}$$

Equation (4.3) then becomes

$$\int_{4\pi}\mathbf{F}_\omega^{(\infty)}(r\mathbf{u})\cdot\mathbf{u}r^2d\Omega = 4\pi k^4\int_D\int_D W_p(\mathbf{r}',\mathbf{r}'',\omega)\,\frac{\sin(k|\mathbf{r}'-\mathbf{r}''|)}{k|\mathbf{r}'-\mathbf{r}''|}\,d^3r'\,d^3r''. \tag{4.5}$$

Relation (4.5), together with the fact that in the far zone the flux vector is in the outward radial direction [as seen from Eq. (4.1)], is precisely the energy conservation law (3.6), specialized to the situation where the surface σ is taken to be a sphere of infinitely large radius centered on a point in the source region. We have thus demonstrated that expression (2.9) and, consequently, also expression (2.14) for the spectrum of the radiated field do not violate energy conservation. This conclusion confirms the correctness of the prediction evident from Eqs. (2.14) and (2.15) that, in general, the spectrum of the radiation field differs from the source spectrum and that it may be different at different points of observation.

V. QUANTUM FORMULATION

The preceding analysis was based entirely on the statistical theory of classical fields. However, it can readily be seen that the same conclusions also follow when the field is quantized. One only needs to replace the classical field variable $\tilde{E}(\mathbf{r},\omega)$ by the positive frequency part $\hat{\tilde{E}}^{(+)}(\mathbf{r},\omega)$ of the electric field operator and the polarization $\tilde{p}(\mathbf{r},\omega)$ of the source by the positive frequency part $\hat{\tilde{p}}^{(+)}(\mathbf{r},\omega)$ of the polarization operator. In place of Eqs. (2.8) and (2.10) we then have

$$\langle \hat{\tilde{p}}^{(-)}(\mathbf{r},\omega)\hat{\tilde{p}}^{(+)}(\mathbf{r},\omega')\rangle = S_p(\mathbf{r},\omega)\,\delta(\omega-\omega'), \quad (5.1a)$$

$$\langle \hat{\tilde{E}}^{(-)}(\mathbf{r},\omega)\hat{\tilde{E}}^{(+)}(\mathbf{r},\omega')\rangle = S_E(\mathbf{r},\omega)\,\delta(\omega-\omega'), \quad (5.1b)$$

$$\langle \hat{\tilde{p}}^{(-)}(\mathbf{r},\omega)\hat{\tilde{p}}^{(+)}(\mathbf{r}',\omega')\rangle = W_p(\mathbf{r},\mathbf{r}',\omega)\,\delta(\omega-\omega'), \quad (5.2)$$

where the angular brackets now denote the quantum-mechanical expectation value. Similarly, Eq. (3.1) for the average flux vector will now be replaced by the formula

$$F_\omega(\mathbf{r})\,\delta(\omega-\omega') = -\frac{i}{2k}\langle \hat{\tilde{E}}^{(-)}(\mathbf{r},\omega)\nabla\hat{\tilde{E}}^{(+)}(\mathbf{r},\omega') - \hat{\tilde{E}}^{(+)}(\mathbf{r},\omega')\nabla\hat{\tilde{E}}^{(-)}(\mathbf{r},\omega)\rangle. \quad (5.3)$$

With these definitions our basic conversation law (3.6) holds.

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