A Genuinely Multi-dimensional Relaxation Scheme for Hyperbolic Conservation Laws

K.R. Arun[1], S.V. Raghurama Rao[2], Phoolan Prasad[1], M. Lukáčová - Medviďová[3]

Abstract

A new genuinely multi-dimensional relaxation scheme is proposed. Based on a new discrete velocity Boltzmann equation, which is an improvement over previously introduced relaxation systems in terms of isotropic coverage of the multi-dimensional domain by the foot of the characteristic, a finite volume method is developed in which the fluxes at the cell interfaces are evaluated in a genuinely multi-dimensional way, in contrast to the traditional dimension-by-dimension treatment. This algorithm is tested on some bench-mark test problems for hyperbolic conservation laws.

1 Introduction

Finite volume methods have been popular for the numerical solution of hyperbolic conservation laws in the last three decades. For multi-dimensional flows, however, the traditional finite volume methods are typically based on a dimension-by-dimension treatment in using upwind discretizations. As a result of this inherently one-dimensional treatment, the discontinuities which are oblique to the coordinate directions are not resolved well. Developing genuinely multi-dimensional algorithms has been a topic of intense research in the last decade and a half. The reader is referred to [1], [2], [3], [4] for some of the developments in this topic.

In this paper, we construct genuinely multidimensional finite volume schemes based on a discrete kinetic approximation for multi-dimensional hyperbolic systems of conservation laws. Let u be a weak solution to the Cauchy problem,

$$\partial_t u + \sum_{\alpha=1}^{m} \partial_{x_\alpha} g_\alpha(u) = 0, \tag{1.1}$$

$$u(x,0) = u_0(x). \tag{1.2}$$

where $u : \mathbb{R}^m \times \mathbb{R}_+ \to \mathcal{U} \subset \mathbb{R}^n$. The flux functions $g_\alpha : \mathcal{U} \to \mathbb{R}^n$ are locally Lipschitz continuous. Let $A_\alpha = Dg_\alpha$, be the flux Jacobian matrices. We assume that our system of conservation laws is strictly hyperbolic, *i.e.*, for any $\omega \in \mathbb{R}^m$, with $|\omega| = 1$, the matrix pencil $A_\alpha := \sum_{\alpha=1}^{m} \omega_\alpha A_\alpha$ has real eigenvalues and a full set of right eigenvectors.

[1]Department of Mathematics, Indian Institute of Science, Bangalore, emails:arunkr@math.iisc.ernet.in, prasad@math.iisc.ernet.in

[2]Department of Aerospace Engineering, Indian Institute of Science, Bangalore, India, email:raghu@aero.iisc.ernet.in

[3]Institute of Numerical Simulation, Hamburg University of Technology, Hamburg, Germany, email: lukacova@tu-harburg.de

We approximate the problem (1.1) by the following.

$$\partial_t f^\epsilon + \sum_{\alpha=1}^{m} \Lambda_\alpha \partial_{x_\alpha} f^\epsilon = \frac{1}{\epsilon}(F(u^\epsilon) - f^\epsilon) \tag{1.3}$$

$$f^\epsilon(x,0) = f_0^\epsilon(x). \tag{1.4}$$

Here ϵ is a very small positive number, Λ_α's are diagonal L×L matrices, P is a constant K×L matrix, the function u^ϵ is defined by $u^\epsilon := Pf^\epsilon$. $F(u)$ is a Lipschitz continuous function defined on $\mathcal{U}$. The above equation (1.3) looks like the Boltzmann equation of Kinetic Theory of gases with the B-G-K model for the collision term, except that the molecular velocity is replaced by discrete values. In this discrete velocity Boltzmann equation, f represents the molecular velocity distribution function. $F(u)$ represents the local Maxwellian and is chosen in such a way that the following consistency conditions are satisfied.

$$\left.\begin{array}{rcl} PF(u) & = & u, \\ P\Lambda_\alpha F(u) & = & g_\alpha(u). \end{array}\right\} \tag{1.5}$$

The discrete velocity Boltzmann equation as a Relaxation System was studied by Natalini [5] and Aregba-Driollet & Natalini [6]. See Jin & Xin [7] for the idea of Relaxation Systems and Relaxation Schemes. The conditions (1.5) imply the consistency of (1.3) with the original non-linear conservation law (1.1). If we multiply (1.3) by P we obtain a conservation law

$$\partial_t u^\epsilon + \sum_{\alpha=1}^{m} \partial_{x_\alpha} \left(P\Lambda_\alpha f^\epsilon\right) = 0, \tag{1.6}$$

which is satisfied by every solution of (1.3).
Fix now the initial function $u_0(x)$ in (1.2). Consider a sequence f^ϵ of solutions to (1.3), with $f_0^\epsilon = F(u_0)$. Assume that the sequence $\{f^\epsilon\}$ is uniformly bounded and there exists a function f^0 such that

$$f^\epsilon \to f^0, \tag{1.7}$$

as $\epsilon \to 0$, in some strong topology. Then setting $u^0 = Pf^0$ we have

$$u^\epsilon \to u^0, \tag{1.8}$$

in the same topology and the following identities holds.

$$f^0 = F(u^0) \tag{1.9}$$

$$\partial_t u^0 + \sum_{\alpha=1}^{m} \partial_{x_\alpha} \left(P\Lambda_\alpha f^0\right) = 0. \tag{1.10}$$

therefore from (1.9) and the first condition of the consistency conditions (1.5) we can obtain

$$P\Lambda_\alpha f^0 = P\Lambda_\alpha F(u^0) = g_\alpha(u^0), \quad \text{for} \quad \alpha = 1, 2, \cdots, m. \tag{1.11}$$

Hence we conclude that u^0 is a weak solution to the Cauchy problem (1.1)-(1.2).
The main motivation to introduce the relaxation model (1.3) comes from the work of Jin and Xin [7]. For instance, consider a single non-linear conservation law in one space dimension,

$$\partial_t u + \partial_x g(u) = 0 \tag{1.12}$$

which the above authors replace by the following semi-linear relaxation equations.

$$\partial_t u + \partial_x v = 0 \tag{1.13}$$

$$\partial_t v + \lambda^2 \partial_x u = \frac{1}{\epsilon}\left(g(u) - v\right). \tag{1.14}$$

In terms of the characteristic variables (f_1, f_2) the above relaxation equations (1.13)-(1.14) can be recast into

$$\partial_t f_1 - \lambda \partial_x f_1 = \frac{1}{\epsilon}(F_1 - f_1) \tag{1.15}$$

$$\partial_t f_2 + \lambda \partial_x f_2 = \frac{1}{\epsilon}(F_2 - f_2) \tag{1.16}$$

Note that (1.3) is an extension of the above relaxation system to systems of conservation laws in multi-dimensions. Our aim is to construct a genuinely multidimensional numerical scheme to (1.3)-(1.4) in the zero relaxation limit $\epsilon = 0$ so that we can eventually recover a genuinely multi-dimensional numerical scheme to the non-linear conservation laws (1.1). The main advantage of the model (1.3) is that it is in the diagonal form. It makes it very easy to develop numerical schemes. Since (1.3) is a discrete velocity BGK Boltzmann equation, the numerical schemes developed based on it will inherit most of the features of the classical kinetic schemes.
It is well known that the relaxation system (1.3) must satisfy certain stability conditions, for its solution to converge to a weak entropy solution of (1.1). This stability condition is obtained using a Chapman-Enskog type expansion of kinetic theory and deriving a first order viscous perturbation of (1.3). In two dimensions, the stability condition is the well known subcharacteristic condition of T.P.Liu [8]. Under some strong conditions such as the monotonicity on the Maxwellians F, for a scalar conservation law in multi-dimensions Aregba-Driollet & Natalini [6] have proved the convergence of Pf^ϵ to the Kruzkov entropy solution of (1.1). To the best of our knowledge, no such convergence result has been reported for systems in multi-dimensions.
The rest of this report is organized as follows. The details of the Chapman-Enskog expansion is given in section-2. We derive the stability condition for a general discrete kinetic system. In section-3 we give the construction of our discrete kinetic approximation. We follow the approach of an orthogonal velocity method given in Aregba-Driollet & Natalini [6]. The relaxation system we use was first derived by Manisha, Raghavendra & Raghurama Rao [9] with the aim of obtaining an isotropic relaxation system and a multi-dimensional numerical scheme based on such a system by using appropriate interpolations to trace the foot of the characteristic for each of the equations of the relaxation system. In section-4 we present our genuinely multi-dimensional finite volume scheme based on characteristics. Section-5 is devoted to numerical experiments and results. Finally in section-6 we give some conclusions and describe the future work to be done.

2 Chapman-Enskog analysis

In this section we discuss the stability conditions for the discrete kinetic approximation (1.3). Since the local equilibrium to the system (1.3) is the original conservation law (1.1), we derive a diffusive first order correction to (1.3) by a Chapman-Enskog type expansion. This is analoge of the compressible Navier-Stokes equations in the kinetic theory. Our expansion follows the work of Aregba-Driollet and Natalini [6]. For rigorous mathematical

treatment of the same the reader is advised to refer Chen, Livermore and Liu [10].
Now onwards we drop the superscript ϵ from f^ϵ for simplicity. Let f be a solution to (1.3), with F as the local Maxwellian. Denote $v_\alpha = P\Lambda_\alpha f$, $\alpha = 1, 2, \cdots m$. From the consistency conditions (1.5), we obtain,

$$\begin{aligned} \partial_t u + \sum_{\alpha=1}^{m} \partial_{x_\alpha} v_\alpha &= 0, \\ \partial_t v_\alpha + \sum_{\beta=1}^{m} \partial_{x_\beta}(P\Lambda_\alpha\Lambda_\beta f) &= \frac{1}{\epsilon}\left(g_\alpha(u) - v_\alpha\right). \end{aligned} \tag{2.17}$$

Consider the following expansion of f as a perturbation series

$$f = F + \epsilon f^{(1)} + O(\epsilon^2). \tag{2.18}$$

Then

$$\begin{aligned} v_\alpha &= g_\alpha(u) - \epsilon\left(\partial_t v_\alpha + \sum_{\beta=1}^{m} \partial_{x_\beta}(P\Lambda_\alpha\Lambda_\beta f)\right) + O(\epsilon^2) \\ &= g_\alpha(u) - \epsilon\left(\partial_t v_\alpha + \sum_{\beta=1}^{m} \partial_{x_\beta}(P\Lambda_\alpha\Lambda_\beta F)\right) + O(\epsilon^2) \end{aligned}$$

Substituting this in (2.17) we obtain,

$$\partial_t u + \sum_{\alpha=1}^{m} \partial_{x_\alpha} g_\alpha(u) = \epsilon \sum_{\alpha=1}^{m} \partial_{x_\alpha}\left(\partial_t v_\alpha + \sum_{\beta=1}^{m} \partial_{x_\beta}(P\Lambda_\alpha\Lambda_\beta F)\right) + O(\epsilon^2). \tag{2.19}$$

Again, we have

$$\partial_t v_\alpha = A_\alpha(u)\partial_t u = -\sum_{\beta=1}^{m} A_\alpha(u)A_\beta(u)\partial_{x_\beta} u + O(\epsilon).$$

Then upto first order terms, from (2.19),

$$\partial_t u + \sum_{\alpha=1}^{m} \partial_{x_\alpha} g_\alpha(u) = \epsilon \sum_{\alpha=1}^{m} \partial_{x_\alpha}\left(\sum_{\beta=1}^{m} B_{\alpha\beta}(u)\partial_{x_\beta} u\right), \tag{2.20}$$

where

$$B_{\alpha\beta} = P\Lambda_\alpha\Lambda_\beta DF(u) - A_\alpha(u)A_\beta(u)$$

therefore, $B = (B_{\alpha\beta})$ is the viscosity matrix. For B to be admissible, we must have the usual positive definiteness condition,

$$\sum_{\alpha,\beta=1}^{m} \langle B_{\alpha\beta}\xi_\alpha, \xi_\beta\rangle \geq 0, \tag{2.21}$$

for all $\xi_\alpha \in \mathbb{R}^n, \cdots, \xi_m \in \mathbb{R}^n$.
Under this criterion, (2.20) is parabolic and this is the required stability criterion.

3 A new relaxation system

In this section we give a precise description of the construction of the relaxation system. For any system of n conservation laws in m space dimensions of the form (1.1), we construct relaxation models of the type given in (1.3). The most important is the construction of the diagonal matrices Λ_α's, the Maxwellians F, and the matrix P for taking the moments. Following Aregba-Driollet and Natalini [6], we use a block structure for obtaining Λ_α's. Take L=Nn and $P = (I_n, I_n, \cdots, I_n)$, with N blocks of the identity matrix I_n of $\mathbb{R}^n$. Each matrix Λ_α is constructed using N diagonal blocks of size n×n.

$$\Lambda_\alpha = \text{diag}\left(C_1^{(\alpha)}, C_2^{(\alpha)}, \cdots, C_N^{(\alpha)}\right), \tag{3.22}$$

where

$$C_\beta^{(\alpha)} = \lambda_\beta^{(\alpha)} I_n, \quad \lambda_\beta^{(\alpha)} \in \mathbb{R} \tag{3.23}$$

with this (1.3) can be written as

$$\partial_t f_\alpha + \sum_{\beta=1}^{m} \lambda_\alpha^{(\beta)} \partial_{x_\beta} f_\alpha = \frac{1}{\epsilon}(F_\alpha - f_\alpha) \quad \text{for} \quad \alpha = 1, 2, \cdots, N \tag{3.24}$$

with

$$u = \sum_{\alpha=1}^{N} f_\alpha \tag{3.25}$$

The Maxwellians are taken as

$$F_\alpha(u) = a_\alpha u + \sum_{\beta=1}^{m} b_{\alpha\beta} g_\beta(u). \tag{3.26}$$

Where the coefficients a_α and $b_{\alpha\beta}$'s are to be chosen according to the consistency conditions (1.5). In particular we use the orthogonal velocity method given Aregba-Driollet and Natalini. We choose Λ_α's in such a way that

$$\Lambda_\alpha = \text{diag}\left(\lambda_{\beta\alpha} I_n\right) \quad \text{where} \quad 1 \le \beta \le N \tag{3.27}$$

The coefficients $\lambda_{\beta\alpha}$'s satisfy the following two conditions,

$$\sum_{\beta=1}^{N} \lambda_{\beta\alpha} = 0, \quad \text{for} \quad \alpha = 1, 2, \cdots, m, \tag{3.28}$$

and

$$\sum_{\gamma=1}^{N} \lambda_{\gamma\alpha}\lambda_{\gamma\beta} = 0, \quad \text{for} \quad \alpha, \beta = 1, 2, \cdots, m \quad \alpha \ne \beta. \tag{3.29}$$

With the above choice of the discrete velocities, $\lambda_{\beta\alpha}$, the Maxwellian $F = (F_1, F_2, \cdots, F_N)$ is obtained to be

$$F_\alpha(u) = \frac{u}{N} + \sum_{\beta=1}^{m} \frac{g_\beta(u)}{a_\beta^2} \lambda_{\alpha\beta}, \quad \text{for} \quad \alpha = 1, 2, \cdots, N. \tag{3.30}$$

where $a_\beta^2 = \sum_{\gamma=1}^{N} \lambda_{\gamma\beta}$. For convenience of the reader, we give examples of the relaxation systems for a single conservation law in one-dimension and two-dimensions.
The one dimensional case. Consider a single conservation in one space dimension,

$$\partial_t u + \partial_x g(u) = 0 \tag{3.31}$$

We introduce the following relaxation system,

$$\partial_t f + \Lambda \partial_x f = \frac{1}{\epsilon}(F - f), \tag{3.32}$$

where

$$\Lambda = \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix} \tag{3.33}$$

and the Maxwellians are given by

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} \frac{u}{2} - \frac{g(u)}{2\lambda} \\ \frac{u}{2} + \frac{g(u)}{2\lambda} \end{bmatrix} \tag{3.34}$$

The two dimensional case. Consider a single conservation law in two dimensions,

$$\partial_t u + \partial_x g_1(u) + \partial_y g_2(u) = 0 \tag{3.35}$$

We propose the relaxation system,

$$\partial_t f + \Lambda_1 \partial_x f + \Lambda_2 \partial_y f = \frac{1}{\epsilon}(F - f) \tag{3.36}$$

where

$$\Lambda_1 = \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \quad \text{and} \quad \Lambda_2 = \begin{bmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \tag{3.37}$$

In an analogous way as the one dimensional case, the Maxwellians can be obtained to be

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} \frac{u}{4} - \frac{g_1(u)}{4\lambda} - \frac{g_2(u)}{4\lambda} \\ \frac{u}{4} + \frac{g_1(u)}{4\lambda} - \frac{g_2(u)}{4\lambda} \\ \frac{u}{4} + \frac{g_1(u)}{4\lambda} + \frac{g_2(u)}{4\lambda} \\ \frac{u}{4} - \frac{g_1(u)}{4\lambda} + \frac{g_2(u)}{4\lambda} \end{bmatrix} \tag{3.38}$$

In this way one can construct a relaxation system for systems of conservation laws in arbitrary space dimensions. The relaxation system for one dimension is exactly the same as the one introduced by Aregba-Driollet and Natalini. But in two dimensions, we differ from the model given by the above authors. The two dimensional relaxation system given above in (3.36) was first introduced by Manisha, Raghavendra and Raghurama Rao [9] to obtain a relaxation system such that the combination of the foot of the characteristic for each equation of the relaxation system covers all the four quadrants in an isotropic way. Our stability condition (2.21) for the case of (3.32) is simply

$$\lambda^2 \geq (\partial_u g(u))^2 \tag{3.39}$$

This is the famous sub-characteristic condition of T.P.Liu (Liu 1987). For the two dimensional relaxation system (3.36) the stability condition is

$$\lambda^2 \geq (\partial_u g_1(u))^2 + (\partial_u g_2(u))^2 \tag{3.40}$$

Note that in the case of two dimensions, our stability criterion is very simple and it differs significantly from that of Aregba-Driollet and Natalini. The condition (3.40) is a realistic extension of the sub-characteristic condition.

4 A numerical scheme based on characteristics

The discrete velocity Boltzmann equation consists of convection terms and a stiff source term. The convection terms are linear whereas the source term is non-linear. We split these two terms via a splitting method. The discrete velocity Boltzmann equation is of the form,

$$\partial_t f + \sum_{\alpha=1}^{m} \Lambda_\alpha \partial_{x_\alpha} f = \frac{1}{\epsilon}(F - f) \tag{4.41}$$

In the splitting method, the above equation (4.41) is solved in two independent steps. Let us denote the convection terms by $C(f)$ and the source terms by $S(f)$. Then (4.41) can be rewritten as

$$\partial_t f = C(f) + S(f). \tag{4.42}$$

We solve the above equation in two steps as,

$$\partial_t f = c(f) \quad \text{(convection step)} \tag{4.43}$$

$$\frac{df}{dt} = \frac{1}{\epsilon}(F - f) \quad \text{(relaxation step)} \tag{4.44}$$

Suppose we have reached the n^{th} iteration and have obtained f^n. Then in the relaxation step we need to solve an ordinary differential equation with the initial condition $f = f^n$. This step can be simplified further by assuming $\epsilon = 0$. Then the relaxation step gives $f^n = F^n$. In other words, the distribution function relaxes instantaneously to the Maxwellian F^n. With this restriction we need to solve only a system of linear advection equations,

$$\partial_t f + \sum_{\alpha=1}^{m} \Lambda_\alpha \partial_{x_\alpha} f = 0. \tag{4.45}$$

These equations are solved using a finite volume scheme using the evolution along the characteristics as described below.
Let Ω be our computational domain and let f_i^n be an approximation to the average of $f(x, t_n)$ over a cell Ω_i of measure $|\Omega_i|$. Then our scheme will be of the form,

$$f_i^{n+1} = f_i^n - \frac{\Delta t}{|\Omega_i|} \int_{\partial\Omega_i} \sum_{\alpha=1}^{m} (\Lambda_\alpha f)^{n+\frac{1}{2}} \nu_\alpha ds, \tag{4.46}$$

Here $(\Lambda_\alpha f)^{n+\frac{1}{2}}$ is generated from a recovered approximation Rf^n which has been evolved to $t_n + \frac{\Delta t}{2}$ along the characteristics. The above formula (4.46) was obtained by integration of (4.45) over $(t_n, t_n + \frac{\Delta t}{2}) \times \Omega_i$ and use of the Gauss theorem as well as the midpoint rule in time on the flux term.

The evolution to the half time step is accomplished through the characteristics constructed at the quadrature points chosen for the integration of the fluxes over the cell interfaces. The obvious quadrature points are the vertices of the cells but our results suggested that this is not a right choice. So we have decided to use the midpoints of the edges also, resulting in Simpson's one third rule. Our algorithm consists of three steps: recovery of a higher order approximation Rf^n from the cell averages f_i^n; the evolution to $t_n + \frac{\Delta t}{2}$ to calculate the fluxes; then an update of the cell averages by (4.46).
Since the exact solution of (4.45) is available, we use this solution to predict the value of the solution at time level $t_n + \frac{\Delta t}{2}$. But this representation requires the values of the solution at non-grid points. In the recovery step, we calculate these values using the recovery functions. In a first order scheme, we have used just piecewise constant data, whereas in the second order scheme we have used a conservative discontinuous bilinear recovery. We illustrate the numerical method for 1D and 2D scalar conservation laws.
One dimensional case
Consider the conservation law,

$$\partial_t u + \partial_x g(u) = 0. \tag{4.47}$$

We approximate the above conservation law by the relaxation system,

$$\partial_t f_1 - \lambda \partial_x f_1 = \frac{1}{\epsilon}(F_1 - f_1) \tag{4.48}$$

$$\partial_t f_2 + \lambda \partial_x f_2 = \frac{1}{\epsilon}(F_2 - f_2) \tag{4.49}$$

In the convection step we need to solve only

$$\partial_t f_1 - \lambda \partial_x f_1 = 0 \tag{4.50}$$

$$\partial_t f_2 + \lambda \partial_x f_2 = 0 \tag{4.51}$$

The finite volume update formula (4.46) for this system reads,

$$f_{1,i}^{n+1} = f_{1,i}^n + \frac{\lambda \Delta t}{\Delta x}\left\{ f_{1,i+1/2}^{n+1/2} - f_{1,i-1/2}^{n+1/2} \right\} \tag{4.52}$$

$$f_{2,i}^{n+1} = f_{2,i}^n - \frac{\lambda \Delta t}{\Delta x}\left\{ f_{2,i+1/2}^{n+1/2} - f_{2,i-1/2}^{n+1/2} \right\}. \tag{4.53}$$

Now, the exact solution of (4.50)-(4.51) is

$$f_1(x, t + \Delta t) = f_1(x + \lambda \Delta t, t) \tag{4.54}$$

$$f_2(x, t + \Delta t) = f_2(x - \lambda \Delta t, t). \tag{4.55}$$

Note that

$$\begin{aligned} f_{1,i+1/2}^{n+1/2} &= f_1(x_{i+1/2}, t_n + \Delta t/2) \\ &= f_1(x_{i+1/2} + \lambda \Delta t/2, t_n). \end{aligned} \tag{4.56}$$

Similarly,

$$f_{1,i-1/2}^{n+1/2} = f_1(x_{i-1/2} + \lambda \Delta t/2, t_n). \tag{4.57}$$

Analogous expressions holds for f_2 also. We now use our recovery functions to evaluate the expressions (4.56)-(4.57). In the first order scheme, we use only piecewise constant recovery functions,

$$f_1(x, t_n) = f_{1,i}^n, \quad \text{in} \quad [x_{i-1/2}, x_{i+1/2}]. \tag{4.58}$$

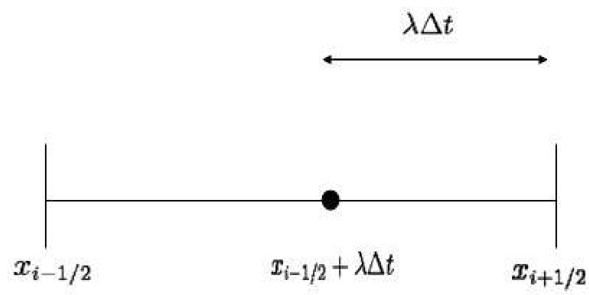

Figure 1: Foot of the characteristic with a negative convection speed

In the second order scheme, we use conservative discontinuous linear recovery functions,

$$f_1(x,t_n) = f^n_{1,i} + (x - x_i)(\partial_x f_1)^n_i, \quad \text{in} \quad [x_{i-1/2}, x_{i+1/2}]. \tag{4.59}$$

For the derivatives, $(\partial_x f_1)^n_i$, we use only the finite difference approximations,

$$(\partial_x f_1)^n_i = \frac{f^n_{1,i+1} - f^n_{1,i}}{\Delta x}. \tag{4.60}$$

In the second order scheme, to prevent the oscillations developing, we have used limiters. We have tested the common limiters such as minmod limiter, van Leer's limiter and superbee limiter. After calculating the fluxes we update the values of f_1 and f_2 using (4.52)-(4.53). The value of the original conserved variable u is recovered as

$$u^n_i = f^n_{1,i} + f^n_{2,i}. \tag{4.61}$$

In the relaxation step, we impose the conditions, $f_1 = F_1$ and $f_2 = F_2$ by virtue of the instantaneous relaxation.

Two dimensional case

Consider a single conservation law in 2D

$$\partial_t u + \partial_x g_1(u) + \partial_y g_2(u) = 0 \tag{4.62}$$

Recall that the relaxation system for the above conservation law is

$$\left.\begin{array}{rcl}
\partial_t f_1 - \lambda\partial_x f_1 - \lambda\partial_y f_1 &=& \frac{1}{\epsilon}(F_1 - f_1) \\
\partial_t f_2 + \lambda\partial_x f_2 - \lambda\partial_y f_2 &=& \frac{1}{\epsilon}(F_2 - f_2) \\
\partial_t f_3 + \lambda\partial_x f_3 + \lambda\partial_y f_3 &=& \frac{1}{\epsilon}(F_3 - f_3) \\
\partial_t f_4 - \lambda\partial_x f_4 + \lambda\partial_y f_4 &=& \frac{1}{\epsilon}(F_4 - f_4).
\end{array}\right\} \tag{4.63}$$

In the convection step we solve first the system of convection equations

$$\left.\begin{array}{rcl}
\partial_t f_1 - \lambda\partial_x f_1 - \lambda\partial_y f_1 &=& 0 \\
\partial_t f_2 + \lambda\partial_x f_2 - \lambda\partial_y f_2 &=& 0 \\
\partial_t f_3 + \lambda\partial_x f_3 + \lambda\partial_y f_3 &=& 0 \\
\partial_t f_4 - \lambda\partial_x f_4 + \lambda\partial_y f_4 &=& 0.
\end{array}\right\} \tag{4.64}$$

The finite volume update formula (4.46) for the individual equations reads,

$$\left.\begin{aligned}
f_{1,i,j}^{n+1} &= f_{1,i,j}^{n} + \tfrac{\lambda\Delta t}{\Delta x}\left\{f_{1,i+1/2,j}^{n+1/2} - f_{1,i-1/2,j}^{n+1/2}\right\} + \tfrac{\lambda\Delta t}{\Delta y}\left\{f_{1,i,j+1/2}^{n+1/2} - f_{1,i,j-1/2}^{n+1/2}\right\} \\
f_{2,i,j}^{n+1} &= f_{2,i,j}^{n} - \tfrac{\lambda\Delta t}{\Delta x}\left\{f_{2,i+1/2,j}^{n+1/2} - f_{2,i-1/2,j}^{n+1/2}\right\} + \tfrac{\lambda\Delta t}{\Delta y}\left\{f_{2,i,j+1/2}^{n+1/2} - f_{2,i,j-1/2}^{n+1/2}\right\} \\
f_{3,i,j}^{n+1} &= f_{3,i,j}^{n} - \tfrac{\lambda\Delta t}{\Delta x}\left\{f_{3,i+1/2,j}^{n+1/2} - f_{3,i-1/2,j}^{n+1/2}\right\} - \tfrac{\lambda\Delta t}{\Delta y}\left\{f_{3,i,j+1/2}^{n+1/2} - f_{3,i,j-1/2}^{n+1/2}\right\} \\
f_{4,i,j}^{n+1} &= f_{4,i,j}^{n} + \tfrac{\lambda\Delta t}{\Delta x}\left\{f_{4,i+1/2,j}^{n+1/2} - f_{4,i-1/2,j}^{n+1/2}\right\} - \tfrac{\lambda\Delta t}{\Delta y}\left\{f_{4,i,j+1/2}^{n+1/2} - f_{4,i,j-1/2}^{n+1/2}\right\}
\end{aligned}\right\} \tag{4.65}$$

The interface fluxes at the half time step are evaluated as follows. Consider the first equation in (4.64). The exact solution of this convection equation is given by

$$f_1(x, y, t + \Delta t) = f_1(x + \lambda\Delta t, y + \lambda\Delta t, t). \tag{4.66}$$

Using this we can evaluate our fluxes in the following way,

$$\begin{aligned}
f_{1,i+1/2,j}^{n+1/2} &= f_1(x_{i+1/2}, y_j, t_n + \Delta t/2) \\
&= f_1(x_{i+1/2} + \lambda\Delta t, y_j + \lambda\Delta t, t_n).
\end{aligned} \tag{4.67}$$

Similarly,

$$f_{1,i-1/2,j}^{n+1/2} = f_1(x_{i-1/2} + \lambda\Delta t, y_j + \lambda\Delta t, t_n). \tag{4.68}$$

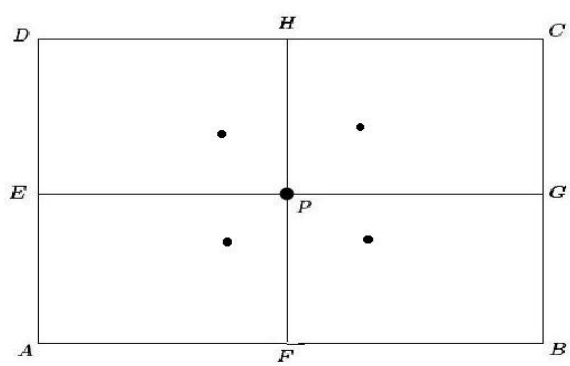

Figure 2: Foot of the characteristic falls isotropically around the point

Analogous expressions holds for the y-fluxes also. We now use the recovery functions to evaluate the expressions on the right side of (4.67)-(4.68). In the first order scheme we take

$$f_1(x, y, t_n) = f_{1,i,j}^{n}, \quad \text{in} \quad [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]. \tag{4.69}$$

and in the second order scheme,

$$\begin{aligned}
f_1(x, y, t_n) &= f_{1,i,j}^{n} + (x - x_i)(\partial_x f_1)_{i,j}^{n} + (y - y_j)(\partial_y f_1)_{i,j}^{n}, \\
&\qquad \text{in} \quad [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}].
\end{aligned} \tag{4.70}$$

where, for the derivatives $\partial_x f_1$ and $\partial_y f_1$ at the point (x_i, y_j, t_n) we have used just the finite difference approximations,

$$\left.\begin{aligned}(\partial_x f_1)^n_{i,j} &= \frac{f^n_{1,i+1,j} - f^n_{1,i,j}}{\Delta x}\\ (\partial_y f_1)^n_{i,j} &= \frac{f^n_{1,i,j+1} - f^n_{1,i,j}}{\Delta y}\end{aligned}\right\} \tag{4.71}$$

The calculations for the other variables f_2, f_3, f_4 are similar. After calculating the fluxes we update the solution using (4.65). The value of the original conserved variable u is recovered via

$$u^{n+1}_{ij} = f^{n+1}_{1,i,j} + f^{n+1}_{2,i,j} + f^{n+1}_{3,i,j} + f^{n+1}_{4,i,j}. \tag{4.72}$$

With the instantaneous relaxation, the relaxation step is simply imposing the conditions

$$\left.\begin{aligned}f^n_{1,i,j} &= F^n_{1,i,j}\\ f^n_{2,i,j} &= F^n_{2,i,j}\\ f^n_{3,i,j} &= F^n_{3,i,j}\\ f^n_{4,i,j} &= F^n_{4,i,j}.\end{aligned}\right\} \tag{4.73}$$

at the staring of the next iteration.

5 Numerical results

The new algorithm developed is tested on a few bench-mark problems for the Burgers equation in 1-D and 2-D.
1-D test cases
These examples are taken from C. Laney [11].
Test Case 1

$$\begin{aligned}&\partial_t u + \partial_x(\frac{u^2}{2}) = 0,\\ &u(x,0) = \left\{\begin{array}{ll} 1 & \text{for} \quad |x| < \frac{1}{3},\\ 0 & \text{for} \quad \frac{1}{3} < |x| \le 1.\end{array}\right.\end{aligned} \tag{5.74}$$

So the initial condition is a square wave. To review briefly, the jump from 0 to 1 at $x = -1/3$ creates an expansion fan, while the jump from 1 to 0 at $x = 1/3$ creates a shock. The analytical solution to this problem is given below.

$$u_{ex}(x,t) = \left\{\begin{array}{ll} 0 & \text{for} \quad -\infty < x \le -\frac{1}{3},\\ \frac{x+1/3}{t} & \text{for} \quad -\frac{1}{3} < x \le t - \frac{1}{3},\\ 1 & \text{for} \quad t - \frac{1}{3} < x \le \frac{t}{2} + \frac{1}{3},\\ 0 & \text{for} \quad \frac{t}{2} + \frac{1}{3} < x < \infty.\end{array}\right. \tag{5.75}$$

where $t < \frac{4}{3}$. The solution is computed for $t = 0.6$. The numerical results are given Fig.5 and Fig.6.

Test Case 2

$$\begin{aligned}&\partial_t u + \partial_x(\frac{u^2}{2}) = 0,\\ &u(x,0) = \left\{\begin{array}{ll} 1 & \text{for} \quad |x| < \frac{1}{3},\\ -1 & \text{for} \quad \frac{1}{3} < |x| \le 1.\end{array}\right.\end{aligned} \tag{5.76}$$

So the initial condition is again a square wave, but with a sonic point. To review briefly, the jump from -1 to 1 at $x = -1/3$ creates an expansion fan, while the jump from 1 to -1 at $x = 1/3$ creates a shock. The analytical solution to this problem is given below.

$$u_{ex}(x,t) = \begin{cases} -1 & \text{for} \quad -\infty < x \le -\frac{1}{3}, \\ \frac{x+1/3}{t} & \text{for} \quad -\frac{1}{3} < x \le t - \frac{1}{3}, \\ 1 & \text{for} \quad t - \frac{1}{3} < x \le \frac{1}{3}, \\ -1 & \text{for} \quad \frac{1}{3} < x < \infty. \end{cases} \tag{5.77}$$

where $t < \frac{2}{3}$. The solution is computed for $t = 0.3$. The numerical results are given in Fig.7 and Fig.8.

2-D test cases

These test cases are taken from Spekreijse [12]. On the square $[0,1] \times [0,1]$, we consider the Burgers equation,

$$\partial_t u + \partial_x \left(\frac{u^2}{2} \right) + \partial_y u = 0.$$

The steady state equation is

$$\partial_y u + u \partial_x u = 0.$$

which is like the inviscid Burgers equation in 1-D. Two different sets of boundary conditions have been considered.

Test Case 1

With the boundary conditions

$$\begin{aligned} u(0,y) &= 1, \quad \text{for} \quad 0 < y < 1, \\ u(1,y) &= -1, \quad \text{for} \quad 0 < y < 1, \\ u(x,0) &= 1 - 2x, \quad \text{for} \quad 0 < x < 1, \end{aligned}$$

the steady solution is given below (see the Fig.3).

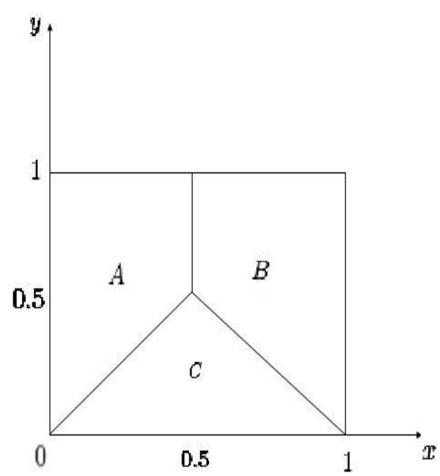

Figure 3: Exact solution for test case 1

$$\begin{aligned} u_{ex}(x,y) &= 1 \quad \text{if } (x,y) \text{ in Region A}, \\ u_{ex}(x,y) &= -1 \quad \text{if } (x,y) \text{ in Region B}, \\ u_{ex}(x,y) &= \frac{1-2x}{1-2y} \quad \text{if } (x,y) \text{ in Region C}. \end{aligned}$$

The regions A and B are separated by a shock, originating at $(x, y) = (0.5, 0.5)$. Computations have been carried out on 32×32, 64×64 and 128×128 grids. The contour plots of the numerical solution are given Fig.9-Fig.14.

Test Case 2
With the boundary conditions

$$\begin{aligned} u(0, y) &= 1.5, \quad \text{for} \quad 0 < y < 1, \\ u(1, y) &= -0.5, \quad \text{for} \quad 0 < y < 1, \\ u(x, 0) &= 1.5 - 2x, \quad \text{for} \quad 0 < x < 1, \end{aligned}$$

the steady solution is (see the Fig.4),

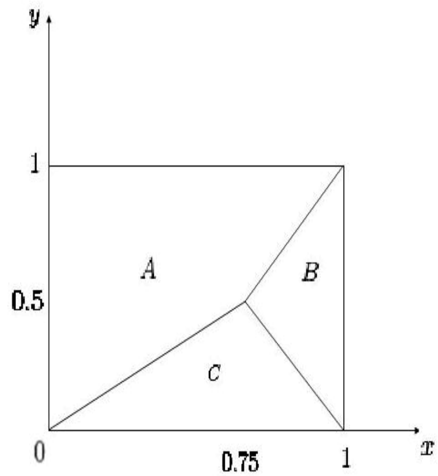

Figure 4: Exact solution for test case 2

$$\begin{aligned} u_{ex}(x, y) &= 1.5 \quad \text{if } (x, y) \text{ in Region A}, \\ u_{ex}(x, y) &= -0.5 \quad \text{if } (x, y) \text{ in Region B}, \\ u_{ex}(x, y) &= \frac{1.5 - 2x}{1.5 - 2y} \quad \text{if in Region C}. \end{aligned}$$

The regions A and B are separated by an oblique shock, originating at $(x, y) = (0.75, 0.5)$. Again, the computations have been carried out on 32×32, 64×64 and 128×128 grids. The contour plots of the numerical solution is given in Fig.15-Fig.20.

6 Conclusions and future work to be done

A genuinely multi-dimensional relaxation scheme is developed, based on a new istropic 2-D relaxation system. The interface fluxes are evaluated in a multi-dimensional way using the characteristics and appropriate piece-wise polynomial approximations in finite volumes. The numerical methods is tested on some typical Bench-mark problems for invscid Burgers eqution in 1-D and 2-D. This algorithms is presentely being tested for the case of hyperbolic vector conservation laws, using Euler equations of Gas Dynamics in multi-dimensions. Future work planned also includes the reduction in numerical dissipation, following the work of Raghurama Rao and Balakrishna [13] and detailed comparisons with the grid-aligned upwind relaxation schemes using a traditional finite volume method.

7 Acknowledgement

The authors sincerely thank the Department of Science and Technology, Goverment of India and DAAD (German Academic Exchange Service) for providing the financial support for collaborative research under the DST-DAAD Project Based Personnel Exchange Programme. K.R.Arun would like to express his gratitude to the Council of Scientific & Industrial Research (CSIR) for supporting his research at the Indian Istitute of Science under the Junior Research Fellowship scheme, grant number-09/079(2084)/2006-EMR-1.

References

[1] P.L. Roe, Discrete Models for the Numerical Analysis of Time-dependent Multi-dimensional Gas Dynamics, Journal of Computational Physics, vol. 63, No. 2, pp. 458-476, 1986.

[2] B. van Leer, Progress in Multi-dimensional Upwind Differencing, ICASE Report No. 92-43, 1992.

[3] Maria Lukacova-Medvidowa, K.W. Morton and G. Warnecke, Mathematics of Computation, vol. 69, No. 232, pp. 1355-1384, 2000.

[4] S.V. Raghurama Rao, Development of New Upwind Methods based on Kinetic Theory, Ph.D. thesis, Department of Mechanical Engineering, Indian Institute of Science, Bangalore, India, 1994.

[5] R. Natalini, A Discrete Kinetic Approximation of Entropy Solutions to Multidimensional Scalar Conservation Laws, Journal of Differential Equations, vol. 148, pp. 292–317, 1999.

[6] D. Aregba-Driollet, and R. Natalini, Discrete Kinetic Schemes for Multidimensional Systems of Conservation Laws, SIAM Journal of Numerical Analysis, vol. 37, no. 6, pp. 1973-2004, 2000.

[7] S. Jin and Z. Xin, The Relaxation Schemes for Systems of Conservation Laws in Arbitrary Space Dimensions, Communications in Pure and Applied Mathematics, vol. 48, 1995, pp. 235–276.

[8] T.P. Liu, Hyperbolic Conservation Laws with Relaxation, Communications in Mathematical Physics, vol. 108, pp. 153-175, 1987.

[9] Manisha, L.N. Raghavendra and S.V. Raghurama Rao, A New Multi-dimensional Relaxation Scheme for Hyperbolic Conservation Laws, in preparation.

[10] G.-Q. Chen, C.D. Levermore, C. D., T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, Communications in Pure and Applied Mathematics, vol. 47, pp. 787-830, 1994.

[11] Culbert B. Laney, Computational Gas Dynamics, Cambridge University Press, 1998.

[12] Stefan Spekreijse, Multigrid Solutions of Monotone Second Order Discretizations of Hyperbolic Conservation Laws, Mathematics of Computation, vol. 49, 1987, pp. 135-155.

[13] S.V. Raghurama Rao and K. Balakrishna, An Accurate Shock Capturing Algorithm with a Relaxation System (ASCARS) for Hyperbolic Conservation Laws, AIAA Paper, AIAA-2003-4115, 2003.

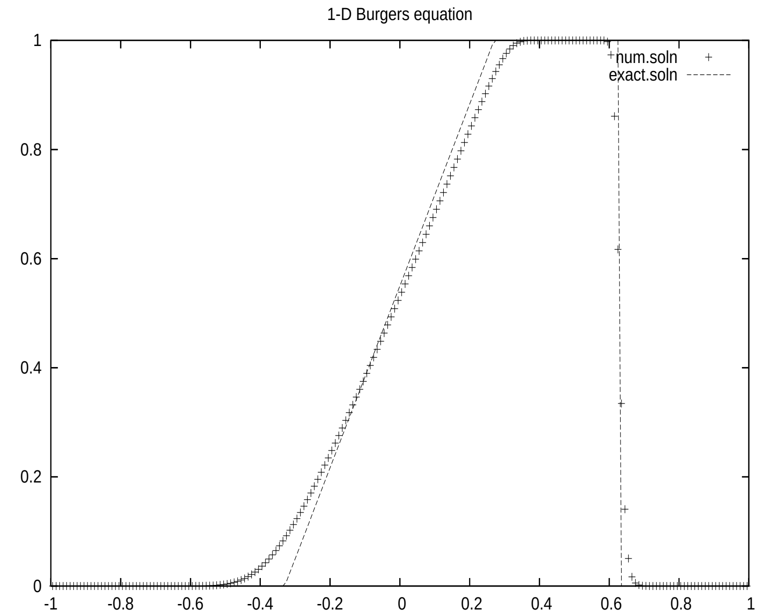

Figure 5: First order scheme with 200 grid points,CFL=0.75

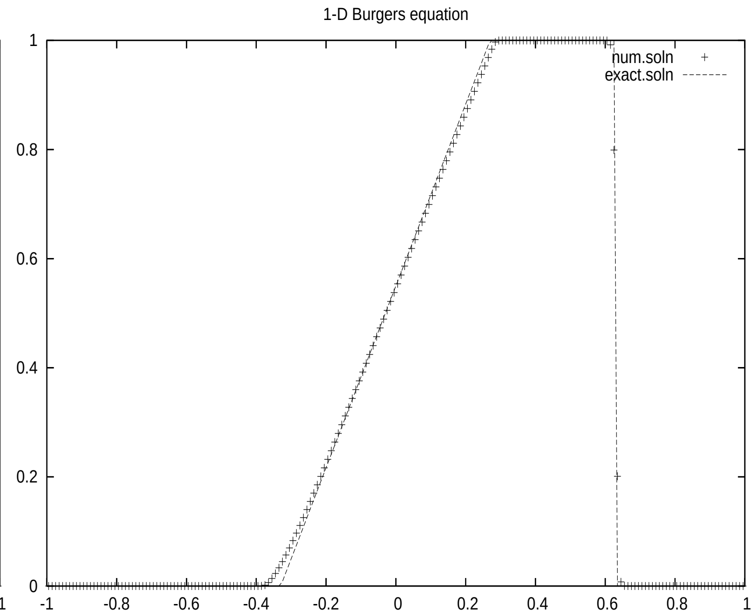

Figure 6: Second order scheme with 200 grid points,CFL=0.15

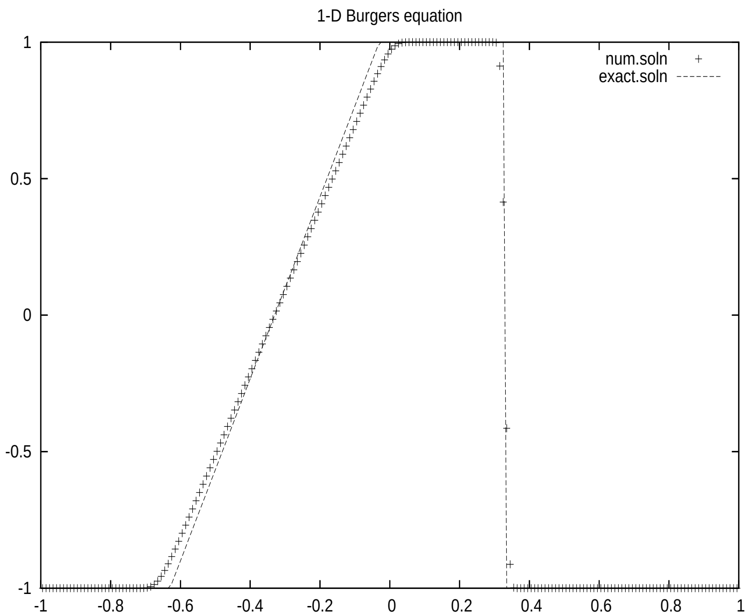

Figure 7: First order scheme with 200 grid points,CFL=0.75

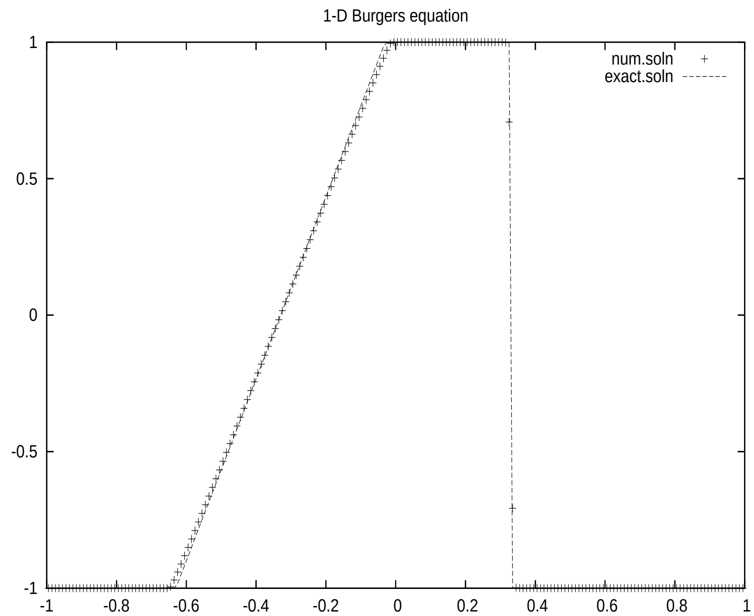

Figure 8: Second order scheme with 200 grid points,CFL=0.15

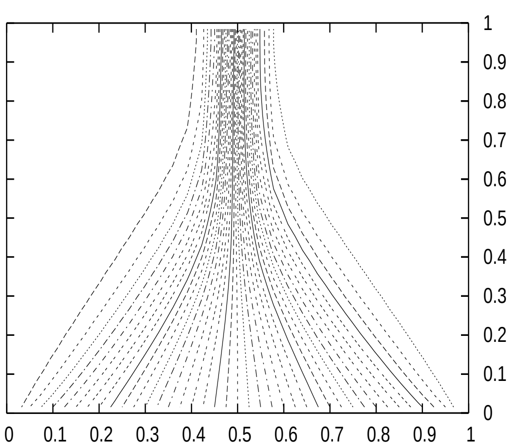

Figure 9: First order scheme with 32×32 grid points,CFL=0.45

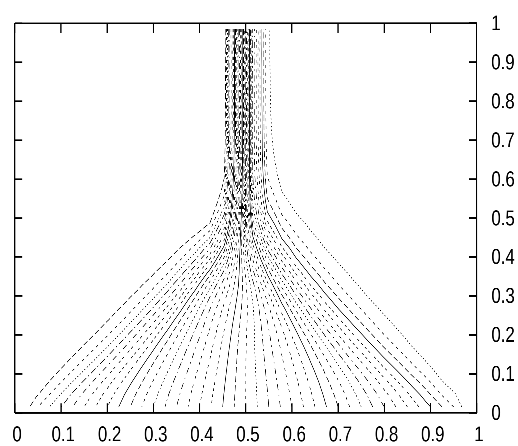

Figure 10: Second order scheme with 32×32 grid points,CFL=0.1

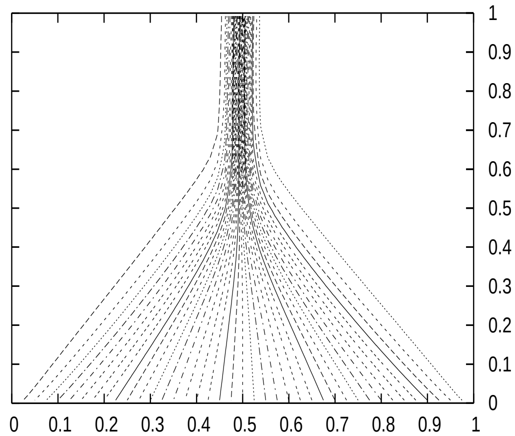

Figure 11: First order scheme with 64×64 grid points,CFL=0.45

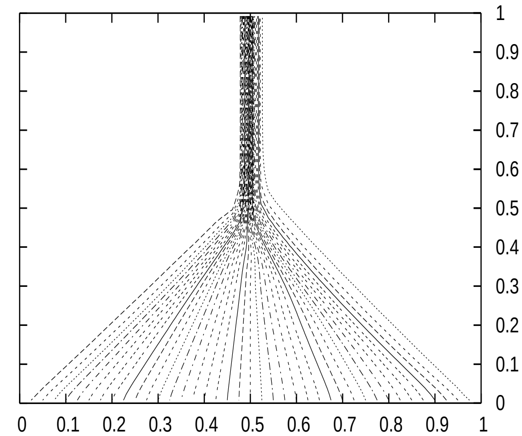

Figure 12: Second order scheme with 64×64 grid points,CFL=0.1

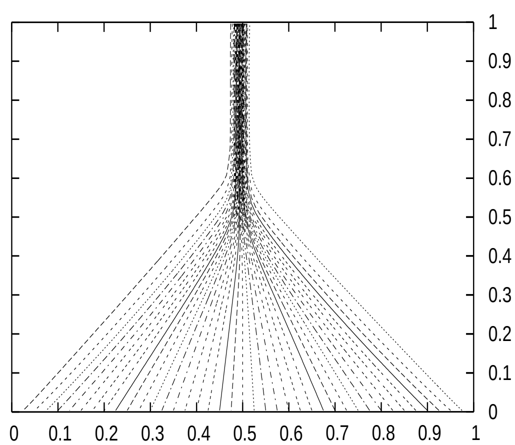

Figure 13: First order scheme with 128×128 grid points,CFL=0.45

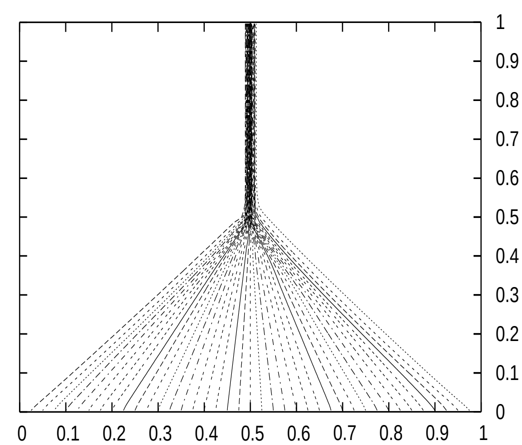

Figure 14: Second order scheme with 128×128 grid points,CFL=0.1

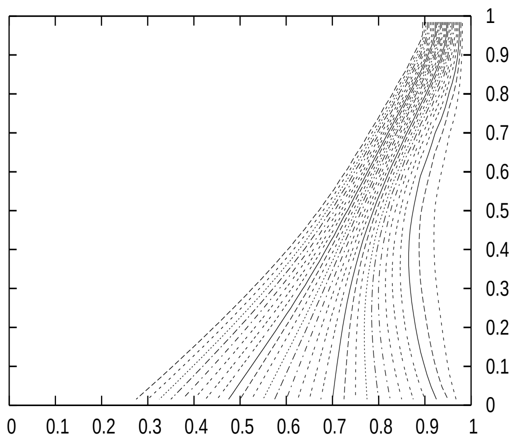

Figure 15: First order scheme with 32×32 grid points,CFL=0.45

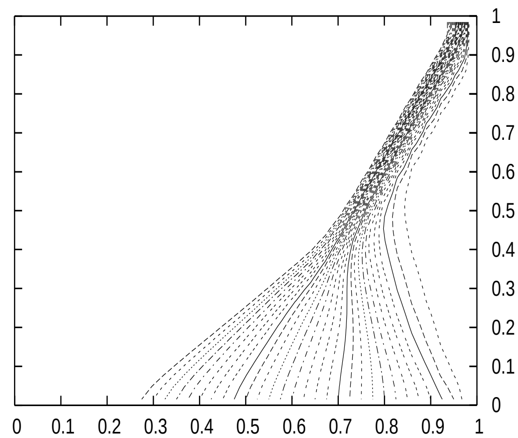

Figure 16: Second order scheme with 32×32 grid points,CFL=0.1

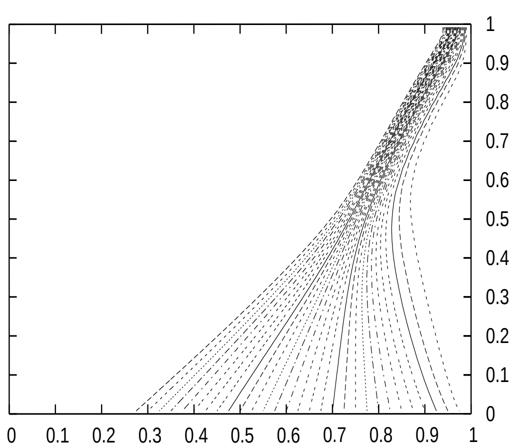

Figure 17: First order scheme with 64×64 grid points,CFL=0.45

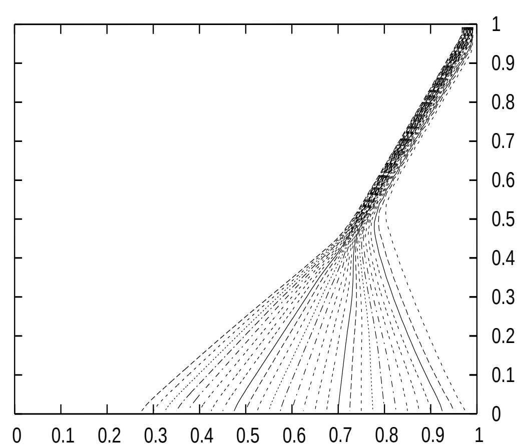

Figure 18: Second order scheme with 64×64 grid points,CFL=0.1

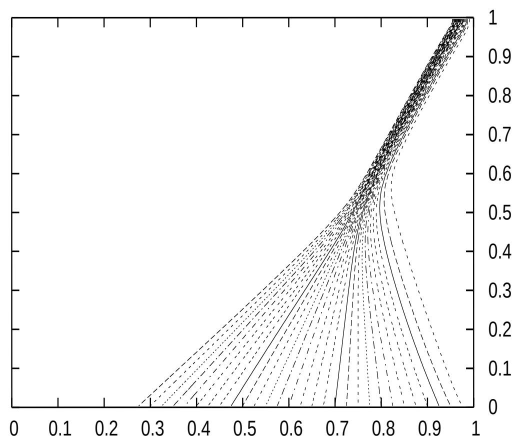

Figure 19: First order scheme with 128×128 grid points,CFL=0.45

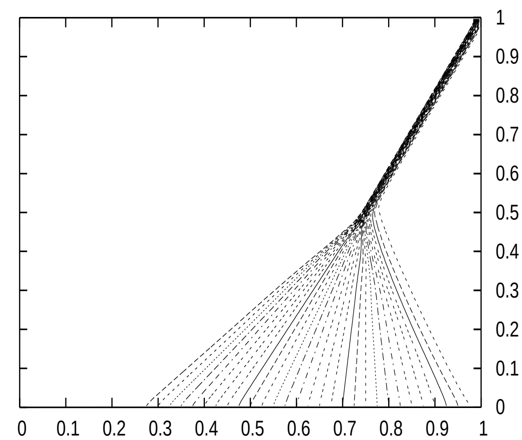

Figure 20: Second order scheme with 128×128 grid points,CFL=0.1