Transonic flow of a fluid with positive and negative nonlinearity through a nozzle

D. Chandrasekar and Phoolan Prasad
Department of Mathematics, Indian Institute of Science, Bangalore 560-012, India

(Received 31 August 1990; accepted 7 September 1990)

The one-dimensional transonic flow of an inviscid fluid, which at large values of the specific heats exhibits both positive ($\overline{\Gamma} > 0$) and negative ($\overline{\Gamma} < 0$) nonlinearity regions $\{\overline{\Gamma} = (1/\rho)[\partial(\rho a)/\partial\rho]_s\}$ and which remains in a single phase, is studied. By assuming that $\overline{\Gamma}$ changes its sign in the small neighborhood of the throat of the nozzle where transonic flow exists and introducing a new scaling of the independent variables, an approximate first-order partial differential equation (PDE) with a nonconvex flux function is derived. It governs both the steady transonic flows and the upstream moving waves near sonic point. The existence of continuous and discontinuous steady transonic flows when the throat area is either a maximum or a minimum is shown. The existence of standing sonic discontinuities and rarefaction shocks in the transonic flow are noted for the first time. Unlike in the classical gas flows, there are two sonic points and continuous transonic flows are possible only through one of them. The numerical evolution of those transonic waves that have both positive and negative nonlinearity in the same pulse is studied and some comments are made on the local stability of the particular steady flows.

I. INTRODUCTION

The study of the thermodynamics of the real fluids at large values of the specific heats c_v and c_p has attracted many researchers due to the increasing applications found in the recent advancements in chemical and nuclear technology. In gasdynamics, it is customary to assume the fundamental derivative[1,2]

$$\overline{\Gamma} = \frac{1}{\bar{\rho}}\left(\frac{\partial(\bar{\rho}\bar{a})}{\partial\bar{\rho}}\right)_{\bar{S}}$$

with the density $\bar{\rho}$, the velocity of sound $\bar{a}$, and the entropy $\bar{S}$ of the fluid, to be positive. But for most of the fluids of common practical interest with large heat capacity, this important quantity becomes negative. In a fluid with negative $\overline{\Gamma}$, only expansion shocks appear. These shocks were encountered in the beginning of this century. Bethe[9] and Zeldovich[4] observed that the expansion shocks violate the entropy condition except for a fluid with the anomalous behavior for which $\overline{\Gamma} < 0$. Since the fluids with $\overline{\Gamma} < 0$ were uncommon they did not give much importance to it.

An important study of the fluids with $\overline{\Gamma} < 0$ was conducted by Thompson and Lambrakis.[2] They have carried out detailed computations with more accurate equations of state to provide specific examples of fluids in which $\overline{\Gamma}$ becomes negative and noted the entropy increase across such shocks. They have also noticed that $\overline{\Gamma}$ vanishes in the single phase of the fluid, and the $\overline{\Gamma} < 0$ region extends sufficiently outside the two-phase region. Borisov *et al.*[5] have experimentally observed the expansion shocks in the relatively simple compound Freon-13. Also Kutateladze *et al.*[6] have recently pointed out that water vapor also admits expansion shocks at high pressures.

Further, recently, Cramer[7] has given an impressive list of seven commercially available fluorocarbons with the Martin–Hou equation of state, which show clearly the existence of negative nonlinearity regions in the single phase of the fluid beyond doubt. Though our analysis is not meant for such phenomena, we remark that these shocks quite frequently connect the vapor and liquid phases of a fluid with large heat capacity under phase transition.[8]

The fundamental derivative $\overline{\Gamma}$ changes from point to point in a disturbance. Garrett[9] has shown experimentally that finite amplitude fourth sound waves in ^{3}He–B exhibit both $\overline{\Gamma} > 0$ and $\overline{\Gamma} < 0$ values in the same disturbance. The properties of these waves seem to be surprisingly different from those for a wave with constant value of $\overline{\Gamma}$, either positive or negative. Thompson and Lambrakis[2] have pointed out that these may lead to the formation of finite amplitude double sonic shocks. Borisov *et al.*[5] have correctly indicated that a partial disintegration of the shock may occur. Recently, Cramer and Kluwick[10] have developed a weak shock theory to study the propagation of the finite amplitude waves exhibiting negative ($\overline{\Gamma} < 0$) and positive ($\overline{\Gamma} > 0$) nonlinearity in a single phase fluid whose undisturbed state lies in the transition zone, a small neighborhood containing the state across which $\overline{\Gamma}$ changes its sign. Cramer *et al.*[11] have further added viscosity and heat conduction to this model and studied the propagation of the dissipative waves with positive and negative nonlinearity. Also Cramer and Sen[12] have studied the propagation of finite amplitude waves in the negative nonlinearity region for a van der Waals fluid. The phenomenon of shock splitting in a single-phase fluid has been studied by Cramer[19] (see also the Ref. 13).

Thompson[1] has investigated the role of the fundamental derivative in nozzle flows. He has found that the continuous flows are possible for an isentropic fluid with either negative (or positive) values of the fundamental derivative only if the throat area is a maximum (or a minimum). Here we present a study of the steady flows of a fluid whose fundamental derivative changes its sign near the throat of the nozzle. We

 0899-8213/91/030427-12$02.00

also study the propagation of nonlinear waves on these steady flows in a small neighborhood at the throat of the nozzle.

The propagation of a nonlinear pulse in the neighborhood of a sonic point can be studied by the general theories of Kulikovskii and Slobodkina[14] and for two-dimensional problems by the theories of Prasad[15] and Ravindran.[16] However these theories are valid only for usual fluids with positive nonlinearity. We present here a new scaling of time, length, and amplitude that gives the approximate equation for the waves moving upstream with positive and negative nonlinearity near the throat of the nozzle. This equation belongs to the class of conservation laws with nonconvex flux function having an inhomogeneous source term. A theory for conservation laws with nonconvex flux functions has been developed by Ballou.[17] The mathematical study of this class of equations is still in progress (see Refs. 18, 19, and 20).

A "singularity separating method" has been developed[20] especially for the numerical solution of this type of conservation laws. We have modified this scheme to incorporate the source term and used it in our model equation to study the behavior of nonlinear waves on the steady flows.

II. BASIC EQUATIONS AND APPROXIMATION IN THE NEIGHBORHOOD OF THE SONIC POINT

Let us consider one-dimensional flow of a single phase, compressible, inviscid fluid through a nozzle of varying cross section. We restrict our attention to the fluids for which the fundamental derivative $\overline{\Gamma} = \overline{\Gamma}(\bar{\rho},\bar{S})$ changes its sign in a small neighborhood close to the sonic point. For a fluid for which $\overline{\Gamma}$ does not change sign, the sonic point where the fluid changes from subsonic to supersonic or vice versa, will occur only at the throat of the nozzle, where the area of cross section is the maximum or minimum.[1] In order to study the behavior of the steady flow and the propagation of a nonlinear wave near the throat of the nozzle, we introduce the following nondimensional variables:

$$x = \frac{\bar{x}}{L}, \quad t = \frac{a_* \bar{t}}{L}, \quad V = \frac{\overline{V}}{a_*}, \quad \rho = \frac{\bar{\rho}}{\rho_*}, \quad P = \frac{\overline{P}}{\rho_* a_*^2},$$

$$S = \frac{\bar{S}}{C_v^*}, \quad h = \frac{\bar{h}}{a_*^2}, \quad T = \frac{\overline{T}}{T_*}, \quad u = \frac{\bar{u}}{a_*}, \quad a = \frac{\bar{a}}{a_*},$$

where $\overline{V}, \bar{\rho}, \overline{P}, \bar{S}, \bar{h}$, and $\overline{T}$ denote, respectively, the particle velocity, density, pressure, entropy, specific enthalpy, and temperature of the fluid and * denotes their values at the sonic point. The x axis is chosen along the axis of the nozzle with the throat at $x=0$, and t is time. Here, u is the velocity of the shock in the flow, L is a characteristic length, and $a = (\partial P/\partial\rho|_S)^{1/2}$ is the velocity of sound. The governing equations for the fluid motion in the nondimensional variables are

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho V A) = 0, \tag{1}$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0, \tag{2}$$

$$\rho T\left(\frac{\partial S}{\partial t} + V\frac{\partial S}{\partial x}\right) = 0, \tag{3}$$

where $A = A(x)$ is the nondimensional area of the cross section of the nozzle. Across a shock discontinuity the following Rankine–Hugoniot conditions should be satisfied:

$$u[\rho] = [\rho V], \tag{4}$$

$$(u - V_l)(u - V_r) = [P]/[\rho], \tag{5}$$

$$[h] = \tfrac{1}{2}(V_l + V_r)[P], \tag{6}$$

$$[S] \geqslant 0, \tag{7}$$

where the subscripts l and r correspond to the state to the left and to the right of the shock. Here, $[Q]$ denotes $Q_r - Q_l$. For our analysis, we express all the state variables of the fluid through ρ and S. In addition, we require the specific heat at constant volume (C_v) and $\partial P/\partial\rho|_T$ both to be non-negative for the thermal and mechanical stability of the fluid.

Since the fundamental derivative changes its sign in a neighborhood close to the sonic point, we set, $(\rho_*/a_*)\overline{\Gamma}(\rho_*,S_*) = O(\epsilon)$, where ϵ is a small non-negative quantity. For the study of the local stability of the transonic flow, we impose a perturbation (wave) whose amplitude is of the order of the fundamental derivative. We shall find later that this wave stays in the transonic region for the time intervals of order $\bar{\epsilon}^{1/2}$ and spreads over a distance of order $\epsilon^{3/2}$.

The compatibility conditions along the characteristic curves $dx/dt = V \pm a$, of the quasilinear, hyperbolic system (1)–(3) are:

$$\rho\left(\frac{\partial V}{\partial t} + (V \pm a)\frac{\partial V}{\partial x}\right) \pm a\left(\frac{\partial \rho}{\partial t} + (V \pm a)\frac{\partial \rho}{\partial x}\right) = \pm\left[\frac{aV\rho A'}{A} \pm \left(\frac{\partial P}{\partial S}\right)S_x\right], \tag{8}$$

where we have used

$$\frac{\partial P}{\partial x} = \left(\frac{\partial P}{\partial \rho}\right)_S\left(\frac{\partial \rho}{\partial x}\right) + \left(\frac{\partial P}{\partial S}\right)_\rho\left(\frac{\partial S}{\partial x}\right) \text{ and } A' = \frac{dA}{dx}.$$

Since along the third family of characteristics $dx/dt = V$ the entropy remains constant, it is clear that the primary source of entropy gradient in the flow is due to the variation in the strength of the shock. Cramer and Kluwick,[10] have shown that the jump in the entropy of such a fluid across a shock is of the fourth order in the strength of the shock.

In this paper, we study the behavior of the flow in a small neighborhood of the throat where the steady flow is almost sonic, i.e., $V - a \simeq 0$. The growth and decay of perturbations in this neighborhood are determined by the waves following the characteristics $dx/dt = V - a$, i.e., upstream propagating waves. The downstream propagating waves move with velocity $V + a$, and hence quickly move away from this neighborhood. They do not affect the nature of the transonic flow.[14,21] Hence we approximate Eq. (8) in a small neighborhood of the characteristics $dx/dt = V - a$ in the (x,t) plane as in the work of Kulikovskii and Slobodkina.[14] However, the method of Kulikovskii and Slobodkina[14] is not applicable when $\overline{\Gamma}$ changes sign. For a state variable, say, for instance, the density, we write $\rho = \rho_0 + \rho'$

with ρ_0 and ρ' representing the value of the density in the steady flow and its perturbation, respectively. We expand the steady solution and the perturbation in the following form:

$$\rho_0 = 1 + \epsilon\rho_{01} + \epsilon^2\rho_{02} + \epsilon^3\rho_{03} + O(\epsilon^4),$$
$$\rho' = \epsilon\rho_1 + \epsilon^2\rho_2 + \epsilon^3\rho_3 + O(\epsilon^4). \quad (9)$$

Hence the total perturbation expansions can be written as

$$\rho = 1 + \epsilon\hat{\rho}_1 + \epsilon^2\hat{\rho}_2 + \epsilon^3\hat{\rho}_3 + O(\epsilon^4),$$
$$V = 1 + \epsilon\hat{V}_1 + \epsilon^2\hat{V}_2 + \epsilon^3\hat{V}_3 + O(\epsilon^4), \quad (10)$$
$$S = 1 + \epsilon^4\hat{S}_4 + O(\epsilon^5),$$

where $\hat{\rho}_1 = \rho_{01} + \rho_1, \rho_2 = \rho_{02} + \rho_2, \hat{\rho}_3 = \rho_{03} + \rho_3$, etc. In the expansion for entropy S, we make use of the fact that the jump in the entropy is of the fourth order, for the fluid under consideration.

As in Cramer and Kluwick,[10] we can use (8) and the shock relations (4)–(6) to argue that the changes in the shock strength is noticeable only over propagation distances and time scales of order ϵ^{-2}. And so $\int_0^t S_x \, d\tau$ evaluated on a characteristic defined by (8) is of the order of $[S]$, i.e., ϵ^4 at most. Thus for the present purpose we can neglect the entropy gradient term appearing in Eq. (8).

Expanding the variables a, P, and $\overline{\Gamma}$ in a Taylor series about the sonic point and using (10) we get

$$a = 1 + \epsilon(-\hat{\rho}_1) + \epsilon^2(\hat{\Gamma}\hat{\rho}_1 + \tfrac{1}{2}\Lambda\hat{\rho}_1^2 + \hat{\rho}_1^2 - \hat{\rho}_2) + O(\epsilon^3), \quad (11)$$

$$p = 1 + \epsilon\hat{\rho}_1 + \epsilon^2(\hat{\rho}_2 - \hat{\rho}_1^2) + \epsilon^3[(\hat{\rho}_3 + \hat{\Gamma}\hat{\rho}_1^2 + \hat{\rho}_1^3(1 + \Lambda/3) - 2\hat{\rho}_1\hat{\rho}_2] + O(\epsilon^4), \quad (12)$$

$$\overline{\Gamma} = \epsilon(\hat{\Gamma} + \Lambda\hat{\rho}_1) + \epsilon^2\left[\Lambda\hat{\rho}_2 + \frac{\hat{\rho}_*^3}{2a_*}\left(\frac{\partial^2\overline{\Gamma}}{\partial\rho^2}\right)_*\hat{\rho}_1^2\right] + O(\epsilon^3), \quad (13)$$

where

$$\hat{\Gamma} = \frac{\rho_*}{\epsilon a_*}\overline{\Gamma}(\rho_*,S_*) \text{ and } \Lambda = \frac{\rho_*^2}{a_*}\left(\frac{\partial\overline{\Gamma}}{\partial\rho}\right)_*.$$

We take an expansion for the velocity of the shock in the form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + O(\epsilon^4). \quad (14)$$

Using the shock relations (4)–(7), we get two values for u_0: $u_0 = 0$ and $u_0 = 2$. Since in the steady flow the shocks have zero velocity, we choose $u_0 = 0$. Further

$$u_1 = \tfrac{1}{2}(\hat{V}_{1l} + \hat{V}_{1r} + \hat{\rho}_{1r} + \hat{\rho}_{1l}),$$
$$u_2 = -\{\hat{\Gamma}[\hat{\rho}_1^2] + (1 + \tfrac{1}{3}\Lambda)[\hat{\rho}_1^3] - 2[\hat{\rho}_1\hat{\rho}_2] + [\hat{\rho}_2](\hat{\rho}_{1l} + \hat{\rho}_{1r}) - \tfrac{1}{4}[\hat{\rho}_1](\hat{V}_{1l} - \hat{V}_{1r} + \hat{\rho}_{1l} + \rho_{1r}) \times (\hat{V}_{1r} - \hat{V}_{1l} + \hat{\rho}_{1l} + \rho_{1r})\}/2[\hat{\rho}_1].$$

Using (10) and (11) in (3)–(7) we obtain

$$[\hat{\rho}_1 + \hat{V}_1] = 0 \quad (15)$$

and

$$[\hat{V}_2 - \hat{\rho}_1^2 + \hat{\rho}_2] = 0. \quad (16)$$

Equations (15) and (16) mean that the Riemann invariants $\hat{\rho}_1 + \hat{V}_1$ and $\hat{V}_2 - \hat{\rho}_1^2 + \hat{\rho}_2$ are continuous across shocks. We note that from Eq. (13) that the value of $\overline{\Gamma}$ in the transonic region is of the order of the amplitude ϵ of the perturbations.

We shall consider a perturbation to the steady flow, which is bounded in space with its amplitude $\hat{\rho}_1$, of the order of the fundamental derivative. A simple analysis of Eq. (8) with (10)–(13) shows that, in order to incorporate the full nonlinear effects of the perturbation and the effect of the curvature of the nozzle, we need to scale the independent variables x and t as follows:

$$\hat{x} = x/\epsilon^{3/2} \text{ and } \hat{t} = \epsilon^{1/2}t. \quad (17)$$

From this it is clear that the perturbations that are of the order of the fundamental derivative stay in the neighborhood of the sonic point for a time interval of order $\epsilon^{-1/2}$ and extend over a distance of order $\epsilon^{3/2}$.

Expanding $A = A(\hat{x})$ about the throat of the nozzle where $dA/d\hat{x} = 0$, we obtain

$$\frac{1}{A}\left(\frac{dA}{d\hat{x}}\right) = \frac{\epsilon^{3/2}}{A_*}\left(\frac{d^2A}{d\hat{x}^2}\right)_*\hat{x} + O(\epsilon^2). \quad (18)$$

Using Eqs. (10)–(13), (17), and (18) in the compatibility condition (8) along the characteristic $dx/dt = V + a$ and equating the terms of various orders, we obtain

$$\frac{\partial(\hat{V}_1 + \hat{\rho}_1)}{\partial\hat{x}} = 0 \text{ and } \frac{\partial}{\partial\hat{x}}(\hat{V}_2 + \hat{\rho}_2 - \hat{\rho}_1^2) = 0. \quad (19)$$

Hence

$$\hat{V}_1 + \hat{\rho}_1 = h(\hat{t}) \text{ and } \hat{V}_2 = \hat{\rho}_1^2 - \hat{\rho}_2 + g(\hat{t}), \quad (20)$$

where $h(\hat{t})$ and $g(\hat{t})$ are arbitrary functions of $\hat{t}$. From Eq. (19) we see that the rate of change of $\hat{V}_1 + \hat{\rho}_1$ along the downstream moving waves in a small neighborhood of the characteristic $dx/dt = V - a$ is zero. Also from Eq. (15) we see that $\hat{V}_1 + \hat{\rho}_1$ is continuous across the shock waves in the flow. In our approximation, the functions h and g, which do not depend on $\hat{x}$, represent the influence of the flow away from the small neighborhood (of the order of $\epsilon^{3/2}$) of the throat on the local waves. Since we are interested in the behavior of only the local waves that stay in this neighborhood for sufficiently long time (of the order of $\epsilon^{-1/2}$), we can set $h(\hat{t}) = 0$, $g(\hat{t}) = 0$. It is now important to realize that the variation of the throat area may also produce reflected or cross waves, which, according to our approximation, are negligible, i.e., smaller than quantities of order ϵ. Hence in the small neighborhood considered, $\hat{V}_1 + \hat{\rho}_1 = 0$ and $\hat{V}_2 = \hat{\rho}_1^2 - \hat{\rho}_2$.

Now using Eqs. (11), (20) and (10) we obtain

$$V - a = -\epsilon^2(\hat{\Gamma}\hat{\rho}_1 + \tfrac{1}{2}\Lambda\hat{\rho}_1^2) + o(\epsilon^2). \quad (21)$$

Using Eqs. (10)–(13), (17), (18) and (21) in the compatibility condition (8) along the characteristics $dx/dt = V - a$, we obtain

$$\frac{\partial\hat{\rho}_1}{\partial\hat{t}} - \left(\hat{\Gamma}\hat{\rho}_1 + \frac{1}{2}\Lambda\hat{\rho}_1^2\right)\frac{\partial\hat{\rho}_1}{\partial\hat{x}} = -\frac{1}{2A_*}\left(\frac{d^2A}{d\hat{x}^2}\right)_*\hat{x}. \quad (22)$$

Equation (22) is the required equation that governs the propagation of the upstream moving waves in the neighborhood of the sonic point. It may be noted, as in the case of the

Kulikovskii and Slobodkina[14] equation for an ordinary gas, this equation governs both the steady transonic flow and the upstream moving disturbances.

In terms of a new set of variables $\bar{\rho}, \bar{x}, \bar{t}$ defined by $\bar{\rho} = (\Lambda/\hat{\Gamma})\hat{\rho}$, $\bar{x} = (\Lambda/\hat{\Gamma}^2)\hat{x}$, and $\bar{t} = \hat{t}$, Eq. (22) becomes

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} - \left(\bar{\rho} + \frac{1}{2}\bar{\rho}^2\right)\frac{\partial \bar{\rho}}{\partial \bar{x}} = -\frac{k\hat{\Gamma}}{2}\bar{x}, \tag{23}$$

where $k = (1/A_*)(d^2A/d\bar{x}^2)_*$. In the characteristic form, Eq. (23) can be written as

$$\frac{d\bar{\rho}}{d\bar{t}} = -K\ \bar{x} \text{ along } \frac{d\bar{x}}{d\bar{t}} = -\left(\bar{\rho} + \frac{1}{2}\bar{\rho}^2\right), \tag{24}$$

with $K = k\hat{\Gamma}/2$. The steady solutions are given by

$$\left(\bar{\rho}_{01} + \frac{1}{2}\bar{\rho}_{01}^2\right)\frac{d\bar{\rho}_{01}}{d\bar{x}} = \frac{k\hat{\Gamma}}{2}\hat{x}, \tag{25}$$

where $\bar{\rho}_{01} = (\Lambda/\hat{\Gamma})\rho_{01}$. From Eq. (12) the shock velocity can be written as

$$\frac{d\bar{x}_S}{d\bar{t}} = -\frac{1}{2}\left((\bar{\rho}_l + \bar{\rho}_r) + \frac{1}{3}(\bar{\rho}_l^2 + \bar{\rho}_l\bar{\rho}_r + \bar{\rho}_r^2)\right). \tag{26}$$

We note here that $k = 0$ reduces Eq. (22) to

$$\frac{\partial \bar{\rho}}{\partial \bar{t}} - \frac{\partial}{\partial \bar{x}}\left(\frac{1}{2}\bar{\rho}^2 + \frac{1}{6}\bar{\rho}^3\right) = 0. \tag{27}$$

This equation governs the propagation of the waves in a tube of constant area of cross section. A study similar to that of Cramer and Kluwick[10] can be made for the understanding of these waves.

We would like to justify the physical validity of the model considered. As mentioned in the Introduction, the fundamental derivative may vanish at a particular thermodynamic state in the single phase of a fluid with sufficiently high values of the specific heats c_v and c_p. Hence there exists a small neighborhood of that state containing the point across which $\bar{\Gamma} = 0$. This neighborhood is such that it does not contain the critical point, but extends outside the two-phase region. In the case of a van der Waals' model this situation prevails[4,12,13] near the two-phase region. When a nozzle is connected to a reservoir with the gas at rest, the state of the fluid in the reservoir can be so adjusted that the state of the fluid obtained near the throat of the nozzle is in the above neighborhood. In the region near the throat, $\bar{\Gamma}$ changes from point to point and both $\bar{\Gamma}$ and its rate of change may either be positive or negative. Hence to get a qualitative understanding of the behavior of the fluid, one must consider all the possible combinations of the signs of the parameters $\hat{\Gamma}$ and Λ.

III. POSSIBLE STEADY FLOWS

The approximate equation (25) governs the steady transonic flows near the throat. Hereafter we use ρ_0 for ρ_{01}, for convenience. For a nozzle with nonzero curvature k, the behavior of the steady flow near the sonic point for different values of the parameters Λ and $\hat{\Gamma}$ can be studied by considering the following cases:

$$\begin{aligned} &(\text{I}) \quad k \neq 0, \quad \Lambda = 0, \quad \hat{\Gamma} \neq 0, \\ &(\text{II}) \quad k \neq 0, \quad \Lambda \neq 0, \quad \hat{\Gamma} = 0, \\ &(\text{III}) \quad k \neq 0, \quad \Lambda \neq 0, \quad \hat{\Gamma} \neq 0. \end{aligned} \tag{28}$$

We discuss below the different steady flows arising out of the different combinations of signs of the parameters. It is important to note that $k < 0$ corresponds to the maximum throat area and $k > 0$ to the minimum throat area. The direction of the fluid motion is from left to right in all ensuing figures. The significance of the arrows in the figures will be explained in the next section.

Case (I): Setting $\Lambda = 0$ in Eq. (25), we get

$$\frac{d\rho_0}{dx} = \left(\frac{k\hat{\Gamma}}{2}\right)\frac{x}{\rho_0}. \tag{29}$$

The integral curves $\rho_0^2 = (k\hat{\Gamma}/2)x^2 + C$, with an arbitrary constant C, represent the one-parameter family of steady flows.

(a) $k > 0$ and $\hat{\Gamma} > 0$: The dominant term of the order of ϵ in Eq. (13) shows that $\bar{\Gamma}(\rho_0, S_0)$ remains positive everywhere in the steady flow. Since the throat area is minimum, the behavior of the fluid is similar to the well-known polytropic gas flow through a nozzle. We may note that for a polytropic gas with constant specific heats (c_v and c_p), $\bar{\Gamma}$ is $(a/\rho)[(\gamma + 1)/2]$, $\gamma = c_p/c_v$. The intensive study of Kantrowitz[21] shows the existence of standing shock waves in the diverging or in the converging part of the nozzle. These results can be seen in the phase plane of (29), as shown by continuous lines in Fig. 1(a). A recent numerical investigation by Embid *et al.*,[22] of the approximate equation modeling such flows, shows the existence of multiple steady states in a nozzle with fixed entry and exit conditions. It may be noted that the nonuniqueness in the position of the shock is due to the local approximation of the equations near the sonic point and in general the position of the shock can be defined uniquely.[23] Various steady flows obtained from the phase plane have been described in the caption of the figure.

(b) $k < 0$ and $\hat{\Gamma} < 0$: Since Eq. (29) remains unchanged, the nature of the singular point is same as in (a). The phase plane is represented in Fig. 1(b). We may note that $\bar{\Gamma}$ remains negative everywhere in the steady flow and the throat area is maximum as noted by Thompson[1] for such fluids. It is known that for a polytropic gas flow through a nozzle with maximum throat area, the singular point is a center, near which no continuous flows are possible. But for the fluid under consideration, we notice the continuous supersonic, subsonic, and transonic flows near the singular point. We also see the existence of rarefaction shocks either in the converging or in the diverging portion of the nozzle to match the entry and the exit conditions of the nozzle.

(c) $k < 0$ and $\hat{\Gamma} > 0$ (or) $k > 0$ and $\hat{\Gamma} < 0$: Both these conditions lead to a singular point that is a center for Eq. (29) and hence there does not exist any continuous flow. For an ordinary fluid this result is well known.[22]

Case (II): In this case Eq. (23) reduces to

$$\frac{d\rho_0}{dx} = \left(\frac{k}{\Lambda}\right)\frac{x}{\rho_0^2} \tag{30}$$

and the integral curves are given by $\frac{1}{3}\rho_0^3 + C = (k/\Lambda)x^2$.

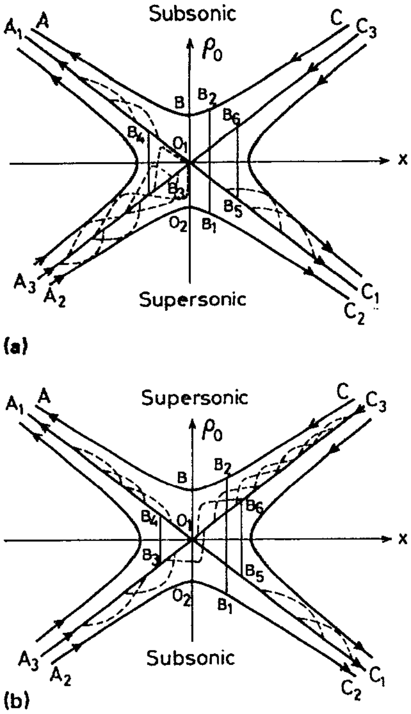

FIG. 1. (a) Flow through a nozzle with minimum throat area, $\Lambda = 0$ and $\hat{\Gamma} > 0$. ABC and $A_2O_2C_2$ are continuous subsonic and supersonic flows, respectively. $A_1O_1C_1$, $A_1O_1C_3$, $A_3O_1C_3$, and $A_3O_1C_1$ are continuous flows having sonic transition. $A_3B_3B_4O_1C_1$, $A_3B_3B_4O_1C_3$, $A_3O_1B_5B_6C_3$, and $A_2O_2B_1B_2C$ are flows with a single compression shock. The positions the shocks are nonunique. (b) Flow through a nozzle with maximum throat area. $\Lambda = 0$ and $\hat{\Gamma} < 0$. The description is same as in Fig. 1(a), except that the supersonic flows are replaced by subsonic ones and vice versa and a compression shock is replaced by a rarefaction shock.

The characteristic velocity V-a is approximately equal to $-\frac{1}{2}\Lambda\rho_0^2$. The integral curves are plotted in Fig. 2 for $k/\Lambda > 0$; the same for $k/\Lambda < 0$ can be obtained by reflecting the curves about the x axis. It is seen that if the fluid starts initially with a supersonic (subsonic) speed, it always remains supersonic (subsonic). The nature of the flow being either supersonic or subsonic depends only on the sign of Λ. Since the Rankine–Hugoniot points are complex conjugates, there does not exist any flow with standing shock waves. It is important to notice that whether the throat area is a maximum or minimum, continuous steady flows are always possible. There is a single flow that becomes sonic at the throat, but continues to be in the same state (supersonic or subsonic). From Fig. 2, it is seen that some of the flows starting initially in a region with $\overline{\Gamma} < 0$ (or $\overline{\Gamma} > 0$) returns to the same region after remaining in the region $\overline{\Gamma} > 0$ (or $\overline{\Gamma} < 0$) over a short distance.

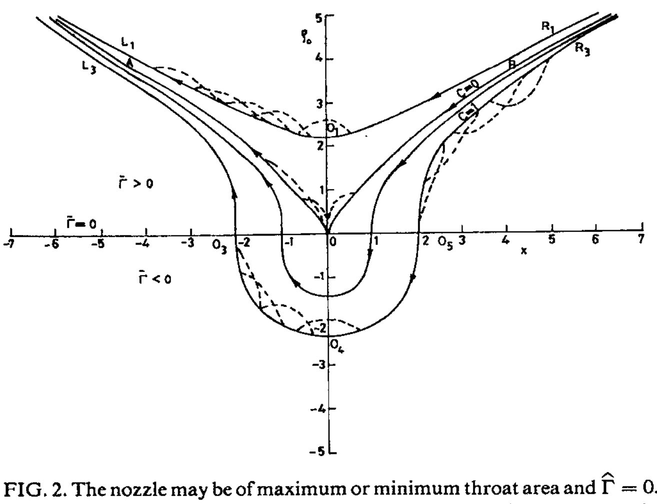

FIG. 2. The nozzle may be of maximum or minimum throat area and $\hat{\Gamma} = 0$. For a nozzle with minimum throat area and $\Lambda > 0$, all the flows are subsonic. For a nozzle with maximum throat area and $\Lambda < 0$, all the flows are supersonic and the direction of the arrows are reversed in this case. All these flows are stable.

Case (III): Equation (25) can be integrated to give the integral curves:

$$k\hat{\Gamma}\bar{x}^2 = \tfrac{1}{3}\bar{\rho}_0^3 + \bar{\rho}_0^2 + C, \tag{31}$$

with an arbitrary constant C giving a one-parameter family of steady flows near the throat. Figure 3 gives a plot of the integral curves for $k = 1$.

For a steady flow we can easily derive the following:

$$[S] = [(\epsilon\hat{\Gamma})^4/6\Lambda^3][\bar{\rho}_0]^3[1 + \tfrac{1}{2}(\bar{\rho}_{0l} + \bar{\rho}_{0r})], \tag{32}$$

$$V - a = -\epsilon^2(\hat{\Gamma}/\Lambda)^2\Lambda(\bar{\rho}_0 + \tfrac{1}{2}\bar{\rho}_0^2), \tag{33}$$

and

$$\overline{\Gamma} = \epsilon\hat{\Gamma}(1 + \bar{\rho}_0) + O(\epsilon^2). \tag{34}$$

We note that (33) can be written as

$$[S] = \frac{1}{6}\left(\frac{\epsilon\hat{\Gamma}}{\Lambda}\right)^2[\bar{\rho}]^2[(V-a)_l - (V-a)_r].$$

Hence we see that the shock stability condition (necessary for a discontinuity to be an admissible shock),

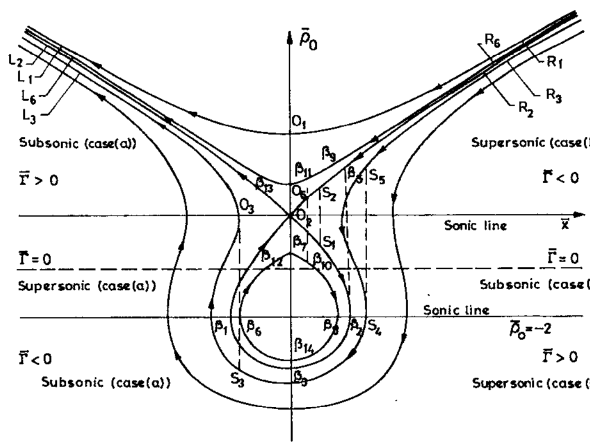

FIG. 3. Case III(a): A nozzle with minimum throat area $k(>0)$, $\overline{\Gamma} > 0$ and $\Lambda > 0$. Arrows indicate the direction of motion of the perturbations. $L_3O_3S_3S_4S_5R_3$ and $L_2O_2\beta_3\beta_2\beta_5R_2$ are flows with two sonic shocks. (O_3S_3, $O_2\beta_3$—rarefaction and S_4S_5, $\beta_2\beta_5$—compression). $L_2O_2S_1S_2R_2$, $\beta_1O_2S_1S_2R_2$, $\beta_6\beta_7\beta_{10}\beta_{11}R_6$, and $\beta_6\beta_{12}\beta_{13}O_6R_6$ are flows with a single standing shock (S_1S_2—compression shock in the diverging section of the nozzle, the position of the shock is nonunique). $L_2O_2\beta_2\beta_5R_2$ and $\beta_6\beta_7\beta_8\beta_9R_6$ are flows with a single sonic discontinuity (compression). Case III (b): A nozzle with maximum throat area ($k < 0$), $\hat{\Gamma} < 0$, and $\Lambda < 0$. Perturbations propagate in the opposite direction given by the arrows and the steady flow is from right to left. $R_3S_5S_4S_3O_3L_3$ and $R_2\beta_5\beta_2\beta_3O_2L_2$ are flows with two right sonic shocks. $R_2O_2\beta_1$ is a transonic flow, it cannot extend until the exit. Closed loops do not represent any flow and no shock solutions are possible.

$$(V-a)_l \geqslant 0 \geqslant (V-a)_r, \tag{35}$$

is sufficient to guarantee that the entropy does not decrease across shocks. However, Eq. (35) is a stronger condition in the sense that $[S]$ may be greater than zero even though Eq. (35) is not satisfied. While dealing with the individual flows through a nozzle we shall see later that (as in Cramer and Kluwick[10]) the equalities on both the sides in (35) can never be satisfied simultaneously. Of course one of the equalities either on the left or right is possibly dependent on the sign of Λ. If the sign of Λ is positive, the left equality is possible, while for negative Λ the right equality is possible.

As pointed out earlier, there are many possible combinations of the signs of the parameters. We shall describe the behavior of the flows in detail for a single combination and the remaining combinations can be studied in the same way.

A. Case (a): $k>0$, $\Lambda>0$, and $\hat{\Gamma}>0$

In this case we have a nozzle with a minimum area at the throat. The fundamental derivative and its rate of change along the isentrope at the critical point are positive. We can study the steady flows with the help of the phase plane shown in Fig. 3.

We may note that an increase in the value of $\bar{\rho}_0$ indicates compression and a decrease in its value indicates expansion. Also a sudden increase in $\bar{\rho}_0$ corresponds to a compression shock and a sudden decrease to a rarefaction (expansion) shock.

We see that there are two singular points of Eq. (25), (0,0) and (0, -2). In Fig. 3 the singular point (0,0) is a saddle point and other point (0, -2) is a center. So, unlike in the flow of an ordinary gas where either a single saddle point singularity or a center is observed, we notice here the occurrence of both types of singularities. A steady flow can attain a sonic state either when it meets the line $\bar{\rho}_0 = 0$ or the line $\bar{\rho}_0 = -2$. The appearance of the second sonic state can be attributed to the change in the sign of the fundamental derivative $\bar{\Gamma}$ across the line $\bar{\rho} = -1$. The multivalued parts of the integral curves in the phase plane cannot represent a real flow and hence should be replaced by shock discontinuities. The steady shocks must satisfy the Rankine–Hugoniot condition

$$\bar{\rho}_{0l} + \bar{\rho}_{0r} + \tfrac{1}{3}(\bar{\rho}_{0r}^2 + \bar{\rho}_{0r}\bar{\rho}_{0l} + \bar{\rho}_{0l}^2) = 0 \tag{36}$$

and the stability condition (35).

We shall use the following definitions in classifying the different discontinuities. If for a discontinuity both the inequalities in Eq. (35) $[(u-a)_l > 0 > (u-a)]$ are satisfied, then the discontinuity is a shock. If for a discontinuity the left or right equality in Eq. (35) holds $[(u-a)_l = 0 > (u-a)_r$ or $(u-a)_l > 0 = (u-a)_r]$ then the discontinuity is a left sonic shock or a right sonic shock. If both the equality signs hold $[(u-a)_l = 0 = (u-a)_r]$, then the discontinuity is a double sonic shock.[11]

Equation (36) represents an ellipse in the $(\bar{\rho}_{0l}, \bar{\rho}_{0r})$ plane as shown in Fig. 4. From the figure we see that steady shocks or sonic shocks of either kind are possible only if $-3 \leqslant \bar{\rho}_{0l} \leqslant 1$ and $-3 \leqslant \bar{\rho}_{0r} \leqslant 1$. The condition (35) shows that the value of $\bar{\rho}_{0l}$ must satisfy $-2 \leqslant \bar{\rho}_{0l} \leqslant 0$ and for each such value of $\bar{\rho}_{0l}$, there are two values of $\bar{\rho}_{0r}$, one in $-3 \leqslant \bar{\rho}_{0r} \leqslant -2$ and the other in $0 \leqslant \bar{\rho}_{0r} \leqslant 1$. Hence for a given state $\bar{\rho}_{0l}$, there is a nonuniqueness in the value of $\bar{\rho}_{0r}$ for a shock or a sonic shock of either kind. In order to resolve the nonuniqueness in the determination of the state to the right of a discontinuity, we need to know the boundary conditions at the exit and entrance of the nozzle. We consult Fig. 3, where the integral curves are represented.

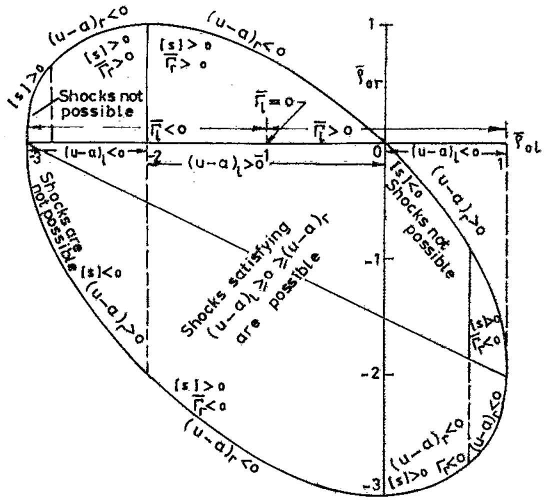

FIG. 4. An ellipse connecting all the Rankine–Hugoniot points and indicates the possible positions of the entropy increasing shocks for case III (a).

It is interesting to note (see also Fig. 3) that only a compression sonic shock connects an upstream state in $\bar{\Gamma} < 0$ to a downstream state in $\bar{\Gamma} > 0$ and only an expansion sonic shock connects an upstream state with $\bar{\Gamma} > 0$ to a downstream state with $\bar{\Gamma} < 0$. It is also seen that only a compression sonic shock or an expansion sonic shock is possible, if the fluid remains in the same state with either $\bar{\Gamma} > 0$ or $\bar{\Gamma} < 0$.

Consider a state upstream represented by a point L_3 on an integral curve along which the flow continues until its slope becomes infinite at the point O_3. After O_3 the flow cannot be continued continuously, since the integral curve folds itself, representing a multivalued solution. For discontinuity, $\bar{\rho}_{0l}$ must satisfy $-2 \leqslant \bar{\rho}_{0l} \leqslant 0$ and hence the only possible value of $\bar{\rho}_{0l}$ is zero at O_3. For $\bar{\rho}_{0l} = 0$, the possible values of ρ_{0r} are 0 and -3, of which 0 must be omitted. Thus the flow jumps to S_3 (with the same value of $\bar{x}$) through a left sonic shock $[(u-a)_l = 0]$. It is important to note that S_3 lies on the same integral curve, since Eq. (36) demands that the value of C in Eq. (31) should be the same. From S_3 the flow is continuous up to S_4, then jumps to S_5 through a left sonic shock and reaches the exit of the nozzle as a subsonic flow again.

Thus we note that given a state upstream say at L_3, the flow through the nozzle is uniquely determined. We may note here that across the discontinuity O_3S_3 the density of the fluid decreases discontinuously while across S_4S_5 the density increases discontinuously. It is interesting to note that $\bar{\Gamma}$ changes its sign from positive to negative across O_3S_3 and negative to positive across S_4S_5. Analogous to this case, Cramer and Kluwick[10] noticed the change of the sign of $\overline{\Gamma}$ across a shock wave. In this flow the fluid speed remains subsonic both at the entrance and the exit of the nozzle.

Consider another flow starting at L_2 in the phase plane, a point near the entrance, such that the integral curve passes through O_2. The flow starts with a subsonic velocity and is accelerated to the sonic speed at O_2. From O_2 there are many possibilities: in the first possibility it has a continuous sonic transition and continues to be supersonic up to β_2 finally reaching R_2 through the jump from β_2 to β_5; in the second one the flow may jump through an expansion sonic shock from O_2 to β_3, then continues as a subsonic flow till β_2, and finally reaches the exit of the nozzle at R_2 through a compression sonic shock $\beta_2\beta_5$. In the other possibilities the flow continues with a continuous sonic transition at O_2 but does not follow the curve up to the second sonic point β_2, instead it jumps from an intermediate state at S_1 to that at S_2 and finally reaches the exit at R_2. The jump from S_1 to S_2 is not a sonic shock. As we remarked in case I(a) there is a nonuniqueness in the position of the compression shock (S_1S_2) when we use the approximate Equation (25) in the neighborhood of the throat. Had we used the original equations, we might have fixed the position of the shock uniquely for the given values of the density (or pressure) at the exit and the entrance as in the case of a polytropic gas.[23]

From the phase plane in Fig. 3, we see that infinity of purely subsonic flows are possible. An example of such a flow is $L_1O_1R_1$. In the case of a polytropic gas through a nozzle with minimum area at the throat, there exist four distinct continuous flows that extend till the exit of the nozzle.[24] Analogous to these we find near the saddle point singularity O_2 four distinct flows: $L_2O_2R_2$, $L_2O_2\beta_2$, $\beta_1O_2R_2$, and $\beta_1O_2\beta_2$. Here, $L_2O_2R_2$ remains subsonic, except at the sonic point O_2. Along $L_2O_2\beta_2$, the flow changes from subsonic to supersonic at O_2, but this flow cannot be continued continuously till the exit of the nozzle as discussed above. $\beta_1O_2R_2$ is a flow that starts at a finite distance in the nozzle with a supersonic velocity, takes continuous sonic transition at O_2 and reaches the exit of the nozzle as a subsonic flow. Here $\beta_1O_2\beta_2$ is a purely subsonic flow that starts at a finite distance in the nozzle and ends at a finite distance as shown in the Fig. 3. We note that though the value of the fundamental derivative $\overline{\Gamma}$ changes from point to point in the fluid, when $\overline{\Gamma}$ is positive everywhere, the fluid accelerates in the diverging part of the nozzle, and decelerates in the converging part of the nozzle, as in the case of a polytropic gas.

The family of integral curves with the parameter value $C<0$ behave in a peculiar way. For each C there are two distinct branches of the integral curve: one remains completely in the subsonic region $(\bar{\rho}>0)$ representing a continuous flow such as $L_6O_6R_6$ and the second is a closed loop around the second sonic point $(0,-2)$. The closed loop, being multivalued, cannot represent a flow. However, parts of the upper or lower portions of the loop represent continuous flows originating and terminating at finite distances, and can be joined to the upper branch of the corresponding integral curves (with the same value of C) through shocks or sonic discontinuities; $\beta_6\beta_7\beta_{10}\beta_{11}R_6$ represents a flow with a compression shock, $\beta_6\beta_{12}\beta_8\beta_9R_6$ represents a flow with a left sonic shock, etc.

Let us consider a flow starting at L_2 with subsonic velocity and, let the exit of the nozzle be at R_2. There exists several possible ways of connecting the two boundary conditions: (i) $L_2O_2\beta_3\beta_2\beta_5R_2$, (ii) $L_2O_2\beta_2\beta_5R_2$, and (iii) $L_2O_2S_1S_2R_2$. We have already described these flows, and a description of this is found in the caption of the figure. This result shows the existence of multiple steady states of transonic flows in the nozzle, as observed by Embid *et al.*[22] for a perfect gas model. Also, this study clearly explains the existence of expansion shocks in the transonic region.

B. Case (b): $k<0$, $\hat{\Gamma}<0$, and $\Lambda<0$

Since the product $k\hat{\Gamma}$ is positive, the phase plane is the same as in case (a). We only note here that as $\hat{x}$ is related to $\bar{x}$ by $\bar{x}=(\Lambda/\hat{\Gamma}^2)\hat{x}$, the exit of the nozzle (which is on the left) corresponds to $\bar{x}=-\infty$ in the figure and the entry is at $\bar{x}=+\infty$ in the figure. Further, due to relation (33), the supersonic regions are $\bar{\rho}_0>0$ and $-\infty<\bar{\rho}_0<-2$ and the subsonic region is the strip $-2<\bar{\rho}_0<0$. From Fig. 4, by making the corresponding changes in the signs, we see that steady shocks are possible only if either $-3\leqslant\bar{\rho}_{0l}\leqslant-2$ in which case, $-2\leqslant\bar{\rho}_{0r}\leqslant0$ or $0\leqslant\bar{\rho}_{0l}\leqslant1$, which requires $-2\leqslant\bar{\rho}_{0r}\leqslant0$. These shocks satisfy the stability condition (35).

We can describe the individual flows as in case (a). We note here that all shocks in the steady flows are the right sonic shocks $((V-a)_r=0)$. For example $R_3S_5S_4S_3O_3L_3$ contains the sonic shocks S_5S_4 (expansion) and S_3O_3 (compression). This analysis clearly shows the existence of continuous and discontinuous steady flows in a nozzle with maximum throat area.

C. Case (c): $k>0$, $\hat{\Gamma}<0$, and $\Lambda>0$

In this case we use $\bar{\rho}=-(\Lambda/\hat{\Gamma})\hat{\rho}$, $\bar{x}=(\Lambda/\hat{\Gamma}^2)\hat{x}$, and $\bar{t}=\hat{t}$. Then the equations corresponding to the Eqs. (31), (33), and (34) become

$$k\hat{\Gamma}\bar{x}^2=\tfrac{1}{3}\bar{\rho}_0^3-\bar{\rho}_0^2+C, \tag{37}$$

$$V_0-a_0=-\epsilon^2(\hat{\Gamma}/\Lambda)^2\Lambda(-\bar{\rho}_0+\tfrac{1}{2}\bar{\rho}_0^2), \tag{38}$$

and

$$\overline{\Gamma}=-\epsilon\overline{\Gamma}(1-\bar{\rho}_0)+O(\epsilon^2). \tag{39}$$

The integral curves given by Eq. (37) are plotted in Fig. 5. The sonic points are (0,2) and (0,0). As in case (a) we can describe various steady flows. We mention a few interesting flows in the caption of the figure.

D. Case (d): $k<0$, $\hat{\Gamma}>0$, and $\Lambda<0$

In this case the phase plane is the same as that in the case (c), except that the direction of the flow of the fluid is from right to left. We mention the corresponding changes of the supersonic and subsonic regions in the figure. We describe a few interesting flows in the caption of the figure.

We have four more possible combinations of the signs of the parameters k, $\hat{\Gamma}$, and Λ, leading to different types of flows. They are (e) $k<0$, $\hat{\Gamma}<0$, and $\Lambda>0$; (f) $k>0$, $\hat{\Gamma}>0$,

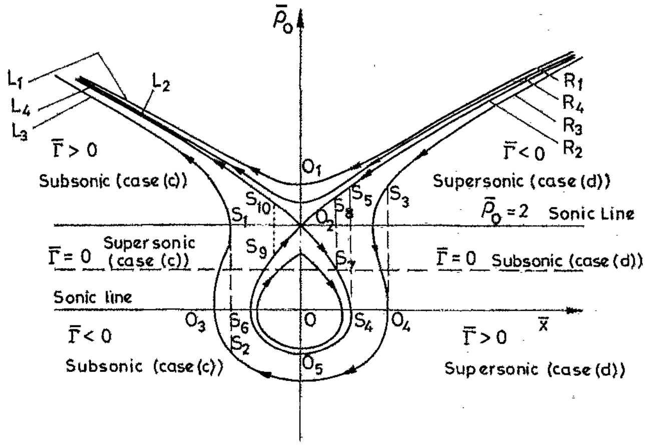

FIG. 5. Case III (c): $k>0, \hat{\Gamma}<0$, and $\Lambda>0$. $L_3S_1S_2O_4S_3R_3$ and $L_2O_2O_5S_4S_5R_2$ are flows with two sonic discontinuities one at the converging part (at the throat) and another at the diverging section of the nozzle. $L_2O_2S_4S_5R_2$ is a flow with a single compression sonic discontinuity. $S_6S_9S_{10}O_2S_4S_5R_2$ and $S_6S_9S_{10}O_2S_7S_8R_2$ are flows with a shock and a sonic discontinuity or two shocks, respectively. Case III (d): $k<0, \hat{\Gamma}>0$ and $\Lambda<0$. The steady flow is from right to left and the perturbations propagate in the opposite directions indicated by the arrow directions. $R_3S_3O_4S_2S_1L_3$, $R_2S_5S_4O_2L_2$ are flows with two right sonic shocks and $R_2O_2S_6$ is a transonic flow terminating before the exit of the nozzle. There are no shock solutions and the closed loops do not represent any flow.

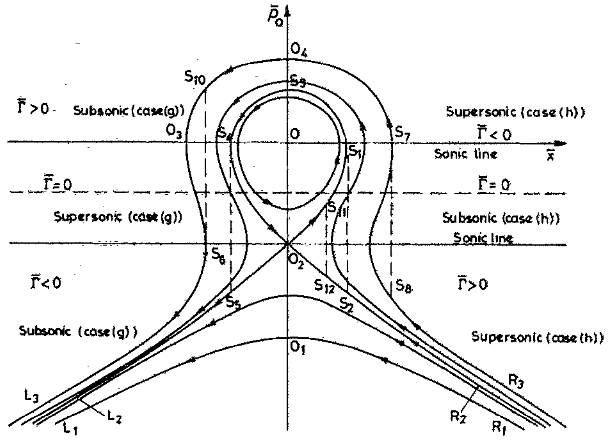

FIG. 7. Case III (g) $k<0, \hat{\Gamma}>0$ and $\Lambda>0$. $L_2O_2S_3S_1S_2R_2$ and $L_3S_6S_{10}O_4S_7S_8R_9$ are flows with left sonic shocks. S_6S_{10}, O_2S_3 are compression discontinuities and S_1S_2, S_7S_8 are expansion discontinuities. $L_2O_2S_{11}S_{12}R_2$, contains $S_{11}S_{12}$, an expansion shock. Parts of the closed loops can also represent discontinuous solutions as shown. Case III (h) $k>0$, $\hat{\Gamma}<0$, and $\Lambda<0$. The steady flow is from right to left along the integral curves. $R_2S_2S_1S_3O_2L_2$ and $R_3S_8S_7O_4S_{10}S_6L_3$ are flows with a right sonic shock. No shock solutions are possible.

and $\Lambda<0$; (g) $k<0, \hat{\Gamma}>0$, and $\Lambda>0$; and (h) $k>0, \hat{\Gamma}<0$, and $\Lambda<0$. The phase planes for the cases (e) and (f) are depicted in Fig. 6 and those of cases (g) and (h) are in Fig. 7. For the economy of space, we describe the flows in the caption of the figures.

The above study clearly shows the existence of transonic flows in a nozzle irrespective of the nozzle having a maximum area or a minimum area at the throat. The above discussion of all possible flows also shows that a flow starting with a supersonic (subsonic) velocity at the entrance of the nozzle reaches the exit of the nozzle only with a supersonic (subsonic) velocity, either continuously or jumping through one or more discontinuities. This result is in contrast with the results for an ordinary gas.

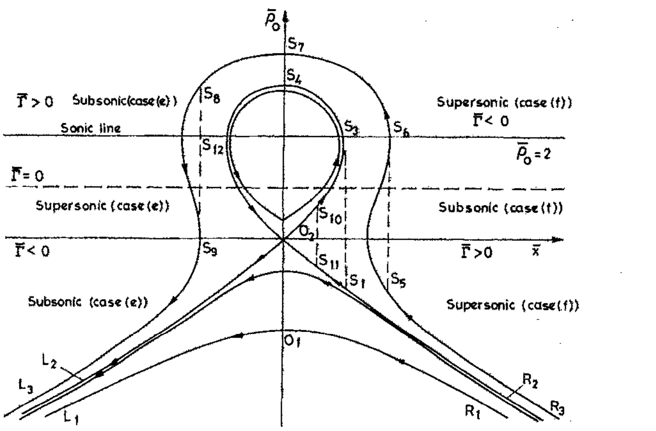

FIG. 6. Case III (e) $k<0, \hat{\Gamma}<0$ and $\Lambda>0$. $L_2O_2S_4S_3S_1R_2$ and $L_3S_9S_8S_7S_6S_5R_3$ are flows with two left sonic shocks; S_9S_8, O_2S_4 are compressions and S_6S_5, S_3S_1 are rarefactions. $L_2O_2S_{10}S_{11}R_2$ and $L_2O_2S_3S_1R_2$ are flows, respectively, with a single expansion shock and an expansion sonic shock. Case III (f) $k>0$, $\hat{\Gamma}>0$, and $\Lambda<0$. The steady flow is from right to left and the perturbations propagate in the opposite direction indicated by the arrows. There are no shock solutions possible and the closed loops do not represent any flow. $R_2S_1S_3S_4O_2L_2$ and $R_3S_5S_6S_7S_8S_9L_3$ are flows with right sonic shock, S_1S_3, S_5S_6 are compression discontinuities and S_4O_2, S_8S_9 are expansion discontinuities.

IV. LOCAL NONLINEAR STABILITY OF STEADY FLOWS

We can study nonlinear propagation, growth, and decay of weak pulses on the steady flows with the help of the Eq. (23). The characteristic ordinary differential equations (24) of the partial differential equation (23), give the same rate of change $d\bar{p}_0/d\bar{x}$ as that given by the steady equation (25) and hence evolution of perturbations can also be studied in the phase plane of Eq. (25). Since we are considering perturbations bounded in space, at any instant, a perturbation of our steady solution can be represented by a closed curve as shown in Fig. 8. In a perturbation, the space rate of change of $\bar{\rho}$ as we move with the wave velocity $-(\bar{\rho}+\frac{1}{2}\bar{\rho}^2)$ is $K\bar{x}/(\bar{\rho}+\frac{1}{2}\bar{\rho}^2)$, which is also the space rate of change of $\bar{\rho}$ as we move along the integral curves of the characteristic equations. Therefore, during the propagation, different points of the boundary curve of the perturbation will move along the integral curves of Eq. (25).

Let s be the area bounded by an arbitrary closed curve in the $(\bar{x},\bar{\rho})$ plane whose points move in accordance with Eq. (24). As the divergence of the vector field given by the right-hand side of Eq. (24) is zero, i.e.,

$$\frac{\partial}{\partial\bar{x}}\left(\frac{d\bar{x}}{d\bar{t}}\right)+\frac{\partial}{\partial\bar{t}}\left(\frac{d\bar{\rho}}{d\bar{t}}\right)=0=\frac{1}{s}\frac{ds}{d\bar{t}}, \tag{40}$$

it follows that during its motion the total area s remains constant and is equal to its initial value s_0. Thus we get the

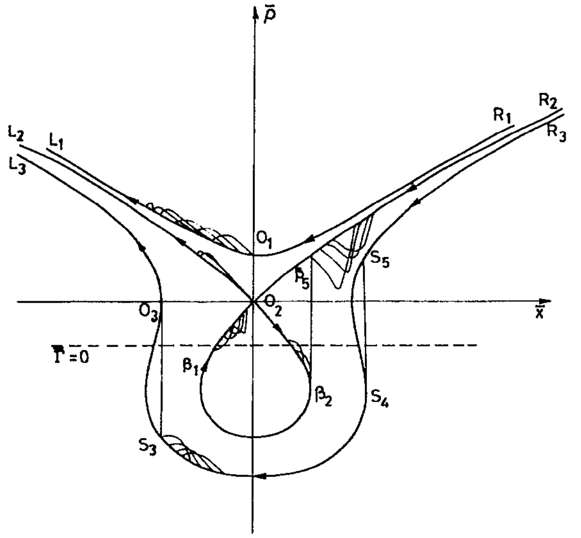

FIG. 8. Stability of the fluid flow through a nozzle with minimum throat area. History of the perturbation of the steady flows for case III (a). This clearly demonstrates the existence of the shocks and the local stability of the flows with shocks.

important pulse area rule[21] that the area occupied by a disturbance in the $(\bar{x},\bar{\rho})$ plane remains constant as the disturbance propagates. Following Kantrowitz[21] we can easily see that the pulse area is conserved even if a weak shock appears in the pulse.[14,25] Using the constancy of the pulse area, we can get a qualitative idea of the history of the pulse in the phase plane. But since Eq. (23) is nonlinear, in order to get the complete history of the pulses, we have to resort to the numerical solution of Eq. (23).

As pointed out in the Introduction, Eq. (23) can be derived from a hyberbolic conservation law,

$$\frac{\partial\bar{\rho}}{\partial\bar{t}}+\frac{\partial}{\partial\bar{x}}\left[-\left(\frac{1}{2}\bar{\rho}^2+\frac{1}{6}\bar{\rho}^3\right)\right]=-\frac{k\hat{\Gamma}}{2}\bar{x}$$

with a nonconvex flux function having an inhomogeneous source term. We use the singularity separating method,[20] a numerical scheme especially designed for such scaler conservation laws with a modification to incorporate the source term on the right.

Equation (23) can be rewritten, by using the steady solution Eq. (25), as

$$\frac{\partial\bar{\rho}_1}{\partial\bar{t}}-\left((\bar{\rho}_0+\bar{\rho}_1)+\frac{1}{2}(\bar{\rho}_0+\bar{\rho}_1)^2\right)\frac{\partial\bar{\rho}_1}{\partial\bar{x}}=\left(\bar{\rho}_1\bar{\rho}_0+\bar{\rho}_1+\frac{1}{2}\bar{\rho}_1^2\right)\frac{d\bar{\rho}_0}{d\bar{x}}. \quad (41)$$

Following the pulse area method of Kantrowitz,[21] we can derive an equation for the slope $\epsilon=\partial\bar{\rho}_1/\partial\bar{x}$ of the boundary of the pulse along an integral curve of Eq. (25). In the characteristic form this equation is

$$\frac{d\epsilon}{d\bar{t}}=\bar{\rho}_1\left[\left(\frac{d\bar{\rho}_0}{d\bar{x}}\right)^2+\left(\frac{d^2\bar{\rho}_0}{d\bar{x}^2}\right)\left(1+\bar{\rho}_0+\frac{1}{2}\bar{\rho}_1\right)+2\epsilon(1+\bar{\rho}_0+\bar{\rho}_1)\left(\frac{d\bar{\rho}_0}{d\bar{x}}\right)+\epsilon^2(1+\bar{\rho}_0+\bar{\rho}_1)\right] \quad (42)$$

along the characteristics give by

$$\frac{d\bar{x}}{d\bar{t}}=-\left((\bar{\rho}_0+\bar{\rho}_1)+\frac{1}{2}(\bar{\rho}_0+\bar{\rho}_1)^2\right).$$

We select a steady flow given by $\bar{\rho}_0$ and impose a perturbation $\bar{\rho}_1(\bar{x},0)$ on it. We study its evolution by numerically integrating Eqs. (41). The growth of the perturbation with time means that the steady-state solution is unstable. On the other hand, the decay of the solution does not imply stability, since the cause of instability need not be connected with the behavior of the flow near the critical point. Nevertheless we shall call the solution to be stable in this case. The time of the formation of a discontinuity and its subsequent position can be obtained by solving[21] Eq. (42) and Eq. (26), respectively. The discontinuity must satisfy the stability conditions

$$\left.\frac{d\bar{x}}{d\bar{t}}\right|_l>\frac{d\bar{x}_S}{d\bar{t}}>\left.\frac{d\bar{x}}{d\bar{t}}\right|_r. \quad (43)$$

As we have seen in the case of the steady solution, the condition (43) is sufficient to guarantee the entropy increase across shocks.

We shall now study the evolution of the perturbations on each of the steady flows discussed in Sec. III. The direction of the arrow on a steady flow indicates the direction of propagation of a perturbation.

Case I(a): The behavior of these steady flows are well known since it corresponds to an ordinary gas. Kantrowitz[21] brought out many interesting features of these flows by his pulse area method.[14] Recently, Liu[26] has rigorously proved that a flow with a standing shock in the diverging section of the nozzle is stable while the steady shock in the converging section is unstable to perturbations. Successive positions of the pulse on a steady flow have been shown by dotted lines in Fig. 1(a).

Case I(b): This case corresponds to the flow of a negative $\bar{\Gamma}$ fluid through a nozzle of maximum throat area. Successive positions of a few pulses have been shown by dotted lines in Fig. 1(b). We see that the continuous acceleration of the fluid through the speed of sound, is neutrally stable to the compression and expansion pulses, since these pulses finally attain a stationary triangular shape at the throat. But for an ordinary fluid this flow is known to be stable, since the pulses move away from the throat without amplification. The continuous deceleration of the fluid through the speed of sound is stable, unlike the neutral stability of such flows in an ordinary gas. We notice that the flows with a standing rarefaction shock in the converging part of the nozzle is stable.

Case II: We get an equation for the unsteady perturbation in the form

$$\frac{\partial\hat{\rho}_1}{\partial\hat{t}}-\left(\frac{1}{2}\Lambda\hat{\rho}_1^2\right)\frac{\partial\hat{\rho}_1}{\partial\hat{x}}=-\frac{k\hat{x}}{2}. \quad (44)$$

Since this equation is also nonlinear, we solve Eq. (44) numerically. Irrespective of the fluid being in a supersonic $(\Lambda<0)$ or subsonic $(\Lambda>0)$ state, the amplitude of the perturbation created at any part of the flow decays while the pulse spreads over larger distance. Hence all the flows are stable, irrespective of the shape of the nozzle ($k<0$ or $k>0$, see Fig. 2).

Case III (a): A large amount of computation for different types of perturbations on different parts of the steady flows was done to decide the local stability of the flows. We sketch the successive positions of a few perturbations in Figs. 8, 9, 10, and 11.

$L_1O_1R_1$: This subsonic flow is stable to all types of perturbations (expansion, compression, or with both phases). Any perturbation to the flow moves upstream and disappears in the flow field (see Figs. 8 and 9).

$L_2O_2\beta_2\beta_5R_2$: The continuously accelerating transonic flow $L_2O_2\beta_2$ through the speed of sound is stable to the perturbations. A perturbation either in L_2O_2 or in $O_2\beta_2$ moves away from O_2 with its amplitude decaying, signifying the stability of the sonic state in the flow. Further, a perturbation of $O_2\beta_2$ catches up with the standing right sonic shock $\beta_2\beta_5$, resulting in a complicated interaction; the later history of this pulse is not known from our computations. Also any perturbation of β_5R_2 catches up with the sonic shock $\beta_2\beta_5$. Hence the final stability of the flow can be determined only after studying the stability of the right sonic shock (see Figs. 8 and 9). We note that an expansion or a compression pulse on β_5R_2 attains a triangular form with the tail or the head of the wave becoming a shock, respectively. A pulse with expansion and compression phases becomes an N wave, either with a single shock connecting both phases if the pulse is headed by an expansion or with leading and trailing shocks if headed by a compression phase.

$L_3O_3S_3S_4S_5R_3$: From Fig. 8, we see that the complete stability of the flow will be known only when the complicated interactions are studied in detail, since a perturbation of R_3S_5 and S_3S_4 move towards the sonic shocks S_4S_5 and O_3S_3, respectively.

$\beta_1O_2R_2$: As it was noted earlier, this flow is the only continuous flow, starting at a finite distance in the nozzle, and decelerating through the speed of sound. A compression wave in the supersonic part β_1O_2, moves towards the sonic point O_2 and gets trapped there with the leading part establishing the flow L_2O_2 and the trailing end becoming a compression shock. An expansion wave also attains a triangular form with its leading edge becoming a shock (compression). It is interesting to note that even if a part of the wave has $\bar{\Gamma} < 0$, this does not show any significant effect on the propagation of the perturbation. A wave containing both expansion and compression phases shows an interesting behavior (Figs. 10 and 11). If the wave is headed by a compression phase, during propagation, both the compression and the expansion phases tend to cancel each other. The excess of the compression phase attains a triangular form, establishing

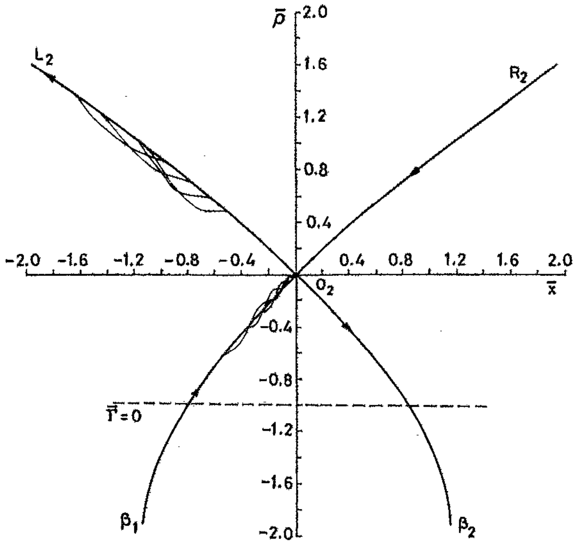

FIG. 10. The flow $\beta_1O_2R_2$ is neutrally stable to perturbations. In a perturbation with both positive and negative values, the excess of the positive perturbation attains a triangular form with a compression shock at the trailing end of the wave. The amplitude of a negative perturbation of L_2O_2 decays.

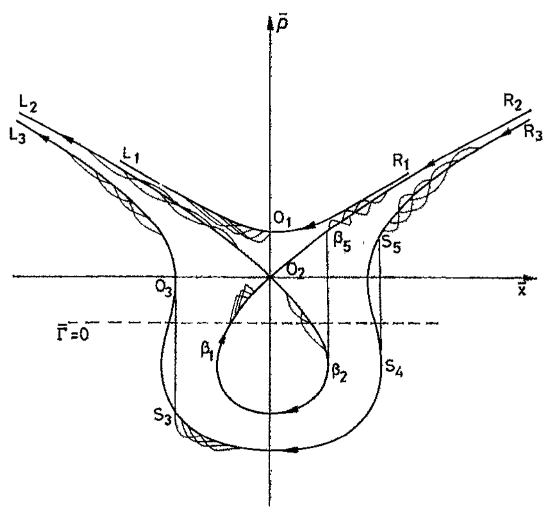

FIG. 9. History of different types of perturbations of the various possible steady flows in a nozzle with a minimum area at the throat.

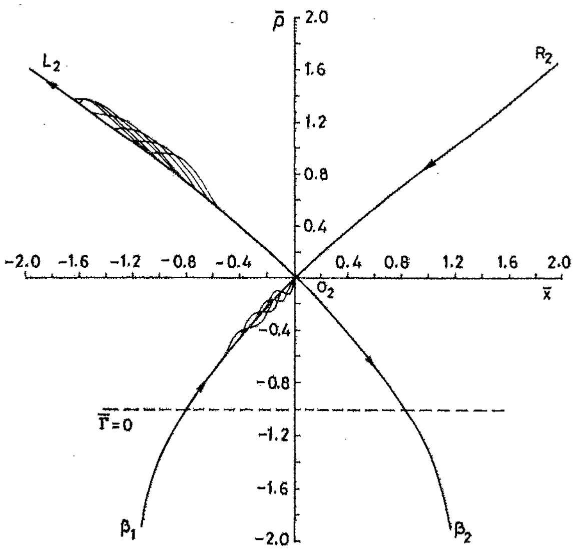

FIG. 11. A perturbation with positive and negative values, becomes an N wave near the sonic point O_2. A positive perturbation of L_2O_2 moves away from the throat and its amplitude decays.

partly (up to O_2) the flow L_2O_2 ahead of the shock. A pulse headed by an expansion phase results in an N wave with shocks at the head and the tail of the wave and the pulse gets trapped at O_2 where it decays. Hence we observe that as in the case of an ordinary fluid, the continuous deceleration through speed of sound is neutrally stable to the perturbations.[15]

$L_2O_2S_1S_2R_2$: Any perturbation of L_2O_2 moves away from the sonic point and decays. A perturbation of O_2S_1 catches up with the shock and also perturbations of S_2R_2 move towards S_1S_2. Hence to know the further history of the flow, we need to study the nonlinear interactions. However, since the area of the perturbations remains constant (also $\hat{\Gamma} > 0$ and the nozzle has minimum throat area), we expect this shock solution to be stable.

Case III (b): The history of the pulses imposed on different parts of the flows has been plotted in Figs. 12, 13(a), and 13(b). We note that the continuous supersonic flow $R_1O_1L_1$ is stable to any perturbation. The flow $\beta_2O_2L_2$, representing the continuous acceleration of the fluid through the speed of sound, is neutrally stable. From the above figures, it is clear that any perturbation of β_2O_2 moves toward O_2, with the expansion phases of the wave becoming shocks and the pulse finally gets trapped at O_2. Also a perturbation of O_2L_2 moves towards O_2. A pulse with both phases becomes an N wave and its description can be made as in case III (a). Hence we notice that this continuous acceleration of the fluid through the speed of sound is neutrally stable.

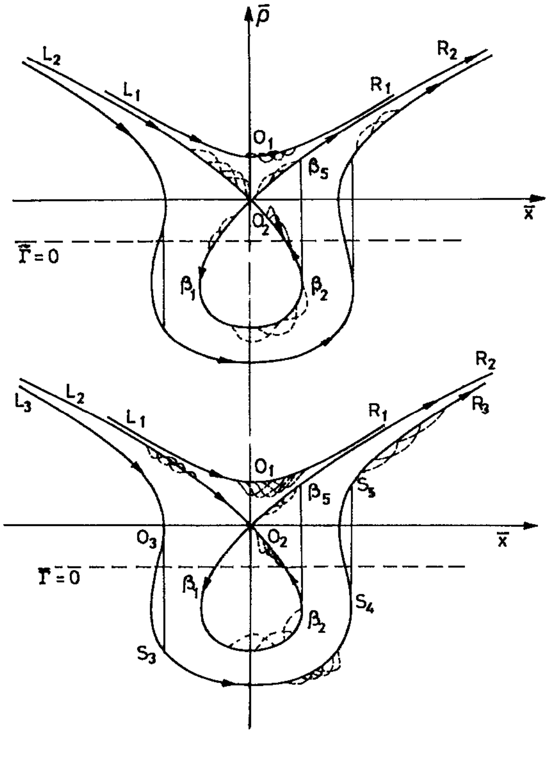

FIG. 12. Stability of the fluid flow through a nozzle with maximum throat area. History of the perturbations of possible steady flows for case III (b). The continuous acceleration of the fluid through the speed of sound is neutrally stable.

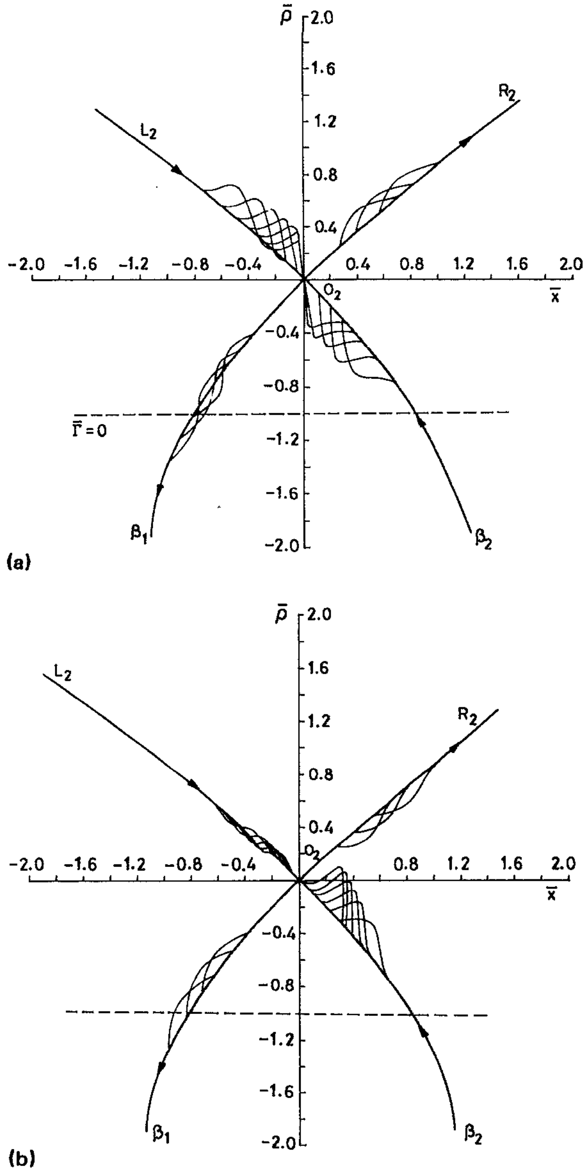

FIG. 13. (a) Both positive and negative perturbations of $\beta_2O_2L_2$ attain triangular form. Perturbations of $O_2\beta_1$ move away from the sonic point and its amplitude decays. (b) A positive perturbation on $O_2\beta_2$ attains a triangular form and catches up with main flow O_2R_2.

The flow $\beta_1O_2R_1$ representing the deceleration of the fluid through the speed of sound is stable, since the perturbations move away from the sonic point, disappearing from the flow field. We can also describe the stability of the flows with the sonic shocks as in case III (a).

Finally, we notice that even though the negative values of $\bar{\Gamma}$ changes from point to point in the fluid, the stability of the flow is not very different from that of a fluid with constant negative $\bar{\Gamma}$. We can easily compare the properties of the flow in this case to that in case I (b). Since the negative $\bar{\Gamma}$ behavior dominates over the positive $\bar{\Gamma}$ behavior, any perturbation with both positive and negative $\bar{\Gamma}$ does not show a behavior significantly different from that of a pulse with negative $\bar{\Gamma}$.

We also carried out numerical computations for the evolution of the perturbations on the various steady flows shown in Figs. 5,6, and 7. We do not go into any further detailed description. However, we only point out that even though $\overline{\Gamma}$ changes sign, the stability results are similar to those with constant $\overline{\Gamma}$ with positive or negative sign according to whether the saddle point is in the region with positive or negative sign of $\overline{\Gamma}$.

V. CONCLUSIONS

The steady transonic flow of a real fluid through a nozzle and its local nonlinear stability to weak pulses has been examined. The specific heat of the fluid has been taken to be large enough to result in regions of both positive and negative nonlinearity, in the single phase of the fluid. We assume that the fundamental derivative $\overline{\Gamma}$ of the fluid vanishes at a state close to the sonic state and hence remains small at the sonic point. The behavior of the flow near the throat (maximum or minimum area), based on a local approximation near the sonic point has been studied for various possible choices of the signs of the fundamental derivative and its rate of change along the isentrope through the sonic point.

The behavior of the steady transonic flows is interestingly different from those of a polytropic gas with constant positive $\overline{\Gamma}$. Continuous transonic flows have been found to exist through nozzles of minimum as well as maximum throat area. Unlike in a fluid with the same sign of $\overline{\Gamma}$ (positive or negative), when $\overline{\Gamma}$ changes sign, there are two sonic points, one at the throat and another close to the throat of the nozzle. It is found that continuous transonic flows are possible only through one of the sonic points, at which $k\hat{\Gamma}$ remains positive. Sonic discontinuities (expansion or compression) are found to connect the upstream and the downstream states in regions with different signs of $\overline{\Gamma}$. Also, expansion (compression) shocks exist in a nozzle with maximum (minimum) throat area. Unlike in the case of a regular fluid or a fluid with $\overline{\Gamma} < 0$, if the flow starts with a subsonic (supersonic) velocity at the entrance of the nozzle, it can reach the exit only with subsonic (supersonic) velocity, except in the case when the exit is not too far away from the throat. This flow may be a continuous one or containing one or more sonic discontinuities or shocks. But there is a single continuous flow that may start (end) at a sufficiently short distance from the throat of the nozzle with minimum (maximum) throat area that results in a supersonic (or subsonic) velocity at the exit different from the initial subsonic (or supersonic) velocity at the entrance.

Computations of the evolution of various possible perturbations of the steady flows have been made to study the local stability of the flows for weak pulses. It is known for a regular fluid that the continuous acceleration of the fluid through the speed of sound is stable whereas the continuous deceleration of the fluid through the speed of sound is neutrally stable. For a flow with $\overline{\Gamma} < 0$ we find that the continuous acceleration is neutrally stable whereas the continuous deceleration is stable. The stability of the steady flows mainly depends on the particular combination of the parameters $\overline{\Gamma}$, Λ, and k. For a flow starting with positive values of $\overline{\Gamma}$ and with minimum throat area, we find that the nature of stability of a continuous flow near the saddle point singularity agrees with that of a regular fluid flow. Also for a flow through maximum throat area and with $\overline{\Gamma}$ negative, the stability of the continuous flows does not differ from that of a fluid flow with constant $\overline{\Gamma} < 0$. The various flows containing sonic discontinuities in both cases may be stable, but a precise statement about the stability can be made only if the nonlinear interactions of the waves with the sonic shocks are studied in detail. The perturbations quite often get trapped at the throat and attain a triangular shape with a shock either at the trailing end or at the leading end. Some of the perturbations with positive and negative area tend to become N waves.

ACKNOWLEDGMENT

The authors sincerely thank the referees whose valuable comments led to significant improvement of the paper.

[1] P. A. Thompson, Phys. Fluids **14**, 1843 (1971).
[2] P. A. Thompson and K. C. Lambrakis, J. Fluid Mech. **60**, 187 (1973).
[3] H. A. Bethe, Office Sci. Res. Dev. Rep. 545 (1942).
[4] Ya. B. Zeldovich and Yu. P. Raizer, *Physics of Shock Waves and High Temperature Hydrodynamic Phenomena* (Academic, New York, 1966), Vol. (1), pp. 63–99.
[5] A. A. Borisov, A. A. Borisov, S. S. Kutateladze, and V. E. Nakorykov, J. Fluid Mech. **126**, 53 (1983).
[6] S. S. Kutateladze, V. E. Nakoryakov, and A. A. Borisov, Annu. Rev. Fluid. Mech. **19**, 577 (1987).
[7] M. S. Cramer, Phys. Fluids A **1**, 1894 (1989).
[8] P. A. Thompson, G. C. Carofano, and Y. G. Kim, J. Fluid Mech. **166**, 57 (1986).
[9] S. Garrett, J. Acoust. Soc. Am. **69**, 137 (1981).
[10] M. S. Cramer and A. Kluwick, J. Fluid Mech. **142**, 9 (1984).
[11] M. S. Cramer, A. Kluwick, L. T. Watson, and W. Pelz, J. Fluid Mech. **169**, 323 (1985).
[12] M. S. Cramer and R. Sen, Phys. Fluids, **29**, 2181 (1986).
[13] M. S. Cramer, J. Fluid Mech. **199**, 281 (1989).
[14] A. G. Kulikovskii and F. A. Slobodkina, PMM J. Appl. Math. Mech. **31**, (4) 623 (1967).
[15] P. Prasad, J. Fluid Mech. **57**, 721 (1973).
[16] R. Ravindran, J. Fluid Mech. **95**, 465 (1979).
[17] D. P. Ballou, Trans. Am. Math. Soc. **152**, 441 (1970).
[18] C. M. Dafermos, Proc. R. Soc. Edinburgh **99** (A), 201 (1985).
[19] K. S. Cheng, J. Diff. Eqn. **61**, 79 (1985).
[20] X. H. Wu and Y. I. Zhu, Comput. Fluids **13**, 473 (1985).
[21] A. Kantrowitz, *Fundamentals of Gasdynamics*, edited by T. Emmons (Oxford U.P., London, 1958), pp. 350–415.
[22] P. Embid, J. Goodman and A. Majda, SIAM, J. Sci. Stat. Comput. **5**, 21 (1984).
[23] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves* (Springer-Verlag, New York, 1948).
[24] P. Prasad, Natl. Acad. Sci. India (P. L. Bhatnagar Commemoration Volume), 413 (1979).
[25] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon, New York, 1979).
[26] T. P. Liu, Commun. Math. Phys. **83**, 243 (1984).