Micromaser spectrum

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We calculate the spectrum of the micromaser for a wide range of pump parameters and propose a multiple microwave field method similar to the Ramsey fringe technique to measure this spectrum. Thus it becomes possible to investigate the phase diffusion in a fundamental system such as the micromaser.

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The micromaser [1] is a unique tool for the investigation of quantum aspects of the interaction of radiation and matter as well as subtleties of the quantum measurement process [2]. The most prominent examples are the measurement of the collapse and the revivals of the atomic population predicted by the Jaynes-Cummings model [3], the generation of nonclassical light with a sub-Poissonian photon statistics [4,5], and number state generation by state reduction [6] or trapping states [7].

In this paper we investigate another feature of this most fundamental maser system: here we present the micromaser spectrum. The decay of the expectation value of the electric field [8]

$$\langle E(t)\rangle \sim \sum_{n=0}^{\infty}(n+1)^{1/2}\rho_{n,n+1}(t) \tag{1}$$

governs this quantity. Hence the micromaser spectrum is different from the above-mentioned effects in two regards: (i) it envolves the off-diagonal elements $\rho_{n,n+1}\equiv\rho_n^{(1)}$ of the radiation-density matrix, rather than the photon statistics, that is, the diagonal elements $\rho_n^{(0)}$ and (ii) it requires their full time dependence rather than their steady-state values.

We introduce an analytical approach to calculate the linewidth D of the micromaser. Two novel features, quite distinct from the familiar Schawlow-Townes linewidth [8], come to light in Fig. 1 (i) the trapping states [7] impress sharp resonances onto D as a function of the pump parameter θ. (ii) For large values of θ the linewidth D decreases, and can even oscillate, a phenomenon alien to the monotonic dependence of the Schawlow-Townes linewidth. We compare these analytical, but only quasirigorous results to an exact numerical treatment [9] of the relevant density-matrix equation [5].

We conclude this letter by proposing a modified Ramsey-type scheme [10–12], shown in the inset of Fig. 2, to observe this linewidth. To measure the decay of the off-diagonal elements $\rho_n^{(1)}$, starting from a well-defined initial value $\rho_n^{(1)}(t=0)$ is our strategy. But how do we prepare $\rho_n^{(1)}$? Atoms prepared in a coherent superposition via a microwave field before entering the micromaser cavity indeed fulfill this task [11]. This in turn leads to a fixed phase of the radiation field in the micromaser cavity [13]. The atoms leave the cavity and move through another microwave field which turns the coherent superposition of their atomic states into a simple population in upper or lower maser level. The decay of the so-measured atomic population beginning after the switch-off of the first field corresponds to the decay of the electric field of the micromaser due to phase diffusion, as per the discussion at the end of this article.

We begin with the equation of motion for the density matrix $\rho_{n,n+k}\equiv\rho_n^{(k)}$ of the maser field, that is, with Eq. (8) of Ref. 5

$$\frac{d}{dt}\rho_n^{(k)}(t)=-r[1-\cos(g\tau\sqrt{n+1})\cos(g\tau\sqrt{n+1+k})]\rho_n^{(k)}-\gamma(n_b+1)\left[n+\frac{k}{2}\right]\rho_n^{(k)}-\gamma n_b\left[n+1+\frac{k}{2}\right]\rho_n^{(k)}$$
$$+r\sin(g\tau\sqrt{n})\sin(g\tau\sqrt{n+k})\rho_{n-1}^{(k)}+\gamma n_b\sqrt{n(n+k)}\rho_{n-1}^{(k)}+\gamma(n_b+1)\sqrt{(n+1)(n+1+k)}\rho_{n+1}^{(k)}\ . \tag{2}$$

Here r is the injection rate of the atoms whose time of flight through and coupling strength with the cavity field are τ and g, respectively. The cavity decay rate we denote by γ and n_b is a mean thermal photon number.

The quantum theory of the laser [8] calculates the off-diagonal elements based upon the ansatz [8,14]

$$\rho_n^{(k)}=F(k)(\rho_n^{(0)}\rho_{n+k}^{(0)})^{1/2}\exp[-\mu(k)t]\ ,$$

where $F(k)$ is some arbitrary function of k. This ansatz assumes that the off-diagonal elements $\rho_n^{(k)}$ resemble the diagonal elements $\rho_n^{(0)}$. In the limit of a large photon number n and small number of thermal photons this is indeed an excellent approximation. However, the micromaser does not meet these conditions.

In this paper we therefore pursue a different approach. Two strategies offer themselves. (i) The equation of motion for $\rho_n^{(k)}$, Eq. (2) couples only the nearest neighbors of $\rho_n^{(k)}$, that is, it couples $\rho_n^{(k)}$ to $\rho_{n+1}^{(k)}$ and $\rho_{n-1}^{(k)}$, but does not introduce a coupling in the k index. We therefore face a three-term differential recurrence relation of complicated but time-independent coefficients. A Laplace transformation casts this equation into an algebraic equation and a scalar continued fraction [15] treatment provides immediately the eigenvalues. This approach we discuss in detail elsewhere [9]. (ii) The second approach, an approximate but analytical one, derives the lowest eigenvalue [16] from a detailed balance condition.

In the present paper we focus on the second approach. When we add and subtract the appropriate terms, Eq. (2) reads

$$\frac{d}{dt}\rho_n^{(k)} = -\tfrac{1}{2}\mu_n\rho_n^{(k)} + c_{n-1}\rho_{n-1}^{(k)} - d_n\rho_n^{(k)} - (c_n\rho_n^{(k)} - d_{n+1}\rho_{n+1}^{(k)}) , \tag{3}$$

where

$$\tfrac{1}{2}\mu_n(k) = 2r\sin^2\left[\frac{g\tau}{2}(\sqrt{n+1+k} - \sqrt{n+1})\right] + \gamma(n_b+1)\left[n + \frac{k}{2} - \sqrt{n(n+k)}\right] + \gamma n_b\left[n+1+\frac{k}{2} - \sqrt{(n+1)(n+1+k)}\right] , \tag{4}$$

together with

$$c_{n-1}(k) \equiv r\sin(g\tau\sqrt{n})\sin(g\tau\sqrt{n+k}) + \gamma n_b\sqrt{n(n+k)} ,$$

and

$$d_n(k) \equiv \gamma(n_b+1)\sqrt{n(n+k)} .$$

The ansatz

$$\rho_n^{(k)} = e^{-D_n(t)}\rho_n^{(k)}(0) = e^{-D_n(t)}\prod_{j=1}^{n}\frac{c_{j-1}}{d_j}\rho_0^{(0)}(0) = e^{-D_n(t)}\rho_0^{(0)}(0)\prod_{j=1}^{n}\left[\frac{n_b}{n_b+1} + \frac{r\sin(g\tau\sqrt{j})\sin(g\tau\sqrt{j+k})}{\gamma(n_b+1)\sqrt{j(j+k)}}\right] \tag{5}$$

suggested by the detailed balance condition $c_n\rho_n^{(k)} = d_{n+1}\rho_{n+1}^{(k)}$ (and hence $c_n\rho_{n-1}^{(k)} = d_n\rho_n^{(k)}$) yields with Eq. (2)

$$\dot{D}_n = \tfrac{1}{2}\mu_n + d_n(1 - e^{-(D_{n-1}-D_n)}) + c_n(1 - e^{-(D_{n+1}-D_n)}) . \tag{6}$$

Thus far the analysis is exact. We now expand the exponents and find to lowest order

$$D_n(t) \cong \tfrac{1}{2}\mu_n t . \tag{7}$$

Here we have assumed that

$$|D_n - D_{n-1}| \cong \tfrac{1}{2}|\mu_n - \mu_{n-1}|t \cong \tfrac{1}{2}\left|\frac{\partial\mu_n}{\partial n}\right| t \ll 1$$

and analogously for $D_{n+1} - D_n$. This condition is certainly satisfied (i) for short times t and (ii) when μ_n, Eq. (4), is a slowly varying function of n. Equations (5) and (7) then yield

$$\rho_n^{(1)}(t) - e^{-D_n(t)}\rho_n^{(1)}(0) \cong \exp[-\tfrac{1}{2}\mu_n(k=1)t]\rho_n^{(1)}(0) \cong \exp(-\tfrac{1}{2}\mu_{n=\langle n\rangle}t)\rho_n^{(1)}(0) , \tag{8}$$

where in the last step we have replaced [17] n in μ_n by the average number of photon $\langle n\rangle$. This relation reduces Eq. (1) to

$$\langle E\rangle \sim e^{-(D/2)t}\sum_{n=0}^{\infty}(n+1)^{1/2}\rho_n^{(1)}(0) , \tag{9}$$

and with the help of Eq. (4), the linewidth D of the maser reads

$$D \equiv \mu_{n=\langle n\rangle}(k=1) = 4r\sin^2\left[\frac{g\tau}{4\sqrt{\langle n\rangle}}\right] + \frac{\gamma(2n_b+1)}{4\langle n\rangle} . \tag{10}$$

Here we have also expanded the square roots for $k=1$.

In Fig. 1 we depict the detailed behavior of this approximate phase diffusion constant D (solid line), as a function of the pump parameter $\theta = \sqrt{N}g\tau$ for $N = r/\gamma = 50$ atoms and $n_b = 10^{-4}$ thermal photons. The sharp resonances in the monotonic increase of D are reminiscent of the trapping states [7]. To bring out this similarity we show in the same figure the average photon number as a function of θ (dotted curve). We note that the phase diffusion is especially large when the maser is locked to a trapping state, that is, when $\langle n\rangle$ is caught in one of those sharp minima. Equation (10) reveals this behavior in the limit of short interaction times or large pho-

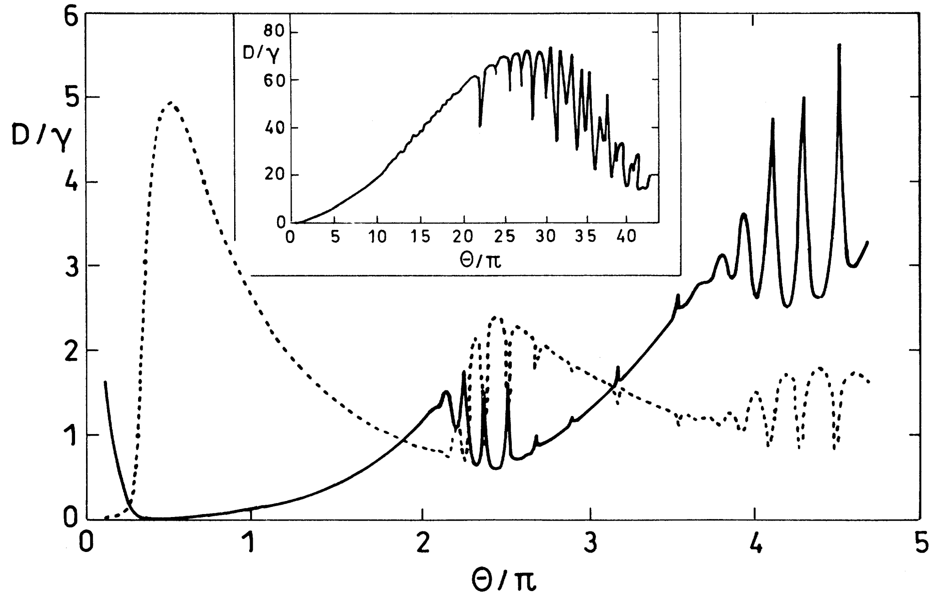

FIG. 1. The relative linewidth D/γ based on Eq. (10) (solid curve) and mean photon number $\langle n \rangle/10$ (dotted curve) as a function of the pump parameter $\theta = N^{1/2} g\tau$ for $N=50$ and mean thermal photon number $n_b = 10^{-4}$. The inset shows the exact relative linewidth D/γ based on a numerical solution of the density matrix Eq. (2) for large pump parameters θ for $N=20$ and $n_b=1$.

ton numbers, that is when $g\tau/4\langle n \rangle^{1/2} \ll 1$. We expand the sine function and arrive at the familiar Schawlow-Townes linewidth [8]

$$D = \frac{\alpha + \gamma(2n_b+1)}{4\langle n \rangle} ,$$

where

$$\alpha = \gamma (N^{1/2} g\tau)^2 = \gamma\theta^2 .$$

The complicated pattern of the micromaser linewidth results from the complicated dependence of $\langle n \rangle$ on the pump parameter indicated in Fig. 1 by the dotted curve which enters in the denominator. We emphasize that the maser linewidth, Eq. (10), goes beyond the standard Schawlow-Townes linewidth. The sine function in Eq. (10) suggests in the limit of large θ values an oscillatory behavior [18] of the linewidth. The exact numerical treatment shown in the inset of Fig. 1 confirms this.

We conclude by outlining a measurement scheme summarized in the inset of Fig. 2 to test these predictions. The proposed experiment consists of two steps: (i) We prepare a maser field of well-defined phase and (ii) we probe the diffusion of this field (see also Ref. [11] for comparison). Before the atoms enter the cavity they are (weakly) driven by a coherent microwave field into a coherent superposition of their ground state $|b\rangle$ and excited state $|a\rangle$. Here the probability amplitude for the lower level is much smaller than that of the upper level so as to not alter the photon number distribution of the micromaser field when (in the second step of the experiment) the microwave field E_1 is shut off. The atoms—so prepared—pass into the maser cavity and produce a phased state of the maser field with off-diagonal elements $\rho_n^{(1)}$, that is, the atoms have established a maser field of definite phase. In the second step of the experiment we switch off the field E_1 and maser field begins to "phase diffuse." In such a case, the atoms enter the maser in level $|a\rangle$, emerge in a superposition of $|a\rangle$ and $|b\rangle$, and then pass through the second microwave field E_2. We may write the density matrix for the atoms after the interaction with field E_2 as

$$\rho_a = \mathrm{Tr}_{\mathrm{maser}}[U_2(\tau_2)U_m(\tau)\rho_a(t_0) \otimes \rho_m(t_0)U_m^{+}(\tau)U_2^{+}(\tau_2)] , \quad (11)$$

where ρ_a and ρ_m denote the density matrix for the atom and maser field, respectively, and U_m and U_2 are the time evolution operators for the maser-atom system and the atoms in the second microwave field E_2, respectively.

Now it is an easy calculation [9] to obtain the joint time evolution operator $U_2 U_m$. We may assume the atoms are in resonance with both fields and that the second one is taken as classical. For the sake of simplicity we arrange the Rabi angle in the second field such that $\Omega_2\tau_2 = 3\pi/4$. Here, Ω_2 and τ_2 denote the Rabi frequency and the interaction time in the second field, respectively.

Suppose we count the atoms emerging from the second field E_2 in the excited state $|a\rangle$. Equation (11) provides the probability

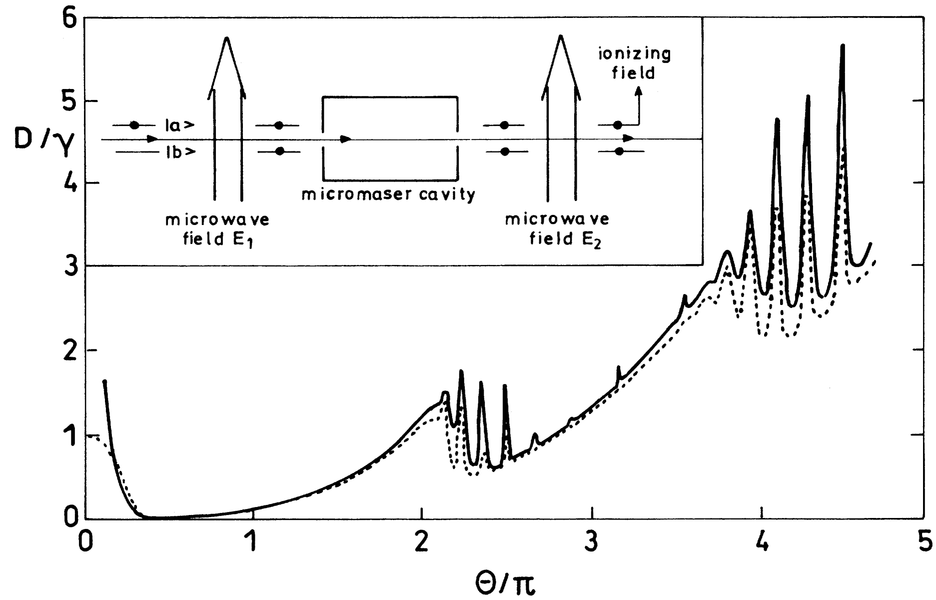

FIG. 2. The approximate analytical relative linewidth D/γ for $N=50$ and $n_b=10^{-4}$ calculated from Eq. (10) (solid curve) compared and contrasted to the linewidth from counting atoms in the excited state in the Ramsey-type measurement (dotted curve) shown in the inset: Experimental setup for the preparation and measurement of nondiagonal elements of the radiation field density matrix. The setup is, in principle, similar to the Ramsey technique. The first field is only applied for an initial period in order to "seed" a phase in the micromaser cavity. The phase of E_1 and E_2 are coupled so that phase diffusion can be measured.

$$P_a=\tfrac{1}{2}+\sum_n \cos(\varphi-\beta)\cos(g\tau\sqrt{n+2})\times\sin(g\tau\sqrt{n+1})\rho_n^{(1)}(t) \quad (12)$$

that an atom is in the state $|a\rangle$ at a time t after the first microwave has been switched off at $t=0$. Note that the time t denotes the interval since the field E_1 was switched off and is not to be confused with the interaction time τ. Here φ and β denote the phase of the field E_2 and the initial phase of the micromaser field, respectively.

When we substitute the ansatz Eq. (8) for $\rho_n^{(1)}(t)$ into Eq. (12) we find

$$P_a=\tfrac{1}{2}+\exp(-\tfrac{1}{2}Dt)\sum_n \cos(\varphi-\beta)\cos(g\tau\sqrt{n+2})\times\sin(g\tau\sqrt{n+1})\rho_n^{(1)}(0)\ . \quad (13)$$

Thus the atomic probability P_a and the off-diagonal elements $\rho_n^{(1)}$ have the same diffusion coefficient D, Eq. (10). For this case the linewidth D obtained via a measurement of the atomic probability P_a is independent of the initial condition $\rho_n^{(1)}(0)$ and is identical to the linewidth of the micromaser field.

In Fig. 2 we compare and contrast the diffusion rate (dotted curve) of the atomic probability P_a, Eq. (12) with the approximate analytical expression for D, Eq. (10) (solid curve). The dotted curve is based on a numerical solution of the density matrix equations, Eq. (2), for $\rho_n^{(1)}$. The initial condition $\rho_n^{(1)}(0)$ we determine from a numerical steady-state solution of the relevant density-matrix equations describing the maser with injected coherence [11]. Figure 2 clearly demonstrates that the linewidth D obtained via the diffusion of P_a is close to the calculated linewidth. Hence by probing atoms as a function of time, after phase diffusion has begun, we measure the maser linewidth.

We conclude by summarizing our main results: In this paper we derive the linewidth of the micromaser. We derive an analytical expression for the corresponding diffusion coefficient and compare it with an exact numerical evaluation of D. Two true quantum phenomena manifest themselves in the dependence of D on the pump parameter: (i) The trapping states create sharp resonances in the familiar Schawlow-Townes result and (ii) the possibility of full Rabi cycles in the maser, that is, the presence of the sine function in Eq. (10), causes D to oscillate as a function of the pumping. Two additional microwave fields, one before and one after the micromaser cavity, that is, a modified Ramsey setup, allows us to measure the phase diffusion coefficient by simple counting the number of atoms in an excited state. This proposed experiment will lead to a precise measurement of the maser-laser linewidth. A detailed discussion of the micromaser spectral properties is planned to be given elsewhere [9].

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[17] This is an excellent approximation when the photon distribution of the maser field enjoys one sharp maximum. For a maser state whose photon statistics employs more maxima a more sophisticated treatment such as a sum over these maxima is necessary.

[18] In this extreme limit the slowly varying condition on μ_n breaks down and Eq. (10) is not strictly valid. However, the numerical calculations do reproduce the qualitative predictions of Eq. (10).