Equivalence of the grand canonical partition functions of particles with different statistics

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It is shown that the grand partition function of an ideal Bose system with single particle spectrum $\epsilon_i = (2n + k + 3/2)\hbar\omega$ is identical to that of a system of particles with single particle energy $\epsilon_i = (n + 1/2)\hbar\omega$ and obeying a particular kind of statistics based on the permutation group.

In the recent literature, one often comes across attempts at establishing the equivalence of dynamical systems having excitations with differing statistics. One celebrated example is the Bose-Fermi equivalence in 1+1 dimensional field theories[1]. Another well-known case appears in 2+1 dimensions, where an interacting theory with a Chern-Simons term of appropriate coefficient can be made equivalent to another theory with excitations of given spin and statistics[2]. Apart from proving the equivalence of the partition functions, one usually constructs suitable operators in a given theory which satisfy the canonical commutation relations of the elementary fields of the model with which equivalence is being sought. In this context, it is interesting to note that often an interacting system with known statistics can be represented by a collection of non-interacting quasi-particles of very different statistics. The so-called exclusion statistics recently advocated by Haldane falls in this category[3]. It has recently been shown that the many-body interacting Calogero-Sutherland system (quantized as bosons or fermions) can be thought of, in the statistical sense, as a collection of non-interacting particles obeying Haldane's exclusion statistics[4]. In many of these cases, the composite operators of the original model with canonical commutation relations appropriate to the new statistics are not known.

In the present work we show that, at the level of the grand partition function, an ideal Bose system with a single particle spectrum (SPS)

$$\epsilon_i = (2n + k + 3/2)\hbar\omega \quad , \tag{1}$$

is completely identical to that of a collection of particles with SPS

$$\epsilon_i = (n + 1/2)\hbar\omega \quad , \tag{2}$$

but with a rather unusual statistics based on the permutation group. Here n and k are positive integers. As is clear from the SPS (1), the Bose case involves degeneracy of the single-particle energy levels. This type of spectrum arises naturally in elementary systems, such as a particle in two dimensions experiencing a harmonic potential with a frequency ratio 1:2 in the x and y directions respectively. The non-relativistic single-particle Hamiltonian can be written as

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2x^2 + 2m\omega^2y^2 \quad . \tag{3}$$

The second case (2) can be thought of as a particle in a harmonic well in one dimension.

We begin with a brief summary of the results obtained in Ref. 5, on which the present work is based. In this work, which makes use of the ideas developed in Ref. 6, it was shown that the the general structure of the canonical partition function for an ideal M-level system corresponding to any quantum statistics based on the permutation group is as follows:

$$Z_N(x_1, \cdots, x_M) = {\sum_\lambda}' s_\lambda(x_1, \cdots, x_M) \quad , \tag{4}$$

where $x_i \equiv \exp(-\beta\epsilon_i)$, $i = 1, \cdots, M$, ϵ_i denote the energies corresponding to the states $i = 1, \cdots, M$, and $\lambda \equiv (\lambda_1, \cdots, \lambda_M)$. Here $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_M$ denotes a partition of N, the number of particles. The prime on the sum on the R.H.S. of (4) is to indicate possible restrictions on the partitions λ of N. As we shall see later, different restrictions on λ correspond to different quantum statistics. The functions $s_\lambda(x_1, \cdots, x_M)$ that appear in (4) denote the Schur functions[7,8] which are homogeneous, symmetric polynomials of degree N in the variables $x_1, \cdots, x_M$. Explicitly, the Schur functions are given by

$$s_\lambda(x_1, \cdots, x_M) = \frac{\det(x_i^{\lambda_j+M-j})}{\det(x_i^{M-j})} \quad ; \quad 1 \leq i, j \leq M \quad . \tag{5}$$

(An alternative definition of the Schur functions in terms of the monomial symmetric functions[7,8] is

$$s_\chi(x_1,\cdots,x_M)=\sum_\lambda K_{\chi\lambda}m_\lambda(x_1,\cdots,x_M)\ , \tag{6}$$

where $K_{\chi\lambda}$ denote the Kostka numbers[7,8].)

The grand canonical partition function is given by

$$\mathcal{Z}(x_1,\cdots,x_M,\mu)=\sum_N \exp(\mu\beta N)Z_N(x_1,\cdots,x_M)\qquad . \tag{7}$$

Using the fact that the Schur functions are homogeneous polynomials of degree N, (7) can be written as

$$\mathcal{Z}(X_1,\cdots,X_M;p)=\sum_N {\sum_\lambda}' s_\lambda(X_1,\cdots,X_M)\qquad , \tag{8}$$

where $X_i\equiv\exp(-\beta(\epsilon_i-\mu))$.

The familiar quantum statistics correspond to the following restrictions on the λ's

(a) Bose statistics : $\lambda=(N,0,\cdots,0)$.

The corresponding grand canonical partition function is given by

$$\mathcal{Z}^B(X_1,\cdots,X_M)=\prod_i\frac{1}{(1-X_i)}\qquad . \tag{9}$$

(b) Fermi statistics: $\lambda=(1,1,\cdots,1,\cdots,0)$

The grand canonical partition function in this case is given by

$$\mathcal{Z}^F(X_1,\cdots,X_M)=\prod_i(1+X_i)\qquad . \tag{10}$$

(c) Parafermi statistics[9] of order p: λ such that $\lambda_1\le p$. Stated differently, $l(\lambda^{'})\le p$ where $\lambda^{'}$ denotes the partition conjugate to λ and $l(\lambda^{'})$ denotes its length, *i.e.*, the number of nonzero $\lambda_i^{'}$'s. (The partition $\lambda^{'}$ conjugate to a partition λ is obtained by interchanging the rows and columns of the Young tableau corresponding to λ.)

In this case, using the results in Ref.7, the grand canonical partition function is found to be explicitly given by

$$\mathcal{Z}^{PF}(X_1,\cdots,X_M;p)=\frac{\det(X_j^{2M+p+1-i}-X_j^i)}{\det(X_j^{2M+1-i}-X_j^i)}\ \ ;\ \ 1\le i,j\le M\quad . \tag{11}$$

(d) Parabose statistics[9] of order p: $l(\lambda)\le p$.

The explicit expression for the grand canonical partition function in this case, to the best of our knowledge, is not known.

The list given above contains only those quantum statistics based on the permutation group which have been extensively studied in the literature. For these statistics one has a second quantized formulation available with a rather simple algebra of creation and annihilation operators. However, one could, in principle, define new kinds of statistics, also based on the permutation group, by putting restrictions on λ in (7) other than those considered above. Some of these possibilities are listed below:

(e) p-q statistics: $l(\lambda)\le p\quad;\quad l(\lambda^{'})\le q$.

No explicit formula for the grand canonical partition function for this case is known.

(f) HST statistics[5]: No restrictions on λ.

In this case the grand canonical partition function is explicitly given by[7]

$$\mathcal{Z}^{HST}(X_1,\cdots,X_M)=\prod_i\frac{1}{(1-X_i)}\prod_{i<j}\frac{1}{(1-X_iX_j)}\quad . \tag{12}$$

This statistics may be viewed as the $p\to\infty$ limit of parabose and parafermi statistics.

(g) λ even; *i.e.,* all λ_i's even.

The corresponding grand canonical partition function is given by[7]

$$\mathcal{Z}(X_1, \cdots, X_M) = \prod_i \frac{1}{(1 - {X_i}^2)} \prod_{i<j} \frac{1}{(1 - X_i X_j)} \quad . \tag{13}$$

(h) λ' even; *i.e.,* all ${\lambda_i}'$ even.

The explicit expression for the grand canonical partition function for this case turns out to be[7]

$$\mathcal{Z}(X_1, \cdots, X_M) = \prod_{i<j} \frac{1}{(1 - X_i X_j)} \quad . \tag{14}$$

The close similarity in the structure of the grand canonical partition function of a Bose system (a) and that for a system obeying the statistics defined in (h) suggests the possibility of interpreting a system with a certain SPS obeying the latter statistics as a Bose system with a different SPS. This possibility is, in fact, realised by the SPS given in (1) and (2), as can be seen from the following.

Consider a Bose system with the single-particle spectrum given by (1). It can easily be seen that the states corresponding to the energies $3\hbar\omega/2$ and $5\hbar\omega/2$ are non-degenerate, the states $7\hbar\omega/2$ and $9\hbar\omega/2$ are doubly degenerate, the states $11\hbar\omega/2$ and $13\hbar\omega/2$ are threefold degenerate, and so on. The grand canonical partition function for this Bose system is therefore given by

$$\begin{aligned}\mathcal{Z}^B = &\left(\frac{1}{1-\alpha e^{-\beta\hbar\omega}}\right)\left(\frac{1}{1-\alpha e^{-2\beta\hbar\omega}}\right)\left(\frac{1}{1-\alpha e^{-3\beta\hbar\omega}}\right)^2\left(\frac{1}{1-\alpha e^{-4\beta\hbar\omega}}\right)^2 \times \\ &\left(\frac{1}{1-\alpha e^{-5\beta\hbar\omega}}\right)^3\left(\frac{1}{1-\alpha e^{-6\beta\hbar\omega}}\right)^3\left(\frac{1}{1-\alpha e^{-7\beta\hbar\omega}}\right)^3 \cdots \quad .\end{aligned} \tag{15}$$

where $\alpha = e^{-\beta(\hbar\omega/2-\mu)}$.

Next, consider a system with the single particle spectrum given by (2) but obeying the statistics specified by (h). Substituting the explicit expressions for ϵ_i into (14), one finds that the grand canonical partition function for this system is exactly the same as that in (15) with α replaced by $\alpha^{'}$ where $\alpha^{'} = e^{-\beta(\hbar\omega-2\mu)}$. Since α (or $\alpha^{'}$) is to be determined by fixing the mean number of particles, one finds that the two systems, at the level of grand canonical partition functions are equivalent. In other words, the grand canonical partition function of the first system can be interpreted in terms of the second system which has a different single particle spectrum and is governed by a different statistics.

To conclude, we have shown the equivalence of two systems with different single particle spectra and obeying different statistics. One of the two systems has a "complicated" single-particle spectrum (in the sense that the SPS has degeneracies) but obeys a "simple" (Bose) statistics while the other system has a "simple" (i.e. non-degenerate) spectrum but obeys a somewhat exotic statistics.

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