THE NUMBER OF REPRESENTATIONS OF A NUMBER AS A SUM OF n NON-NEGATIVE nTH POWERS

By S. CHOWLA (*Waltair, India*) and
S. S. PILLAI (*Annamalainagar, India*)

[Received 21 October 1935]

1. LET $r_{s,n}(N)$ denote the number of representations of N as a sum of s non-negative nth powers. Hardy and Littlewood conjectured that for fixed n ($\geqslant 3$),

$$r_{n,n}(N) = O(N^{\epsilon}) \text{ for any positive } \epsilon. \tag{1}$$

We shall prove the following result in the opposite direction.

THEOREM*. *For fixed n ($\geqslant 5$) we have*

$$r_{n,n}(N) = \Omega(\log\log N)$$

i.e. there is a positive number c_1, depending only on n, such that

$$r_{n,n}(N) > c_1 \log\log N,$$

is true for infinitely many N.

2. In the sequel we shall denote by $c_2, c_3, \ldots, c_{19}$, *positive constants depending only on n.*

Notation. Let ρ be a primitive qth root of unity,

$$S_\rho = \sum_{h=0}^{q-1} e^{2\pi i \frac{a}{q} h^n} = \sum_{h=0}^{q-1} \rho^{h^n} \quad [(a,q)=1], \qquad A(q) = q^{-s} \sum_{\rho} (S_\rho)^s e^{-2\pi i \frac{a}{q} N},$$

$$S(N;n;s) = \sum_{q=1}^{\infty} A(q) = \prod_p \chi_p \quad (s \geqslant 5),$$

where $\chi_p = 1 + A(p) + A(p^2) + \ldots$ and p runs through all primes; p_r denotes the rth prime. $S(N;n;s)$ is the singular series in Waring's problem. See Landau, *Vorlesungen über Zahlentheorie*, Band 1, 281–302, where the first four of the following lemmas are proved.

LEMMA 1. (i) $S_\rho = p^{l-1} \quad (q = p^l;\ p \nmid n;\ 2 \leqslant l \leqslant n)$;

(ii) $|S_\rho| \leqslant (n-1)\sqrt{p} \quad (q = p)$;

(iii) $S_\rho = p^{n-1} S_{\rho^{p^n}} \quad (q = p^l;\ l > n)$.

(*Clearly ρ^{p^n} is a primitive p^{l-n}th root of unity.*)

* For $n = 5$ Chowla has previously proved a sharper result. See the paper by Sastry in *J. of London Math. Soc.* 9 (1934), 242–6. For a weaker form of the theorem of this paper see *Proc. Indian Ac. Sc.* (A), 2 (1935), 397–401.

We shall use $p^\theta \| m$ to denote that $p^\theta | m$ but $p^{\theta+1} \nmid m$. Then we have

LEMMA 2. *If* (i) $p^{\beta n+\sigma} \| N$ *where* $\beta \geqslant 0, n > \sigma \geqslant 0$, *and* (ii) $p^\theta \| n$, *and* $\gamma = \theta+1$ $(p > 2)$, *or* $\gamma = \theta+2$ $(p = 2)$, *then* $A(p^l) = 0$ *for* $l > l_0$ *where*

$$l_0 = \max(\beta n+\sigma+1,\ \beta n+\gamma).$$

LEMMA 3. $1+A(p)+...+A(p^l) = p^{-(s-1)l} M(p^l, N)$ *where* $M(p^l, N)$ *is the number of solutions of* $\sum_{m \leqslant s} h_m^n \equiv N \pmod{p^l}$ *and* $0 \leqslant h_m < p^l$.

LEMMA 4. *For* $s = n+1$ $(n \geqslant 4)$ *there exists a number* c_2, *such that*

$$\chi_p > 1-p^{-\frac{3}{2}} \quad \textit{for } p \geqslant c_2.$$

3. LEMMA 5. *Let* $n \geqslant 5$ *and* $N \equiv p_r^n \pmod{p_r^{2n}}$ *so that* $p_r^n \| N$. *Then there exists a number* c_3 *such that, for* $s = n+1$, $r > c_3$,

$$\chi_{p_r} = 1+\frac{1}{p_r}+\frac{b}{p_r^{\frac{3}{2}}} \qquad (|b| < 1).$$

Proof. First of all we find c_4 so large that $(p_r, n) = 1$ for all r not less than c_4. For such r, since $p_r^n \| N$, Lemma 2 shows that

$$\chi_{p_r} = 1+A(p_r)+...+A(p_r^{n+1}). \tag{2}$$

By Lemma 1 (i), for $s = n+1$,

$$A(p_r^m) = p_r^{m-n-1}-p_r^{m-n-2} \quad (2 \leqslant m \leqslant n). \tag{3}$$

By Lemma 1 (ii), for $s = n+1$,

$$|A(p_r)| \leqslant \frac{c_5 p_r^{\frac{1}{2}(n+1)} p_r}{p_r^{n+1}} < \frac{c_6}{p_r^2} \quad (n \geqslant 5). \tag{4}$$

By Lemma 1, (iii) and (ii), for $s = n+1$,

$$|A(p_r^{n+1})| \leqslant \frac{c_7 (p_r^{n-\frac{1}{2}})^{n+1} p_r^{n+1}}{p_r^{(n+1)(n+1)}} = \frac{c_7}{p_r^{\frac{1}{2}(n+1)}} < \frac{c_8}{p_r^2} \qquad (n \geqslant 3). \tag{5}$$

From (2), (3), (4), (5) we immediately obtain Lemma 5.

LEMMA 6. *For* $r \leqslant c_3$ *(of Lemma 5) let* $p_r^{\theta_r} \| n$. *Let* $Y = 2^{\theta_1+2} p_2^{\theta_2+1} ... p_m^{\theta_m+1}$ *where* $m = [c_3]$. *Then for all* $N \equiv 1 \pmod{Y}$, *and* $s = n+1$,

$$\prod_{r < c_3} \chi_{p_r} > c_9.$$

Proof. By Lemmas 2 and 3, if $s = n+1$, $N \equiv 1 \pmod{Y}$,

$$\chi_2 = 1+A(2)+...+A(2^{\theta_1+2}) = 2^{-(\theta_1+2)n} M(2^{2+\theta_1}, N), \tag{6}$$

$$\begin{aligned} \chi_{p_r} &= 1+A(p_r)+...+A(p_r^{\theta_r+1}) \\ &= 2^{-(\theta_r+1)n} M(p_r^{\theta_r+1}, N) \quad (2 \leqslant r \leqslant m). \end{aligned} \tag{7}$$

But, since $N \equiv 1 \pmod{2^{\theta_1+2} p_2^{\theta_2+1} ... p_m^{\theta_m+1}}$, we have

$$M(2^{\theta_1+2}, N) \geqslant 1, \qquad M(p_r^{\theta_r+1}, N) \geqslant 1 \quad (2 \leqslant r \leqslant m). \tag{8}$$

From (6), (7), (8) we obtain Lemma 6 since θ_r $(r \leqslant m)$ depends only on n.

LEMMA 7. $$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-C}}{\log x},$$

where C is Euler's constant.

The proof is well known.

LEMMA 8. $$\prod_{p \leqslant x} \left(1 + \frac{1}{p}\right) > c \log x,$$

where c is an absolute positive constant.

Proof. LEMMA 7.

Let $$z = c_{10} \log x, \tag{9}$$

where later we make $x \to \infty$ and c_{10} will be subjected to certain restrictions. It follows from Lemma 5 that, if $N \equiv p_r^n \pmod{p_r^{2n}}$ for $c_3 \leqslant r \leqslant z$, then

$$\prod_{c_3 < r \leqslant z} \chi_{p_r} > c_{11} \log(z \log z) > c_{12} \log\log x, \tag{10}$$

by Lemma 8. Now let us take $z > c_2$ (of Lemma 4). Then*

$$S(N; n; n+1) = \prod_{r \leqslant c_3} \chi_{p_r} \prod_{c_3 < r \leqslant z} \chi_{p_r} \prod_{r > z} \chi_{p_r}$$

$$> c_9 (c_{12} \log\log x) c_{13} \quad \text{by Lemma 6, (10), and Lemma 4}$$

$$> c_{14} \log\log x, \tag{11}$$

where N is subject to the restrictions

$$N \equiv 1 \pmod{2^{\theta_1+2} p_2^{\theta_2+1} \dots p_m^{\theta_m+1}} \tag{12}$$

where $m = [c_3]$ *and the θ's depend only on n, and*

$$N \equiv p_r^n \pmod{p_r^{2n}} \quad (c_3 < r \leqslant z). \tag{13}$$

Now, from (9), for $x > c_{19}$,

$$x^{nc_{10}} < 2^{\theta_1+2} \prod_{2 \leqslant r \leqslant m} \{p_r^{\theta_r+1}\} \prod_{c_3 \leqslant r \leqslant z} \{p_r^{2n}\} < x^{3nc_{10}}. \tag{14}$$

Hence, if $$3nc_{10} < 1, \tag{15}$$

the number of N's satisfying (12), (13) *and*

$$1 \leqslant N \leqslant x, \tag{16}$$

is greater than $c_{15} x^{1-4nc_{10}}$.

* Here c_{11}, c_{12}, c_{13}, c_{14} will depend on our choice of c_{10}; c_{10} itself is subject to (15) and (17).

Again, we have from Theorem 5 of P.N. VI of Hardy and Littlewood*

LEMMA 9. *There exists a positive number c_{16}, depending only on n, such that, if* (1) *is true, then*

$$r_{n+1,n}(N) = ES(N;n;n+1)N^{\frac{1}{n}}+O\left(N^{\frac{1}{n}}\right)$$

(where E is positive and depends only on n) is false for at most $x^{1-c_{16}}$ values of $N \leqslant x$.

Let us choose c_{10} in (9) so that (15) is true and

$$4nc_{10} < c_{16}. \tag{17}$$

Hence the number of solutions (in N) of (12), (13), (16) is greater than

$$c_{15}\, x^{1-c_{16}+\delta},$$

where, by (17), $\delta = c_{16}-4nc_{10} > 0$. Since $x \geqslant N$ it now follows† from (11) and Lemma 9 that

LEMMA 10. *There exists a positive number c_{17}, depending only on n, such that, if* (1) *is true, then*

$$r_{n+1,n}(N) > c_{17}(\log\log N)N^{\frac{1}{n}},$$

is true for infinitely many N.

From Lemma 10 it follows immediately that

LEMMA 11. *If* (1) *is true, there exists a positive number c_{18}, depending only on n, such that*

$$r_{n,n}(N) > c_{18}\log\log N$$

is true for infinitely many N.

If possible, let

$$r_{n,n}(N) = o(\log\log N), \tag{18}$$

then, obviously, (1) is true, and hence by Lemma 11,

$$r_{n,n}(N) > c_{18}\log\log N \textit{ for infinitely many } N. \tag{19}$$

But (19) contradicts (18). Hence (18) is false and our theorem is proved.

* *Math. Zeits.* 23 (1925), 1–37.

† Since $c_{15}\, x^{1-c_{16}+\delta} > x^{1-c_{16}}$ for $x > x_0(n)$.