A REMARKABLE PROPERTY OF THE "SINGULAR SERIES" IN WARING'S PROBLEM AND ITS RELATION TO HYPOTHESIS K OF HARDY AND LITTLEWOOD.*

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§1. Let n be a positive integer $\geqslant 5$. Let ρ denote a primitive qth root of unity. We write

$$S_\rho = \sum_{h=0}^{q-1} \rho^{h^n} = \sum_{h=0}^{q-1} e^{2\pi i \frac{a}{q} h^n}$$

where $(a, q) = 1$. Further

$$A(q) = A(N, n, q, s) = q^{-s} \sum_\rho e^{-2\pi i \frac{a}{q} N} (S_\rho)^s$$

where ρ runs over the $\phi(q)$ primitive qth roots of unity [$\phi(q)$ is Euler's totient function]. The "singular series" under consideration is for $s \geqslant 5$,

$$S(N) = S(N; n; s) = \sum_{q=1}^{\infty} A(q) = \prod_p \chi_p$$

where p runs through all primes and

$$\chi_p = 1 + A(p) + A(p^2) + \cdots$$

We shall show that

Theorem 1.

$$S(N, n; n+1) \neq O(1)$$

i.e., for fixed n *and an arbitrary* A *we can find infinitely many positive integers* N *such that*

$$(1) \quad S(N, n; n+1) > A.$$

In fact we prove that *the number of numbers* $N \leqslant x$ *with the property* (1) *is greater than* Cx (*for large* x), *where* C = C(A, n) *is a positive constant depending only on* A *and* n.

Let $r_{s,n}(N)$ denote the number of representations of N as a sum of s nth powers $\geqslant 0$. Then the last result helps us to show that

Theorem 2.

$$r_{n,n}(N) \neq O(1)$$

i.e., for fixed n *and an arbitrary* A *we can find infinitely many* N *such that* $r_{n,n}(N) > A$.

* Dedicated to my friend Sivasankaranarayana Pillai.

Theorem 2 gives us some idea of the behaviour of $r_{n,n}(N)$ for large N. An opposite kind of result is the famous unproved

Hypothesis K. $r_{n,n}(N) = O(N^{\epsilon})$ *for any* $\epsilon > 0$.

In what follows the numbers c (or c_1, c_2, etc.) denote positive constants which depend only on n.

§2. Let us assume the existence of a $c = c(n)$ such that

(I) $r_{n,n}(N) < c$.

If (I) is true, then clearly

(II) $r_{n+1,n}(N) < cN^{\frac{1}{n}}$

since (I) is true, Hypothesis K is true. On the latter assumption we have (from Hardy-Littlewood's P.N. VI in *Math. Ztschr.*, Vol. 23)

$$(2) \quad r_{n+1,n}(N) = E N^{\frac{1}{n}} S(N ; n ; n+1) + \sigma(N)$$

where

$$(3) \quad \sum_{m \leqslant x} [\sigma(m)]^2 = O(x^{1+\frac{2}{n}-c})$$

and E is a positive constant involving only on n.

From (2) and (3) it follows that

$$(4) \quad r_{n+1,n}(N) = E S(N; n ; n+1) N^{\frac{1}{n}} + o(N^{\frac{1}{n}})$$

is true for " almost all " N.

§3. *Notation.*

a/b

means that a is a divisor of b ; $a \nmid b$ is the negation of a/b.

$p^{\theta} \parallel m$

means that p^{θ}/m but $p^{\theta+1} \nmid m$.

$M(p^l) = M(p^l, N)$ is the number of solutions of $h_1^n + \cdots + h_s^n \equiv N (\text{mod } p^l)$ where $0 \leqslant h_m < p^l$. Clearly $M(p^l, N) > 0$ if $N \equiv 1 \ (\text{mod } p^l)$.

The first six of the lemmas below are from Landau's *Vorlesungen uber. Zahlentheorie*, Band 1, pages 284, 294, 297, 297, 302, 281 respectively.

Lemma 1. *Let* $p^{\beta n+\sigma} \parallel N$ *where* $\beta \geqslant 0$, $0 \leqslant \sigma < n$; *let* $p^{\theta} \parallel n$ *and* $\gamma = \theta+1$ if $p > 2$, $\gamma = \theta+2$ *if* $p=2$. *Then* $A(p^l) = 0$ *for* $l > l_0$ *where*

$l_0 = \text{Max}\,(\beta n + \sigma + 1, \beta n + \gamma)$.

Lemma 2. *If* $q = p^l$, $p \nmid n$, $2 \leqslant l \leqslant n$ *then*

$S\rho = p^{l-1}$

Lemma 3. *If* $q = p$,

$|S\rho| \leqslant (n-1)\sqrt{p}$.

Lemma 4. *For* $q = p^l$, $l > n$, *we have*

$$S_\rho = p^{n-1} S_{\rho^{p^n}}$$

(clearly, ρ^{p^n} is a primitive p^{l-n}th root of unity)

Lemma 5. *For* $p > c_1$, $s = n + 1$ $(n \geqslant 4)$

$$\chi_p > 1 - p^{-\frac{5}{4}}$$

Lemma 6.

$$1 + A(p) + A(p^2) + \cdots + A(p^l) = p^{-(s-1)l} M(p^l, N).$$

§4. Let p_r denote the rth prime. In the sequel we shall employ the above lemmas only in the case $s = n + 1$.

Lemma 7. *We can find a number* z_1 *so large that* $(p_r, n) = 1$ *for all* $r \geqslant z_1$ *and further,*

$$\chi_{p_r} = 1 + \frac{1}{p_r} + \frac{b}{p_r^{\frac{3}{2}}} \ (n \geqslant 5) \qquad (|b| < 1)$$

for all $r \geqslant z_1$, *where* N *is such that*

$$p_r^n \| N \qquad (r \geqslant z_1).$$

Proof.—Since $r \geqslant z_1$, p_r is prime to n for sufficiently large z_1 and it follows from lemma 1 that

$$(5) \quad \chi_{p_r} = 1 + A(p_r) + A(p_r^2) + \cdots + A(p_r^{n+1}).$$

From lemma 2,

$$(6) \quad S_\rho(q = p_r^m,\ 2 \leqslant m \leqslant n) = p_r^{m-1},$$

$$(7) \quad (S_\rho)^{n+1} = p_r^{m(n+1)-n-1},$$

$$(8) \quad \rho^{-N} = 1,$$

$$(9) \quad A(q) = \frac{1}{q^{n+1}} \sum_\rho (S_\rho)^{n+1} \rho^{-N}$$

$$= (p_r^m - p_r^{m-1}) p_r^{-n-1}$$

$$= p_r^{m-n-1} - p_r^{m-n-2} \quad (2 \leqslant m \leqslant n).$$

From (5) and (9),

$$(10) \quad \chi_{p_r} = 1 + \frac{1}{p_r} - \frac{1}{p_r^n} + A(p_r) + A(p_r^{n+1}).$$

By lemma 3,

$$(11) \quad A(p_r) \leqslant \frac{c}{p_r^{n+1}} p_r^{\frac{n+1}{2}} p_r = \frac{c}{p_r^{\frac{n+1}{2} - 1}} = \frac{c}{p_r^2}$$

for $n \geqslant 5$; by lemmas 3 and 4,

$$(12) \quad A(p_r^{n+1}) \leqslant c \frac{(p_r^{n-\frac{1}{2}})^{n+1} p_r^{n+1}}{p_r^{(n+1)(n+1)}}$$

$$= \frac{c}{p_r^{\frac{n}{2}+\frac{1}{2}}} = \frac{c}{p_r^2} \ (n \geqslant 3).$$

(by applying lemma 5, since $z_2 > c_1$, to the third product in (19)), where N is restricted by the conditions

$$(20) \quad N \equiv 1 (\text{mod } 2^{\theta_1+2} p_2^{\theta_2+1} \dots p_m^{\theta_m+1})$$

and

$$(21) \quad p_r^n \| N \quad \text{for } z_1 \leqslant r \leqslant z_2.$$

It is clear that for given n and A there is a finite proportion of numbers N satisfying (20) and (21), *i.e.*, the number of solutions of (20), (21) and

$$(22) \quad N \leqslant x$$

is greater than $d\,x$ (for large x) where d is positive and depends only on A and n. Hence we have

Theorem 3. *Let* n *be a fixed integer* $\geqslant 5$. *Let* A *be an arbitrarily large number. Then we can find a positive number* d, *depending only on* A *and* n, *such that the number of solutions (in* N*) of*

$$S(N; n; n+1) > A,$$
$$1 \leqslant N \leqslant x,$$

is greater than d x *for large* x.

From Theorem 3, and (4) which is true for almost all N, we easily deduce that

Theorem 4. *Let* n $\geqslant$ 5 *be fixed. Let* A *be arbitrarily large. Then if Hypothesis K is true there exists a number* d = d (A, n) > 0 *such that the number of solutions (in* N*) of*

$$(23) \quad r_{n+1,n}(N) > A\, N^{\frac{1}{n}}$$

and $1 \leqslant N \leqslant x$

is greater than d x *for large* x.

Now if (I) were true, then (II) would be true. Further Hypothesis K would be true since (I) is supposed true, and hence by Theorem 4, there would be infinitely many N satisfying (23), which contradicts (II). Hence (I) is false and Theorem 2 is true.

Note.—Results similar to Theorem 2, *e.g.*, that $r'_{k,k}(N) \geqslant 2$ for infinitely many N($3 \leqslant k \leqslant 9$) where $r'_{k,k}(m)$ is the number of distinct and primitive representations (permutation of the bases not allowed) of m as a sum of k kth powers $\geqslant 0$, have been proved by Wright in his paper "On Sums of kth Powers," *Jour. London Math. Soc.*, 1935, **10**, 94-99.