THE CLASS-NUMBER OF REAL QUADRATIC FIELDS

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Let $h = h(d)$ denote the class-number of the real quadratic field $F = R(\sqrt{d})$ of fundamental discriminant d; let $\epsilon = \frac{t + u\sqrt{d}}{2} > 1$ be its fundamental unit. Further let $\chi(\nu)$ denote the character belonging to F, and $d = pm$ where p is an odd prime, and m is an integer. Further $[x]$ denotes the greatest integer contained in x, and $\left(\frac{n}{p}\right)$ is Legendre's symbol of quadratic residuacity.

Then we have the following results.

THEOREM 1. *If* $p > 3$, $m > 1$,

$$-2h\frac{u}{t} \equiv \sum_{0 < \nu < d} \frac{\chi(\nu)}{m\nu}\left[\frac{\nu}{p}\right] \quad (\text{mod } p).$$

In the case $p = 3$ *there is an additional factor* $(1 + m)$ *on the left side of this equation.*

A special case of Theorem 4 is

THEOREM 2. *If* $d = p \equiv 5(8)$, *we have*

$$+ 4\frac{u}{t}h \equiv - \sum_{1}^{p/4} \frac{1}{n}\left(\frac{n}{p}\right) \quad (\text{mod } p).$$

Write

$$\chi(\nu) = \left(\frac{\nu}{p}\right)X(\nu),$$

so that $X(\nu)$ *is a real primitive character (mod* m*). Then we have*

THEOREM 3. *If*

$$\sum_{-1}^{\infty} \frac{C_n x^n}{n!} = \frac{\sum_{t=1}^{m} X(t)e^{tx}}{e^{mx} - 1}.$$

Then if $p > m$, $p \neq 3$,

$$-2h\frac{u}{t} \equiv C_{(p-3)/2} \pmod{p}.$$

Finally let A denote the product of all the quadratic residues of p lying between 0 and p, B the product of the quadratic non-residues of p lying between 0 and p. Then

THEOREM 4. *If* $d = p \equiv 1(4)$, *then*

$$2h\frac{u}{t} \equiv \frac{A+B}{p} \pmod{p}.$$