

## Detection of gravitational waves using a network of detectors

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**Abstract.** We formulate the data analysis problem for the detection of the Newtonian coalescing-binary signal by a network of laser interferometric gravitational wave detectors that have arbitrary orientations, but are located at the same site. We use the maximum likelihood method for optimizing the detection problem. We show that for networks comprising of up to three detectors, the optimal statistic is just the matched network-filter. Alternatively, it is simply a linear combination of the signal-to-noise ratios of the individual detectors. This statistic, therefore, can be interpreted as the signal-to-noise ratio of the network. The overall sensitivity of the network is shown to increase roughly as the square-root of the number of detectors in the network. We further show that these results continue to hold even for the restricted post-Newtonian filters. Finally, our formalism is general enough to be extended, in a straightforward way, to address the problem of detection of such waves from other sources by some other types of detectors, eg., bars or spheres, or even by networks of spatially well-separated detectors.

**Keywords.** Gravitational radiation; interferometric detector; coalescing binary; data analysis.

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### 1. Introduction

The existence of gravitational waves (GW), which is predicted in the theory of general relativity, has long been verified ‘indirectly’ through the observations of Hulse and Taylor [1]. The inspiral of the members of the binary pulsar system named after them has been successfully accounted for in terms of the back reaction due to the radiated gravitational waves [1,2]. However, detection of such waves with man-made ‘antennas’ has not been confirmed so far. Nevertheless, this problem has received a lot of attention this decade, especially, due to arrival of laser-interferometric detectors, which are touted to have the sensitivity required for detecting such waves.

In the past, a sizable amount of research has been done on the problem of detecting gravitational waves using a single bar or interferometric detector. However, very little work has been devoted on developing techniques to optimally analyze the data from a network of such detectors to seek the presence of coalescing binary signal. As has been argued in the past (see, eg., ref. [3]), for a given false-alarm probability, the threshold for detection is lowered as the number of detectors is increased. This increases the probability of detection by a network rather than a single detector, provided the observer accepts only coincidences.

One of the early papers which came close to discussing the problem of detection of these waves using a network was that of Finn and Chernoff [4]. This paper observed that since the orientations of the two LIGO detectors were very similar, their joint sensitivity was larger than any one of them. Another work which dealt with the issue of detection using a network was that of Bhawal and Dhurandhar [5]. The main aim of this paper was to find the optimal recycling mode of operation of the planned laser interferometric detectors for which a meaningful coincidence detection of broadband signals could be performed. However, none of these earlier papers addressed the issue of how a network of detectors with arbitrary orientations can be optimally used as a ‘single’ detector of sensitivity higher than that of any of its subsets of individual detectors. One of our main aims is to show precisely how this can be achieved. In the process, we will arrive at a network statistic based on the individual detector outputs that can be used to ascertain the presence of a signal in them with a given level of confidence.

We note that the use of a network has nevertheless received considerable attention in the context of the parameter estimation problem. Some of the notable works that address this issue are refs. [6–10]. The prime motivation in the use of networks in this regard is that the larger the number of detectors, smaller the errors in estimated values of the binary parameters. However, the starting point in these approaches is the assumption that the problem of detection has already been addressed and the detector specific chirp filters that result in ‘super-threshold’ cross-correlations with the individual detector outputs, have been picked.

Here, we formulate the data analysis problem in the case of the coalescing binary signal for a network of, say,  $N$  number of laser interferometric gravitational wave detectors that have arbitrary orientations, but are located at the same site. The noise in each detector is assumed to be additive and Gaussian. Also, the noises in different detectors are taken to be independent of one another. We use the maximum likelihood method for optimizing the detection problem.

The paper is organized as follows. In §2, we define the various coordinate frames, such as the detector frame and the wave frame that we use in our calculations. We describe some known representations of the Newtonian signal corresponding to gravitational waves from a coalescing binary. In §3, we present a new representation for this signal in terms of the complex expansion coefficients of the wave and the detector tensor in a basis of STF tensors of rank 2. Section 4 shows how the detection problem can be optimally addressed using the maximum likelihood method. In §5, we present the analysis for the improvement of the sensitivity of a network as a function of  $N$ . Finally, in §6, we discuss how our results continue to hold for the restricted post-Newtonian waveforms. We also mention how our formalism can be extended to address the detection problem for a network of spatially well separated detectors.

We use the following convention for symbols in this paper. Variables characterizing the network are displayed in the Sans Serif font. A parenthetic index in the superscript or subscript of a variable identifies a particular detector. Network- or individual detector-based variables that are complex are denoted by uppercase letters, whereas the lower case letters are reserved for real variables. Note that quantities such as the gravitational constant,  $G$ , though written in upper case, are not complex since they do not represent any inherent characteristic of the network or an individual detector. Also, we define the complex inner product as  $\langle A\Delta_{(I)}, Bf \rangle = A^*B\langle \Delta_{(I)}, f \rangle$ . By our convention, all the quantities featuring in the above expression, except  $f$ , are complex. Moreover,  $\Delta_{(I)}$  denotes a variable that

characterizes the  $I$ -th detector in the network, where  $I$  is a natural number. Also,  $f$  denotes a network-based variable.

## 2. Preliminaries

We describe the various coordinate frames in terms of which we will analyze the different polarizations of an incoming wave. Let  $(X, Y, Z)$  be the orthogonal Cartesian coordinates connected with a weak plane gravitational wave traveling along the positive  $Z$ -direction;  $X$  and  $Y$  denote the axes of the polarization ellipse of the wave. Let  $(x, y, z)$  form a right-handed coordinate system that describes a fiducial detector (henceforth referred to as the ‘fide’ or the ‘network frame’). Let us define the Euler angles  $\theta$  and  $\phi$  to give the incoming direction of the wave, and  $\psi$  to denote the angle between one semi-axis of the ellipse of polarization and the node direction. The orthogonal matrix transformation from the wave frame to the fide is thus defined by the Euler angles  $\{\phi, \theta, \psi\}$ . The orthogonal matrix transformation from the fide to the frame of the  $I$ -th detector is defined by the Euler angles  $\{\alpha_{(I)}, \beta_{(I)}, \gamma_{(I)}\}$ .

A gravitational wave is represented by metric tensor fluctuation,  $h_{ij}$ , about the vacuum. In the transverse trace-free gauge, its non-vanishing components in the wave-frame are  $h_{xx} = -h_{yy} \equiv h_+$ ,  $h_{xy} = h_{yx} \equiv h_\times$ . Here,  $h_+$  and  $h_\times$  are the two polarizations of the waveform. In the Newtonian approximation, they are:

$$h_+(t) = \frac{2\mathcal{N}a^{-1/4}(t)}{r} \frac{1 + \cos^2 \epsilon}{2} \cos[\chi(t) + \delta] , \quad (1a)$$

$$h_\times(t) = \frac{2\mathcal{N}a^{-1/4}(t)}{r} \cos \epsilon \sin[\chi(t) + \delta] . \quad (1b)$$

Above,  $\mathcal{N} \equiv [2G^{5/3} \mathcal{M}^{5/3} (\pi f_a)^{2/3} / c^4]$ ,  $r$  is the luminosity distance from the earth to the binary,  $\mathcal{M}$  is the ‘chirp’ mass defined by  $\mathcal{M} = (1 + z)\mu^{3/5}m^{2/5}$ , where  $m = m_1 + m_2$  is the total mass of the binary,  $\mu$  is the reduced mass,  $z$  is the cosmological redshift of the binary, and  $c$  is the speed of light in vacuum. The angle  $\epsilon$  is the angle of inclination of the binary, i.e., the angle between the line of sight and the vector normal to the orbit of the binary, and  $\delta$  is an initial phase of the orbital motion. The frequency of the gravitational wave is twice the orbital frequency and is given by

$$f(t; t_a, \mathcal{M}) = \left(\frac{c^3}{G}\right)^{5/8} \frac{1}{\pi} \left(\frac{5}{256 \mathcal{M}^{5/3} [t_c(t_a, \mathcal{M}) - t]}\right)^{3/8} , \quad (2)$$

where  $t_a$  is the time of arrival of the signal (such that  $f(t_a) \equiv f_a = 10\text{Hz}$ ) and  $t_c$  is the time at which coalescence occurs. Inverting the above equation after setting  $f(t_a; t_a, \mathcal{M}) = f_a$ , we get the time of coalescence:

$$t_c(t_a, \mathcal{M}) = t_a + \frac{5}{256 \mathcal{M}^{5/3} (\pi f_a)^{8/3}} \left(\frac{c^3}{G}\right)^{5/3} . \quad (3)$$

Finally,  $\chi(t) \equiv 2\pi \int_{t_a}^t f(t') dt'$  and  $a(t) \equiv 1 - (t - t_a)/\xi$ , where  $\xi = 3.00 (\mathcal{M}/M_\odot)^{-5/3} (f_a/100 \text{ Hz})^{-8/3}$  sec, is the chirp parameter. Note that a total of

eight independent parameters, viz.,  $\{r, \delta, \theta, \phi, \psi, \epsilon, t_a, \xi\}$  are required to specify this signal. The ranges of the four angles are as follows:  $\theta \in (0, \pi)$ ,  $\phi \in (0, 2\pi)$ ,  $\psi \in (0, 2\pi)$ , and  $\epsilon \in (0, \pi)$ .

It can be shown that the signal at the fiducial detector is [11]

$$s(t) = 2\kappa a(t)^{-1/4} \cos\left(\nu_a \xi \left(1 - a(t)^{5/8}\right) + \delta - \eta\right) , \quad (4)$$

where

$$\nu_a = 320\pi \left(\frac{f_a}{100 \text{ Hz}}\right) \text{ Hz} , \quad \kappa \equiv \mathcal{N}\zeta/r , \quad (5a)$$

$$\zeta(\epsilon) \equiv \left[\frac{(1 + \cos^2 \epsilon)^2}{4} + \cos^2 \epsilon\right]^{1/2} , \quad \tan \eta = \frac{2 \cos \epsilon}{1 + \cos^2 \epsilon} , \quad (5b)$$

with  $\eta \in (-\pi/4, \pi/4)$ . The signal at the  $I$ -th detector can be expressed in terms of the quantities defined above as

$$s_{(I)}(t) = o_{(I)+} h_{(I)+}(t) + o_{(I)\times} h_{(I)\times}(t) . \quad (6)$$

where  $h_{(I)+}$  and  $h_{(I)\times}$  are the two polarizations of the wave arriving at the  $I$ -th detector. Also,  $o_{(I)+}$  and  $o_{(I)\times}$  are the beam pattern functions, which depend on  $\{\phi, \theta, \psi\}$  and the orientation angles  $\{\alpha_{(I)}, \beta_{(I)}, \gamma_{(I)}\}$ . The problem with the above representation for the signal is that it mixes up the factors dependent on the detector specific Euler angles  $\{\alpha_{(I)}, \beta_{(I)}, \gamma_{(I)}\}$  and those dependent on the angles  $\{\phi, \theta, \psi\}$ . We now give a different representation of the signal where this problem does not occur. It will prove useful to address the detection problem for a network.

Consider the complex null vector  $\mathbf{M} \equiv 1/\sqrt{2}(\mathbf{e}_X + i\mathbf{e}_Y)$ , where  $\mathbf{e}_X$  and  $\mathbf{e}_Y$  are real unit vectors in the  $X$  and  $Y$  directions, respectively, of the wave-frame. Then the wave tensor  $w_{ij}$  is defined as

$$w_{ij} = h_+ \text{Re}(M_i M_j) + h_\times \text{Im}(M_i M_j) , \quad (7)$$

which is a real symmetric trace-free (STF) tensor. The components of  $\mathbf{M}$  in the detector axis are [6]:  $\mathbf{M} = 1/\sqrt{2}(\cos \phi - i \cos \theta \sin \phi, \sin \phi + i \cos \theta \cos \phi, i \sin \theta) \exp(-i\psi)$ . A detector can also be represented as an STF tensor. For an interferometer with arms along the directions  $\mathbf{n}_1$  and  $\mathbf{n}_2$  (both being unit vectors), the detector tensor  $\mathbf{d}$  is given by

$$d_{ij} = n_{1i} n_{1j} - n_{2i} n_{2j} . \quad (8)$$

The response amplitude of the detector or, equivalently, the signal is just the scalar product of the wave and detector STF tensors,

$$s = w^{ij} d_{ij} , \quad (9)$$

where it is implicit that the Einstein summation convention holds over the repeated upper and lower indices  $i$  and  $j$ . In the following analysis, we will specifically consider

a network of interferometric detectors. However, the generalization to bar detectors is straightforward.

Since we extensively deal with STF tensors of rank 2, enunciating some frequently used properties of such objects is in order [1a]. Since such tensors have five independent elements, they can be expanded in terms of (location-independent) ‘STF- $l$ ’ tensors,  $\mathcal{Y}_{lm}^{ij}$ , with rank  $l = 2$ . They are related to the spherical harmonics as follows:

$$Y_{2m}(\theta, \phi) = \mathcal{Y}_{2m}^{ij} n_i n_j, \quad \text{where } m = \pm 2, \pm 1, 0, \quad (10)$$

where  $\mathbf{n} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta)$ . There are five independent  $3 \times 3$  complex matrices,  $\mathcal{Y}_{2m}^{ij}$ , obeying the normalization

$$\mathcal{Y}_{2m}^{ij} \mathcal{Y}_{ij}^{2m'*} = \frac{15}{8\pi} \delta_m^{m'}. \quad (11)$$

In this paper, we will also be interested in the behavior of these STF tensors under action of an element  $g(\alpha, \beta, \gamma)$  of the rotation group  $\text{SO}(3)$ , where  $(\alpha, \beta, \gamma)$  are the Euler angles. Consequently, we mention that under such an action, the spherical harmonics obey the following transformation law:

$$Y_{2m'}(\theta', \phi') = \sum_{m=-2}^2 T_{-m'}^{-m}(\alpha, \beta, \gamma) Y_{2m}(\theta, \phi), \quad (12)$$

where  $T_{-m'}^{-m}$  are the Gel'fand functions of rank 2 [12,6].

As shown in Ref. [6], in the detector frame the wave tensor  $w_{ij}$  can be expanded in terms of the STF-2 tensors as:

$$w^{ij}(t) = \sqrt{\frac{2\pi}{15}} [h_+(t)(T_2^n + T_{-2}^n) - i h_\times(t)(T_2^n - T_{-2}^n)] \mathcal{Y}_{2n}^{ij}, \quad (13)$$

where the expansion coefficients are combinations of the Gel'fand functions, which depend on the parameters  $(\phi, \theta, \psi)$ . For interferometric detectors with arms making an angle of  $2\Omega$ , the only non-vanishing detector-tensor components in its own frame are  $d_{12} = d_{21} = \sin 2\Omega$ . The following analysis, where we will deal with the case  $\Omega = \pi/4$ , can be easily generalized to other values of  $\Omega$ .

Consider the  $I$ -th detector of a network. Using the wave- and detector-tensor components in eq. (9), it was shown in Ref. [6] that the signal takes the form:

$$s_{(I)}(t) = I_{(I)n}^m T_m^s(\phi, \theta, \psi) T_{(I)s}^{n*}(\alpha_{(I)}, \beta_{(I)}, \gamma_{(I)}), \quad (14)$$

Note that  $T_{(I)m}^n$  is to be distinguished from  $T_m^n$  in that  $T_m^n = T_m^n(\phi, \theta, \psi)$ , whereas  $T_{(I)m}^n = T_m^n(\alpha_{(I)}, \beta_{(I)}, \gamma_{(I)})$ . Above,

$$I_{(I)n}^m = \pm \frac{i}{2} (h_{+(I)} - i h_{\times(I)}) \equiv \pm J_{(I)} \quad \text{for } m = 2, \quad n = \pm 2, \quad (15a)$$

$$I_{(I)n}^m = \pm \frac{i}{2} (h_{+(I)} + i h_{\times(I)}) = \mp J_{(I)}^* \quad \text{for } m = -2, \quad n = \pm 2. \quad (15b)$$

The signal given in (14) has the advantage of keeping factors dependent on the two sets of Euler angles separate.

### 3. A new representation for the signal

To address the detection problem, we will be directly dealing with a construct known as the likelihood ratio (LR) (to be defined in §4.). The LR is a non-linear functional of the signal. To keep this functional form simple we develop a new representation for the signal based on (14).

In terms of  $J_{(I)}$ , the signal given in eq. (14) takes the form

$$s_{(I)} = 2\text{Re}(iJ_{(I)}^* \Gamma_{(I)}) \quad , \quad (16)$$

where  $\Gamma_{(I)} = i \left( T_{(I)s}^{2*} - T_{(I)s}^{-2*} \right) T_{-2}^s$ . Next, we define:

$$\begin{aligned} h_{c(I)} &= a_{(I)}^{-1/4}(t) \cos \chi_{(I)}(t) \equiv g_{(I)} s_{(I)0} \quad , \\ h_{s(I)} &= a_{(I)}^{-1/4}(t) \sin \chi_{(I)}(t) \equiv g_{(I)} s_{(I)\pi/2} \quad . \end{aligned} \quad (17)$$

where  $g_{(I)}$  is the maximum signal-to-noise ratio for the  $I$ -th detector obtainable by using an optimal filter:

$$g_{(I)}^2 \equiv \int df \frac{|\tilde{h}_{c,s}^{(I)}(f)|^2}{s_h^{(I)}(f)} \quad , \quad (18)$$

$s_h^{(I)}(f)$  being the noise power-spectral-density (p.s.d.) of the  $I$ -th detector. Above,  $s_{(I)0}$  and  $s_{(I)\pi/2}$  can be represented as the real and imaginary parts of a complex quantity:

$$S_{(I)}(t) = s_{(I)0} + i s_{(I)\pi/2} \quad , \quad (19)$$

which has a norm equal to 2.

Let us now express  $\Gamma_{(I)} = \gamma_{(I)0} + i\gamma_{(I)\pi/2}$ , where  $\gamma_{(I)0}$  and  $\gamma_{(I)\pi/2}$  are, respectively, the real and imaginary parts of  $\Gamma_{(I)}$ . Define

$$W_{(I)} \equiv g_{(I)} \left( \gamma_{(I)0} \cos \eta - i\gamma_{(I)\pi/2} \sin \eta \right) \equiv w_{(I)} e^{-i\omega_{(I)}} \quad , \quad (20)$$

where the last expression is the polar form of  $W_{(I)}$ . Note that the only signal parameters on which  $W_{(I)}$  depends are  $\{\phi, \theta, \psi, \eta\}$ . Armed with these definitions, the signal in eq. (16) can be re-expressed as

$$s_{(I)}(t) = 2\kappa \text{Re} \left( W_{(I)}^* R_{(I)} \right) \quad , \quad (21)$$

where we have defined

$$R_{(I)}(t) = r_{(I)0} + i r_{(I)\pi/2} \equiv S_{(I)} e^{i\delta} \quad , \quad (22)$$

which is just the rotated  $S_{(I)}$ . Here  $r_{(I)0}$  and  $r_{(I)\pi/2}$  are, respectively, the real and imaginary parts of  $R_{(I)}$ . Equation (21) is a new representation for the signal that we will find useful in obtaining the maximum likelihood ratio below.

We end this section by arriving at a relation between the complex variable  $W_{(I)}$  and the detector tensor. First, by using eqs (13) and (22) we obtain the inner products between the wave tensor  $w^{ij}$  and the real and imaginary components of  $R_{(I)}$ :

$$2\kappa g_{(I)}\alpha^{ij} = \langle w^{ij}(t), r_{(I)0} \rangle \quad \text{and} \quad 2\kappa g_{(I)}\beta^{ij} = \langle w^{ij}(t), r_{(I)\pi/2} \rangle \quad (23)$$

where we have defined two new STF tensors  $\alpha_{ij}$  and  $\beta_{ij}$ . These are real functions of  $\{\phi, \theta, \psi, \eta\}$ .

By comparing the different representations (9) and (21) of the signal we obtain

$$W_{(I)} = \Delta^{ij} (g_{(I)} d_{(I)ij}) , \quad \text{where} \quad \Delta^{ij} \equiv (\alpha^{ij} + i\beta^{ij}) , \quad (24)$$

$\Delta^{ij}$  is a complex STF tensor dependent on  $\{\phi, \theta, \psi, \eta\}$ , when expressed in the fide frame. Note that

$$|W_{(I)}|^2 = |\alpha^{ij} (g_{(I)} d_{(I)ij})|^2 + |\beta^{ij} (g_{(I)} d_{(I)ij})|^2 . \quad (25)$$

Above, up to an  $r$ -dependent factor,  $|W_{(I)}|^2$  can be interpreted as the total power transferred to the  $I$ -th detector. More appropriately, it is the gain factor associated with the  $I$ -th detector. We have resolved it as a sum of the fractions of power transferred to the detector by the two polarizations, respectively. It can be shown that, up to a factor of  $g_{(I)}$ ,  $W_{(I)}$  is just a direction cosine that is dependent on the set of angles  $\{\phi, \theta, \psi, \eta\}$ . This is what one would expect from the above interpretation of  $W_{(I)}$  as a gain factor.

#### 4. Addressing the detection problem for a network

The signal from a coalescing binary will typically not stand above the broadband noise of the interferometric detectors; the concept of an absolutely certain detection does not exist in such a case. Only probabilities can be assigned to the presence of an expected signal. In the absence of prior probabilities, such a situation demands a decision strategy that maximizes the detection probability for a given false alarm probability. This is termed as the Neyman–Pearson criterion (see, eg., ref. [14]). Such a criterion implies that the decision must be based on the value of a statistic called the likelihood ratio (LR). It is defined as the ratio of the probability that a signal is present in an observation to the probability that it is not.

For a network of detectors we obtain this statistic as follows. We assume that the noise at each detector is additive, Gaussian, and both statistically as well as algebraically independent of the noise in any other detector in the network. Under these conditions, the network LR, denoted by  $\lambda$ , is just a product of the individual detector LR's. Similarly, the logarithmic likelihood ratio (LLR),  $\ln \lambda$ , can be verified to have the same form as for an individual detector, namely, [14,9].

$$\ln \lambda = \langle \mathbf{s}, \mathbf{x} \rangle_{NW} - \frac{1}{2} \langle \mathbf{s}, \mathbf{s} \rangle_{NW} , \quad (26)$$

where the normalized set of signals are denoted by a single network vector

$$\mathbf{s}(t) = (s_{(1)}(t), s_{(2)}(t), \dots, s_{(N)}(t)) , \quad (27)$$

$N$  being the number of detectors in the network. The subscript  $NW$  denotes that the inner product is defined on the *network* space. Similarly, the individual detector outputs  $x_{(I)}(t)$  are combined to form the network vector

$$\mathbf{x}(t) = (x_{(1)}(t), x_{(2)}(t), \dots, x_{(N)}(t)) . \quad (28)$$

Thus, in terms of the individual detector signals, the LLR is

$$\begin{aligned} \ln \lambda &= \sum_{I=1}^N \langle s_{(I)}, x_{(I)} \rangle_{(I)} - \frac{1}{2} \sum_{I=1}^N \langle s_{(I)}, s_{(I)} \rangle_{(I)} \\ &= \mathbf{b} \sum_{I=1}^N \langle z_{(I)}, x_{(I)} \rangle_{(I)} - \frac{1}{2} \mathbf{b}^2 , \end{aligned} \quad (29)$$

where

$$\mathbf{b} \equiv 2\kappa \sqrt{\sum_{I=1}^N w_{(I)}^2} , \quad (30)$$

is the norm of  $\mathbf{s}$  and  $z_{(I)} = s_{(I)}/\mathbf{b}$ . The aim now is to maximize the LLR over all eight parameters to obtain the maximum (logarithmic) likelihood ratio (MLR). It is the MLR that must be compared with a threshold value to ascertain the presence or absence of signal in the detector output, with a given level of confidence.

We now analytically maximize the above expression with respect to as many of the eight parameters as possible. Note that the luminosity distance  $r$  appears only through  $\mathbf{b}$  in LLR. Maximizing it with respect to  $\mathbf{b}$  yields

$$\ln \lambda|_{\mathbf{b}} = \frac{1}{2} \left( \sum_{I=1}^N \langle z_{(I)}, x_{(I)} \rangle_{(I)} \right)^2 = \frac{1}{2} \left| \operatorname{Re} \left( e^{-i\delta} \sum_{I=1}^N Q_{(I)} C_{(I)}^* \right) \right|^2 , \quad (31)$$

where we have defined

$$Q_{(I)} \equiv \frac{W_{(I)}}{\sqrt{\sum_{I=1}^N w_{(I)}^2}} \quad \text{and} \quad C_{(I)}^* \equiv \langle S_{(I)}, x_{(I)} \rangle_{(I)} . \quad (32)$$

Next we maximize the LLR in eq. (31) with respect to  $\delta$  for, apart from the phase factor, none of the other terms there depend on it. This gives

$$\ln \lambda|_{\mathbf{b}, \delta} = \frac{1}{2} \left| \sum_{I=1}^N Q_{(I)} C_{(I)}^* \right|^2 , \quad (33)$$

which is a function of six parameters, namely,  $\{\phi, \theta, \psi, \eta, t_a, \xi\}$ . Note that when all the detectors are ‘closely’ located, it is only the  $Q_{(I)}$ ’s that depend on four angles  $\{\phi, \theta, \psi, \eta\}$ ; the  $C_{(I)}$ ’s then depend only on  $\{t_a, \xi\}$ , with all the times of arrival being equal. We will refer to this situation as the ‘same-site’ approximation. When the detectors are spatially well separated, the  $C_{(I)}$ ’s will depend on  $\{\phi, \theta\}$  as well.

To obtain the MLR, we need to maximize over these remaining parameters. At this stage it is useful to define the surrogate statistic (SS),  $\lambda' \equiv \ln \lambda|_{\mathbf{b}, \delta}$ . For a network comprising of a total of  $N$  detectors located within a fraction of a wavelength, the SS is maximum when  $Q \propto C$ . where  $C$  is a network vector with the  $C_{(I)}$ ’s as its components. Therefore,



once  $\mathbf{C}$  is known, the maximization procedure determines  $\mathbf{Q}$  through the above condition and the fact that  $\mathbf{Q}$  has a unit norm. However, the  $\mathbf{Q}$  so determined will, in general, yield an overdeterministic set of equations for the four parameters  $\{\phi, \theta, \psi, \eta\}$ . On the other hand, if this set of equations can be solved to yield a physically realizable solution for the parameters, then the maximized LLR will have a simpler form:

$$\lambda'_{|\hat{\phi}, \hat{\theta}, \hat{\psi}, \hat{\eta}} = \frac{1}{2} \sum_{I=1}^N c_{(I)}^2 \equiv \frac{\Lambda}{2}, \quad (34)$$

where  $c_{(I)}$  is the magnitude of  $C_{(I)}$ . Above,  $\Lambda$  is a function of two parameters, namely,  $\{t_a, \xi\}$ . Although  $\Lambda$  is a real quantity, we follow the established convention in literature to denote the LLR by an uppercase letter!

It can be shown that the condition  $\mathbf{Q} \propto \mathbf{C}$  is always realised for two detectors. Numerical calculations suggest that this result holds for three detectors as well [17]. However, for networks with a larger number of detectors we numerically find that this condition is not always realisable and one is forced to maximize  $\lambda'_{|\hat{\delta}, \hat{\delta}}$ , as given in eq. (33), over the four angles. Thus, for networks comprised of up to three detectors, the application of eq. (34) appears to be valid. We will limit our discussion to only such cases below. Hence, only the maximization of  $\Lambda$  over the two parameters,  $\{t_a, \xi\}$ , remains to be done. This is performed numerically along the lines of Sathyaprakash and Dhurandhar (see ref. [11]).

### 5. Network sensitivity

To infer the presence of a signal from the outputs of the members of a network, one compares the value of the statistic  $\Lambda$  in eq. (34) with a predetermined detection threshold  $\Lambda_0$ . As we show below, the value of  $\Lambda_0$  can be obtained (via the Neyman–Pearson decision criterion [14]) from the false alarm probability,  $Q_0$ , associated with the event of detection of such a signal. For  $\Lambda < \Lambda_0$ , presence of a signal in the data is ruled out, whereas if  $\Lambda > \Lambda_0$ , then the detection of a signal in the data is announced.

We now analyze the improvement in the sensitivity of a network over that of a single detector. Apart from the assumptions about detector noise mentioned in §4, we further assume that it is stationary, which implies that for the  $I$ -th detector we have

$$\langle \tilde{n}_{(I)}(f) \tilde{n}_{(I)}^*(f') \rangle = s_{h(I)} \delta(f - f'), \quad (35)$$

where  $s_{h(I)}$  is the noise p.s.d. of that detector and the angular brackets imply ensemble average. In general, different detectors may have non-identical noise p.s.d. We will assume that the noise in any detector is white, i.e.,  $s_{h(I)}$  does not vary with frequency. Note that it has zero mean, i.e.,  $\langle \tilde{n}_{(I)}(f) \rangle = 0$ , and its standard deviation is  $s_{h(I)}$ .

Equation (34) shows that  $\Lambda$  is a sum of squares of independent random variables with Gaussian probability distribution functions (PDF). Thus,  $\Lambda$  itself must have the so-called  $\chi^2$  probability distribution. Hence, the PDF of  $\Lambda$  under the hypothesis that the signal is present,  $H_1$ , is:

$$p_1(\Lambda) = \frac{1}{2} \left( \frac{\Lambda}{b^2} \right)^{(N-1)/2} \exp[-(\Lambda + b^2)/2] I_{N-1}(b\sqrt{\Lambda}), \quad (36)$$

where  $\Lambda > 0$  and  $I_{N-1}(x)$  is the modified Bessel function of order  $(N - 1)$ . Note that  $b^2$  is proportional to the total energy received from the source. When  $b\sqrt{\Lambda} \gg 1$ , the above expression approximates to

$$p_1(\sqrt{\Lambda}) = \frac{1}{\sqrt{2\pi}} \exp[-(\sqrt{\Lambda} - b)^2/2]. \tag{37}$$

On the other hand, under the hypothesis that the signal is absent,  $H_0$ , the PDF of  $\Lambda$  is

$$p_0(\Lambda) = \frac{(\Lambda/2)^{N-1} e^{-\Lambda/2}}{2(N-1)!}, \tag{38}$$

which one can obtain from eq. (36) by taking the limit  $b \rightarrow 0$ .

We are now in a position to calculate the false-alarm probability:

$$Q_0 = \int_{\Lambda_0}^{\infty} p_0(\Lambda) d\Lambda = 1 - I(\Lambda_0/2\sqrt{N}, N-1), \tag{39}$$

where we have made use of the incomplete gamma-function

$$I(u, p) = \int_0^{u\sqrt{p+1}} x^p e^{-x} dx / p!. \tag{40}$$

For a given false-alarm probability  $Q_0$ , eq. (39) allows us to obtain the detection threshold  $\Lambda_0$  of our statistic. Plots of  $Q_0$  versus  $\Lambda_0$  for different values of  $N$  are given in figure 1. From these plots it can be inferred that  $\Lambda_0$  increases slowly with  $N$ .

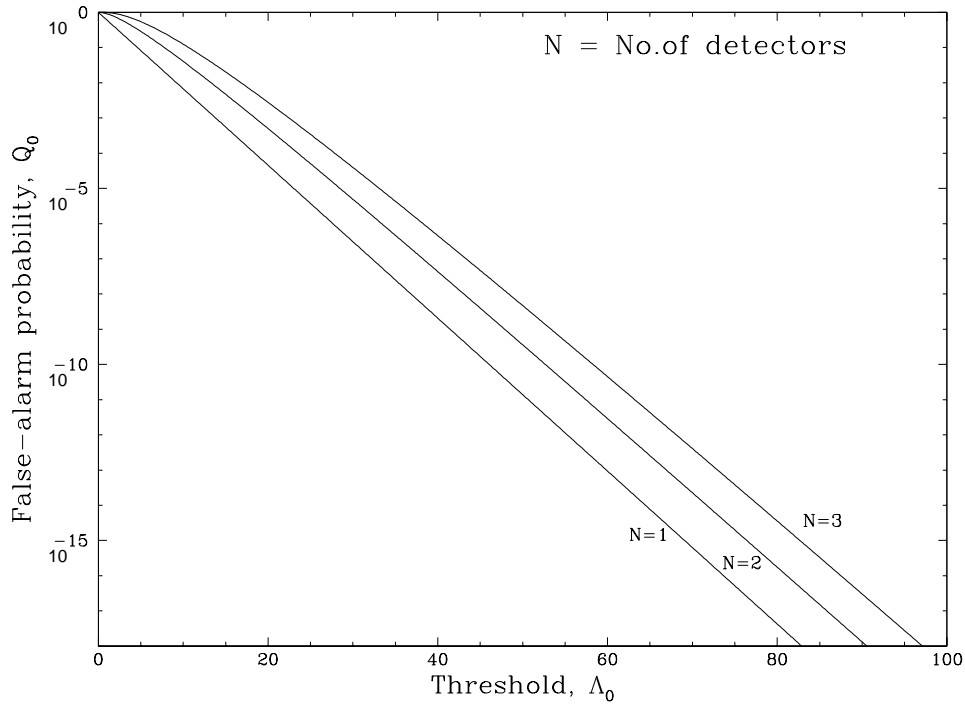
The detection probability  $Q_d$  can be obtained by computing the area under the function  $p_1(\sqrt{\Lambda})$  for  $\sqrt{\Lambda} > \sqrt{\Lambda_0}$ . For  $b\sqrt{\Lambda} \gg 1$ , it is

$$Q_d = \int_{\sqrt{\Lambda_0}}^{\infty} p_1(\sqrt{\Lambda}) d(\sqrt{\Lambda}), \tag{41}$$

where  $p_1$  is given by eq. (37). We now show that for a given  $Q_d$  and  $Q_0$ , the distance  $r$  up to which a network can probe increases with  $N$ . This is tantamount to saying that the sensitivity of a network increases as a function of  $N$ . For simplicity, assume that the detectors are oriented in such a way that  $b^2$  is proportional to  $N$ . This is the case when, e.g., the  $w_{(I)}$ 's are all identical. Let  $Q_d = 0.5$ , i.e.,  $\sqrt{\Lambda_0} = b$ . As  $N$  increases,  $\Lambda_0$  increases for a given  $Q_0$  (see figure 1). However, given the fact that  $b \propto \sqrt{N}/r$ , we have  $r \propto \sqrt{N}/\Lambda_0$ . Thus, given a specific binary, the ratio of sensitivity of a network to that of a single detector behaves as

$$\frac{r(N)}{r(N=1)} = \sqrt{N} \left( \frac{\Lambda_0(N=1)}{\Lambda_0(N)} \right)^{1/2}, \tag{42}$$

which can be computed by using  $\Lambda_0(N)$  from figure 1. These ratios, which are presented in table 1. for  $N = 2, 3$ , clearly show that the sensitivity of such a network increases a little slower than  $\sqrt{N}$ . This implies an increase in the survey volume accessible to a network, which, in turn, implies an increased event rate.



**Figure 1.** The plot of  $Q_0$  as a function of the detection threshold  $\Lambda_0$  for different number,  $N$ , of closely located detectors in a network.

**Table 1.** The ratio of the sensitivity of a network with  $N = 2, 3$ , respectively, relative to a single detector for  $Q_d = 0.5$ , and corresponding to different false-alarm probabilities.

No. of detectors, $N$	False-alarm probability, $Q_0$			
	$0.33 \times 10^{-10}$	$0.67 \times 10^{-12}$	$0.17 \times 10^{-12}$	$0.33 \times 10^{-14}$
2	1.33	1.33	1.33	1.34
3	1.55	1.56	1.57	1.58

## 6. Conclusion

The problem of detecting inspiral waveforms from coalescing binaries via pattern-matching can be made more accurate by including post-Newtonian corrections in the corresponding filters. In this regard, it has been shown [15] that it is both necessary and sufficient to work with the *restricted* post-Newtonian chirp. The description of the resulting waveform involves an extra parameter, apart from the set of eight parameters described above. However, as was shown by Sathyaprakash [16], for the astrophysically

relevant range of parameters, the effective dimensionality of the parameter space remains unchanged. Hence, even after the inclusion of the restricted post-Newtonian corrections in the filters, the detection problem can be addressed in the same manner as described in §4.

When the detectors are spatially well separated, the cross-correlations,  $C_{(I)}$ 's, will be dependent on the times of arrival,  $t_{a(I)}$ 's, which are different from one another. Since a specific  $t_{a(I)}$  depends on the location angles  $\{\phi, \theta\}$ , so will  $C_{(I)}$ . Hence the maximization of the SS over the four angles  $\{\phi, \theta, \psi, \eta\}$  that was performed for the same-site approximation above, can no longer be implemented in the present case. It can be shown that the SS can be recast in such a way that its dependence on the complementary angles  $\{\psi, \eta\}$  is isolated. This aids in the analytic maximization of the SS over these two angles. The maximization over the remaining four parameters  $\{\phi, \theta, t_a, \xi\}$  can then be effected numerically on a four-dimensional parameter-space grid. Details of these calculations will be presented elsewhere [17].

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