Necessary and Sufficient Classicality Conditions on Photon Number Distributions

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We exploit classical results on the Stieltjes moment problem to obtain completely explicit necessary and sufficient conditions for the photon number distribution of a radiation field mode to be classical. These conditions are given in two forms - respectively local and global in the individual photon number probabilities. Equivalence of the two approaches is demonstrated. Detailed quantitative statements on oscillations in the photon number probabilities are also presented.

I. INTRODUCTION

The study of nonclassical aspects of radiation from a variety of viewpoints is of considerable and continuing interest. On the conceptual side one can distinguish between phase sensitive signatures of nonclassicality and phase insensitive ones. The most familiar of the former is quadrature squeezing [1,2], while in the latter we have amplitude squeezing or subpoissonian photon statistics [3,4]. Higher order squeezing [5] as well as conditions on the factorial moments [6] of the photon number distribution (PND) have also been presented as sufficient conditions for nonclassicality. In some cases, the single mode treatments have been generalised to the two or general multimode situations [7].

From another point of view, a finer classification of states of quantised radiation according to increasing nonclassicality has been given, stressing the connection to the specific classes of observables being measured [8]. The case of Gaussian-Wigner distributions for a single mode field has been analysed in detail to give a concrete illustration of these ideas [9].

In the present paper we give a complete treatment of phase insensitive measurements on the single mode quantized radiation field, and develop *necessary and sufficient* conditions for nonclassicality of the field. Our focus is on the PND, for any given state, which contains all information concerning all phase insensitive measurements. There are several previous partial results in this direction. The best known is the Mandel Q-parameter criterion for distinguishing globally between sub and super poissonian statistics. This is *global* in the sense that the evaluation or measurement of the parameter Q in any given case requires knowledge of the probability for finding n photons in the state, *for all* n. We may also mention the important generalisation of the Mandel Q parameter achieved by Agarwal and Tara [6]. Our principal motivation is to look for *local* conditions on the PND, involving only a *finite* number of photon probabilities, which will serve as signatures to distinguish between classical and nonclassical states. This approach or viewpoint allows us to give a rather detailed analysis of oscillations [10,11] in the PND, and to see precisely which kinds of oscillations stem from nonclassicality and which do not. As for the fact that we are able to obtain *necessary and sufficient* conditions for classicality, the principal tool we employ is the solution to the Stieltjes moment problem [12] in the classical theory of moments - when can a given sequence of moments arise from a well-defined probability distribution? The Stieltjes problem deals with the case when the probability density is defined over the half line $[0, \infty)$. In our context, this corresponds to the range of the intensity variable of light.

The material of the paper is arranged as follows. Section II collects basic definitions related to the diagonal coherent state representation of a single mode field, and discusses the classical - nonclassical divide from several angles. The final one chosen is that suited to phase insensitive measurements. Both the probabilities appearing in the PND, and the factorial moments of the PND, are recognised as moments of two auxiliary distributions over $[0, \infty)$, one with a clear physical meaning and the other a formal one. In Section III we give the basic method for obtaining local classicality conditions on the PND and obtain explicitly a minimal three term condition as an example. This condition involves the photon number probabilities p_n for three successive values of n, in contradistinction to the condition in terms of the Mandel Q parameter which involves three successive (the zeroth moment which equals unity in every state, and the first and second) moments of p_n. The keen reader will notice indications of an interesting duality already at this initial stage. We then put together a series of remarks and preliminary results based on these minimal conditions, which give a good orientation towards understanding the features of the PND from both local and global points of view. In particular we are able to discriminate between oscillations in the PND which genuinely reflect nonclassicality and others which do not.

Section IV develops the complete necessary and sufficient conditions for classicality of the PND, by mapping the present problem on to the Stieltjes moment problem which is very well known and extensively studied. All the conditions obtained here are local in the sense mentioned above. From the necessary and sufficient conditions we then extract the conditions for classicality which go one step beyond the minimal ones discussed in Section II; this helps us sharpen some of the conclusions reached earlier.

Section V gives an alternative or dual approach based on the factorial moments of the PND. This again is cast into the form of a Stieltjes problem, and the necessary and sufficient conditions for classicality are similar to the previous ones in structure. Section VI compares and connects the two treatments, and Section VII contains concluding remarks.

II. BASIC DEFINITIONS, PRELIMINARY REMARKS AND RESULTS

We consider a single mode quantized radiation field with photon creation and annihilation operators $\hat{a}^\dagger, \hat{a}$ obeying the standard commutation relation

$$[\hat{a}, \hat{a}^\dagger] \equiv \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1. \tag{2.1}$$

A general (pure or mixed) state of the field is described by a hermitian nonnegative density operator $\hat{\rho}$ with unit trace. According to the diagonal coherent state representation theorem, $\hat{\rho}$ can be expanded as an integral over projections on to the coherent states:

$$\hat{\rho} = \int \frac{d^2z}{\pi} \phi(z) |z\rangle\langle z| \ , \tag{2.2}$$

the integration being over the entire complex plane. Here the coherent state $|z\rangle$ is the normalized right eigenstate of $\hat{a}$ with (generally complex) eigenvalue z, related to the number operator eigenstates (Fock states) $|n\rangle$ in the standard manner:

$$\begin{aligned} |z\rangle &= \exp\left(-\frac{1}{2}|z|^2\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle \\ &= \exp\left(-\frac{1}{2}|z|^2 + z\hat{a}^\dagger\right)|0\rangle \ , \\ |n\rangle &= \left(\hat{a}^\dagger\right)^n |0\rangle/\sqrt{n!} \ ; \\ \hat{a}|z\rangle &= z|z\rangle \ , \ \hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle \ . \end{aligned} \tag{2.3}$$

The weight function $\phi(z)$ in eqn.(2.2) is real and normalised to unit integral:

$$\int \frac{d^2z}{\pi} \phi(z) = 1 \ . \tag{2.4}$$

However it is in general not pointwise nonnegative [13], and can be quite a singular quantity, namely a member of a certain precisely defined class of distributions over the plane.

Any (hermitian) observable can always be written as a function of $\hat{a}^\dagger$ and $\hat{a}$ in normal ordered form, $F\left(\hat{a}^\dagger, \hat{a}\right)$ say [14]. Its expectation value in the state $\hat{\rho}$ is then given by

$$\begin{aligned} \langle F\left(\hat{a}^\dagger, \hat{a}\right)\rangle &= \mathrm{Tr}\left(\hat{\rho} F\left(\hat{a}^\dagger, \hat{a}\right)\right) \\ &= \int \frac{d^2z}{\pi} \phi(z) F(z^*, z) \ . \end{aligned} \tag{2.5}$$

If in particular $F\left(\hat{a}^\dagger, \hat{a}\right)$ is phase invariant, then its expectation value does not require all the "information" contained in $\phi(z)$, and a simpler angle averaged auxiliary distribution $\mathcal{P}(I)$ suffices:

$$\begin{aligned} F\left(\hat{a}^\dagger e^{i\alpha}, \hat{a}e^{-i\alpha}\right) &= F\left(\hat{a}^\dagger, \hat{a}\right) \Rightarrow \\ \langle F\left(\hat{a}^\dagger, \hat{a}\right)\rangle &= \int_0^\infty dI \mathcal{P}(I) F\left(I^{1/2}, I^{1/2}\right) \ , \\ \mathcal{P}(I) &= \int_0^{2\pi} \frac{d\theta}{2\pi} \phi\left(I^{1/2} e^{i\theta}\right) \ , \\ \int_0^\infty dI \mathcal{P}(I) &= 1. \end{aligned} \tag{2.6}$$

We can regard $\mathcal{P}(I)$ as a real normalised marginal radial distribution function obtained from the complete $\phi(z)$. Note that $F(\hat{a}^\dagger, \hat{a})$ being phase invariant is the same as it being a function of the number operator $\hat{a}^\dagger\hat{a}$.

An important set of phase invariant F's leads to the photon number distribution (PND) or photon number probabilities $\{p_n\}$ in the state $\hat{\rho}$:

$$\begin{aligned} F_n\left(\hat{a}^\dagger, \hat{a}\right) &= : \ e^{-\hat{a}^\dagger\hat{a}} \frac{\left(\hat{a}^\dagger\hat{a}\right)^n}{n!} \ : = |n\rangle\langle n| \Rightarrow \\ p_n &= \langle F_n\left(\hat{a}^\dagger, \hat{a}\right)\rangle \\ &= \langle n|\hat{\rho}|n\rangle \\ &= \int_0^\infty dI \mathcal{P}(I) e^{-I} I^n/n! \quad , n = 0, 1, 2, \ldots \ . \end{aligned} \tag{2.7}$$

(The colons denote normal ordering). These p_n are always well-defined for any bonafide $\hat{\rho}$ and always obey the laws for a discrete probability distribution:

$$p_n \geq 0, \quad \sum_{n=0}^{\infty} p_n = 1. \tag{2.8}$$

With the two quantities $\phi(z)$ and $\mathcal{P}(I)$ in hand, we can set up a three-fold classification of states $\hat{\rho}$ of steadily increasing nonclassicality. Conventionally $\hat{\rho}$ is said to be classical if $\phi(z)$ itself is a probability distribution, that is, pointwise nonnegative and nowhere more singular than a delta function; otherwise it is nonclassical. In the former case it follows that $\mathcal{P}(I)$ also can be interpreted as a probability distribution for the intensity. In the latter case, a further refinement is possible based on the properties of $\mathcal{P}(I)$ and we arrive at the following scheme:

$$\begin{aligned} \hat{\rho} \text{ classical} &\Leftrightarrow \phi(z), \mathcal{P}(I) \geq 0 \; ; \\ \hat{\rho} \text{ weakly nonclassical} &\Leftrightarrow \phi(z) \not\geq 0 \; , \; \mathcal{P}(I) \geq 0 \; ; \\ \hat{\rho} \text{ strongly nonclassical} &\Leftrightarrow \phi(z) \not\geq 0 \; , \; \mathcal{P}(I) \not\geq 0 \; . \end{aligned} \tag{2.9}$$

This is an exhaustive and mutually exclusive classification. The special role or significance of the PND $\{p_n\}$ can now be expressed as follows: *even if $\mathcal{P}(I)$ is not a well-defined probability distribution and we are in the strongly nonclassical regime, the quantity $\mathcal{P}(I)e^{-I}$ always has well-defined finite moments $n!p_n$ for all n.* For ease in the following we introduce a special symbol for these moments,

$$\begin{aligned} q_n = n!p_n &= \int_0^\infty dI \tilde{\mathcal{P}}(I) \; I^n \; , \\ \tilde{\mathcal{P}}(I) &= \mathcal{P}(I) e^{-I} \; , \end{aligned} \tag{2.10}$$

the dependence on $\hat{\rho}$ being left implicit. On the other hand, the PND cannot discriminate between the classical and the weakly nonclassical cases in eqn.(2.9), in both of which $\mathcal{P}(I) \geq 0$. Indeed, given a PND $\{p_n\}$ leading to a pointwise nonnegative $\mathcal{P}(I)$ (see below), if no phase sensitive quantities are to be measured we may take

$$\phi(z) = \mathcal{P}(|z|^2), \tag{2.11}$$

and check that via eqn.(2.2) we get a physically acceptable density operator $\hat{\rho}$.

An interesting illustration of the weakly nonclassical case of eqn.(2.9) is the one-parameter family of states that results when a coherent state $|z_0\rangle$ evolves through a Kerr medium for a time interval t. Indeed, for suitable t the state that results is the Yurke-Stoler state [15]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|z_0\rangle \pm i| - z_0\rangle) \; . \tag{2.12}$$

This being the superposition of two coherent states has a $\phi(z)$ more singular than a tempered distribution. Nevertheless when angle averaged it leads to the same $\mathcal{P}(I)$ (and hence the same PND) as for the single coherent state $|z_0\rangle \; : \; \mathcal{P}(I) = \delta(I - |z_0|^2)$. The Hamiltonian of the Kerr medium being a function of $\hat{a}^\dagger \hat{a}$ leaves the diagonal matrix elements $p_n = \langle n|\hat{\rho}|n\rangle$ unaffected, but changes only the phases of $\langle m|\hat{\rho}|n\rangle$ for $m \neq n$, and $\mathcal{P}(I)$ depends only on the diagonal elements of $\hat{\rho}$ and not on the off-diagonal elements.

In the remainder of this work we shall be exclusively concerned with the PND $\{p_n\}$, or more generally with expectation values of phase insensitive observables alone. For these purposes we shall combinedly refer to the classical and weakly nonclassical cases of eqn.(2.9) as *classical* ($\mathcal{P}(I) \geq 0$); and to the strongly nonclassical in eqn.(2.9) as *nonclassical* ($\mathcal{P}(I) \not\geq 0$). That is, for phase insensitive observables, we define:

$$\begin{aligned} \hat{\rho} \text{ classical} &\Leftrightarrow \mathcal{P}(I) \geq 0 \; ; \\ \hat{\rho} \text{ nonclassical} &\Leftrightarrow \mathcal{P}(I) \not\geq 0 \; . \end{aligned} \tag{2.13}$$

Our principal aim now will be to develop *necessary and sufficient* conditions on the PND $\{p_n\}$ for $\hat{\rho}$ to be classical; we may then say for brevity that we have a classical PND.

The recent experiment of Munroe et al [16] best illustrates our considerations based on phase-insensitive measurements on the one hand and our classical-nonclassical divide (2.13) based on $\mathcal{P}(I)$ on the other. They show that the phase averaged quadrature amplitude distribution $\overline{P}(\xi)$ measured by their optical homodyne detection apparatus is invertibly related to the PND. We shall show presently that the relationship (2.7) between $\mathcal{P}(I)$ and the PND is invertible. It thus follows that their probability $\overline{P}(\xi)$ and our quasiprobability $\mathcal{P}(I)$ are invertibly related: their experiment extracts no more, and no less, information than contained in $\mathcal{P}(I)$. In other words, the classification (2.13), rather than (2.9), is the one relevant for such experiments.

The distribution character of $\phi(z)$ passes over to a corresponding property for $\mathcal{P}(I)$ - namely, it too is a member of a precisely defined class of distributions over $[0, \infty)$. While it is clear that the sequence $\{p_n\}$ cannot possibly capture all the information contained in $\phi(z)$ as all the off-diagonal matrix elements $\langle m|\hat{\rho}|n\rangle \; , \; m \neq n$, are ignored, we can easily show that the PND $\{p_n\}$ and the distribution $\mathcal{P}(I)$ determine each other uniquely. To recover $\mathcal{P}(I)$ from $\{p_n\}$ we define a generating function $\Lambda(K)$, $0 \leq K < \infty$, to represent the latter:

$$\begin{aligned} \Lambda(K) &= \sum_{n=0}^{\infty} (-K)^n p_n / n! \\ &= \int_0^\infty dI \mathcal{P}(I) \; e^{-I} J_0(2\sqrt{IK}) \; . \end{aligned} \tag{2.14}$$

There is a great deal of freedom in the way we set up $\Lambda(K)$; we have chosen it so that, among other things, on the basis of eqn.(2.8) it is entire analytic in K. One can then invert eqn.(2.14) by using the Fourier - Bessel integral theorem to get

$$\mathcal{P}(I) = e^{I} \int_0^\infty dK\ \Lambda(K) J_0(2\sqrt{KI})\ . \qquad (2.15)$$

Several signatures of nonclassicality of the PND are well known. The most familiar is the Mandel Q-parameter criterion which distinguishes (in a global sense) between super and subpoissonian PND's:

$$\begin{aligned} Q &= ((\Delta n)^2 - \langle n\rangle)/\langle n\rangle \\ &= (\langle n^2\rangle - \langle n\rangle^2 - \langle n\rangle)/\langle n\rangle\ , \\ \langle n\rangle &= \sum_{n=0}^{\infty} n\ p_n = \int_0^\infty dI\mathcal{P}(I) I\ , \\ \langle n^2\rangle &= \sum_{n=0}^{\infty} n^2 p_n = \int_0^\infty dI\mathcal{P}(I) I(I+1)\ ; \end{aligned}$$

$$\begin{aligned} &\text{classical PND} \Rightarrow Q \geq 0\ ,\ \text{superpoissonian statistics}\ , \\ &Q < 0 \Rightarrow \text{nonclassical PND,subpoissonian statistics}\ . \end{aligned} \qquad (2.16)$$

However one can see that there are (uncountably many) states $\hat{\rho}$ for which Q is undefined, on account of either $\langle n^2\rangle$ or both $\langle n\rangle$ and $\langle n^2\rangle$ being divergent. This can happen because Q involves p_n for all n, and the rate of decrease of p_n as $n \to \infty$ may be insufficient for convergence of either $\langle n\rangle$ or $\langle n^2\rangle$. A similar situation obtains with the factorial moments of the PND, namely

$$\begin{aligned} \gamma_m &= \langle \hat{a}^{\dagger m} \hat{a}^m \rangle \\ &= \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} p_n \\ &= \int_0^\infty dI\mathcal{P}(I)\ I^m\ ,\quad m = 0, 1, , 2, \ldots\ . \end{aligned} \qquad (2.17)$$

In terms of these, the Mandel parameter is

$$Q = \left(\gamma_2 - \gamma_1^2\right)/\gamma_1\ . \qquad (2.18)$$

A well-known feature of the factorial moments is the following [17]:

$$\begin{aligned} \text{Classical PND} \Rightarrow \gamma_{m'}\gamma_m \leq \gamma_{m'+m} \leq \left(\gamma_{2m'}\gamma_{2m}\right)^{1/2}\ , \\ m', m = 0, 1, 2, \ldots\ . \end{aligned} \qquad (2.19)$$

However, as with Q, γ_m also involves p_n for all but a finite number of values of n, and is undefined for the vast majority of states $\hat{\rho}$.

This discussion shows that in general, in contrast to $\tilde{\mathcal{P}}(I) = \mathcal{P}(I)\ e^{-I}$, the moments γ_m of $\mathcal{P}(I)$ may not all exist. States $\hat{\rho}$ for which all γ_m, or even all factorial moments upto some fixed maximum order, are finite are naturally quite severely restricted.

One may enquire about the properties of the ordinary moments of the PND, namely

$$\begin{aligned} \delta_m &= \langle \left(\hat{a}^\dagger \hat{a}\right)^m \rangle \\ &= \sum_{n=0}^{\infty} n^m p_n \\ &= \int_0^\infty dI\mathcal{P}(I)\ e^{-I} \sum_{n=0}^{\infty} n^m I^n/n! \\ &= \int_0^\infty dI\ \mathcal{P}(I)\ e^{-I}\ \left(I\ \frac{d}{dI}\right)^m\ e^I\ , \\ & m = 0, 1, 2, \ldots . \end{aligned} \qquad (2.20)$$

While they may seem to behave qualitatively like the γ_m, they are however not expressible as the moments of some function of I simply related to $\mathcal{P}(I)$ (as the p_n and the γ_m are, eqns.(2.7,10,17)). Therefore we do not deal with them hereafter.

III. LOCAL CLASSICALITY CONDITIONS ON p_n

The above discussion motivates the search for *local* consequences of classicality on the PND, that is, inequalities involving only a finite number of the p_n's which are always well defined and which must be obeyed if the PND is classical. Based on eqn.(2.7), the general form of such conditions can be surveyed as follows. Consider a finite degree polynomial $f(x)$ in a real nonnegative variable x, with the property of itself being real nonnegative:

$$f(x) = \sum_{n=0}^{N} c_n\ x^n \geq 0, \quad \text{for } 0 \leq x < \infty\ . \qquad (3.1)$$

The nontrivial case here is when some coefficients c_n are negative; for example $f(x)$ could be the square of a polynomial with coefficients of both signs. Then we find in view of (2.10):

$$\mathcal{P}(I) \geq 0 \Rightarrow \int_0^\infty dI\mathcal{P}(I) e^{-I} f(I) = \sum_{n=0}^{N} c_n\ q_n \geq 0\ . \qquad (3.2)$$

One thus finds that for a classical PND, every polynomial obeying eqn.(3.1) leads to one inequality (3.2); so these are necessary local conditions for classicality.

One can easily check that there are no such local conditions involving only two consecutive p_n's. The simplest or minimal local conditions involve three consecutive p_n's, and are obtained by choosing for $f(x)$ an integral power of x times a quadratic in the form of a perfect square:

$$\mathcal{P}(I) \geq 0\ ,\ f(x) = x^{n-1}(1-ax)^2, a \text{ real } \Rightarrow$$
$$q_{n-1} - 2aq_n + a^2 q_{n+1} \geq 0\ ,\ \text{all real } a \Rightarrow$$
$$q_n^2 \leq q_{n-1}q_{n+1}\ ,\ n = 1, 2, \ldots . \qquad (3.3)$$

Written in term of the true probabilities p_n this reads:

$$\text{PND classical} \Rightarrow p_n^2 \leq \left(1 + \frac{1}{n}\right) p_{n-1}p_{n+1},$$
$$n = 1, 2, \ldots . \qquad (3.4)$$

We shall call this *sequence of local conditions the minimal or three-term classicality conditions* on the PND.

We now list a series of direct consequences of the above discussions and of the minimal local conditions (3.3) for a PND to be classical.

(a) For a classical PND, the (nonnegative) distribution $\mathcal{P}(I)$ is either concentrated at $I = 0$, $\mathcal{P}(I) = \delta(I)$, or else it has positive definite weight for some values or ranges of I strictly greater than zero. The former corresponds to the vacuum state:

$$\mathcal{P}(I) = \delta(I) \Rightarrow p_n = q_n = \delta_{n,0}\ ,$$
$$\hat{\rho} = |0\rangle\langle 0|\ . \qquad (3.5)$$

In the latter situation we have

$$\mathcal{P}(I) \geq 0,\ \mathcal{P}(I) \neq \delta(I) \Rightarrow p_n > 0,$$
$$n = 0, 1, 2, \ldots . \qquad (3.6)$$

Both (3.5) and (3.6) are immediate consequences of (2.7) or, equivalently, (2.10). Therefore we conclude: (i) if $p_0 = 0$, the PND is definitely nonclassical; (ii) if the PND is classical and the state is not the vacuum, every p_n is strictly positive; equally well a non-vacuum classical state cannot be orthogonal to any Fock state; (iii) conversely,for a nonvacuum state, the vanishing of any one (or more) of the p_n's implies nonclassicality.

These conclusions can also be obtained recursively from the minimal conditions (3.3): if $q_{n_0} = 0$ for some $n_0 \geq 1$, repeated use of these inequalities leads to $q_n = 0$ for all $n \geq 1$, the state being assumed to be classical.

Hereafter we omit the vacuum state from consideration.

(b) As a corollary to the above, we see that if $\hat{\rho}$ is any state and $\hat{\rho}'$ is obtained by adding some number m of photons to $\hat{\rho}$,

$$\hat{\rho}' = \mathcal{N}^{-1}\hat{a}^{\dagger m}\hat{\rho}\hat{a}^m\ ,\ m \geq 1\ , \qquad (3.7)$$

then $\hat{\rho}'$ is always nonclassical. The normalisation constant $\mathcal{N}$ needs to be finite for (3.7) to make sense as a state. A direct calculation gives:

$$\mathcal{N} = \text{tr}(\hat{a}^m\hat{a}^{\dagger m}\hat{\rho}) = \sum_{n=0}^{\infty}\frac{(n+m)!}{n!}p_n\ ;$$
$$p'_n = \begin{cases} 0, & n \leq m-1\ , \\ \mathcal{N}^{-1}\frac{n!}{(n-m)!}p_{n-m}, & n \geq m\ , \end{cases} \qquad (3.8)$$

where $\hat{\rho}$ determines the PND $\{p_n\}$ and $\hat{\rho}'$ determines $\{p'_n\}$. If the series for $\mathcal{N}$ converges, then $\hat{\rho}'$ is a well defined state; and then the vanishing of $p'_0, p'_1, \ldots, p'_{m-1}$ establishes its nonclassicality. This proves that *all photon added states are nonclassical.* Nonclassicality of photon added coherent states [18] and photon added thermal states [6,19] ($\mathcal{N}$ is finite in both cases) have already been studied in great detail

(c) For any classical PND, since $q_0 = p_0 > 0$, we see that $q_0^{-1}\tilde{\mathcal{P}}(I)$ is a nonnegative function normalised to unit integral. Therefore it can be treated mathematically as though it were a probability distribution over $[0, \infty)$. However its physical interpretation is quite different from that of $\mathcal{P}(I)$ given earlier.

(d) Since $\{q_n\}$ is a geometric sequence for a Poissonian distribution, we see that the minimal classicality conditions (3.3) are saturated in this case for every n. Thus we can interpret these conditions as requiring that the sequence $\{p_n\}$ be *locally* Poissonian or superpoissonian *at each* n. For a classical PND each q_n (respectively p_n) has a lower bound determined by the values of the two previous q's (respectively p's):

$$q_n \geq q_{n-1}^2/q_{n-2}\ ,\ p_n\ \geq\ \left(1 - \frac{1}{n}\right)p_{n-1}^2/p_{n-2}\ ,$$
$$n \geq 2\ . \qquad (3.9)$$

If even at one value of $n \geq 1$ we have

$$q_n^2 > q_{n-1}q_{n+1}\ , \qquad (3.10)$$

so that the PND is locally subpoissonian at that value of n, the state is definitely nonclassical.

(e) The inequalities (3.3) also imply that for a classical PND we can never have $q_n > q_{n+1}\ ,\ q_{n-1}$ for any $n \geq 1$. That is, for classical states *local maxima in the sequence* $\{q_n\}$ *are ruled out.* This implies in particular that no oscillations in $\{q_n\}$ are possible for such states. Thus we arrive at the important conclusion: *oscillation in* $\{q_n\}$ *is a sure sign of nonclassicality.*

(f) On the other hand, the minimal conditions (3.3) permit a local minimum in $\{q_n\}$ but no more than one. This is because if we have minima at n_1 and again at $n_2 \geq n_1 + 2$, there would necessarily be a local maximum in between, which is disallowed for classical states.

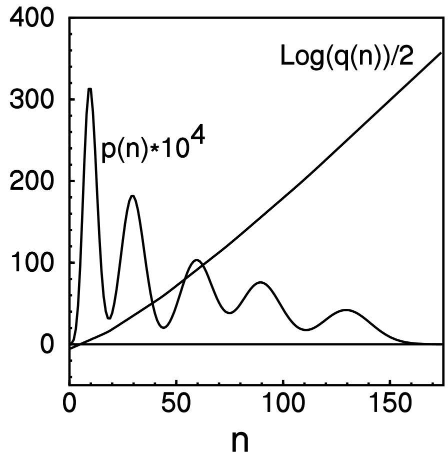

FIG. 1. "Classical" oscillation in the photon number distribution, for an incoherent mixture of coherent states with density matrix $\rho = \lambda_1|\alpha_1\rangle\langle\alpha_1| + \lambda_2|\alpha_2\rangle\langle\alpha_2| + \lambda_3|\alpha_3\rangle\langle\alpha_3| + \lambda_4|\alpha_4\rangle\langle\alpha_4| + \lambda_5|\alpha_5\rangle\langle\alpha_5|$ with $|\alpha_1|^2 = 10$, $|\alpha_2|^2 = 30$, $|\alpha_3|^2 = 60$, $|\alpha_4|^2 = 90$, $|\alpha_5|^2 = 130$ and $\lambda_1 = \lambda_2 = 0.25$, $\lambda_3 = 0.2$, $\lambda_4 = 0.18$ $\lambda_5 = 0.12$. The symbols $p(n)$, $q(n)$ stand, respectively, for p_n, q_n of the text. It may be noted that q_n exhibits no oscillations for this classical state.

(g) We can now combine (e) and (f) above to state the following. Ever since the important work of Schleich and Wheeler on interference in phase space [10], the statement that oscillation in $\{p_n\}$ is a signature of nonclassicality has almost become a folklore. Indeed, oscillations in $\{p_n\}$ are known as nonclassical oscillations [20]. The striking virtue of this characterization is that it is *local* in n, in the spirit of our present approach, as against other characterizations based on the (factorial) moments of $\{p_n\}$. We have shown in Fig.1 the sequence $\{p_n\}$ for a suitably chosen incoherent superposition of coherent states. This state is classical by construction, yet it exhibits oscillations in $\{p_n\}$, showing that the above characterization needs quantification of some sort while retaining the attractive feature of being local in n. Our minimal local conditions (3.3) can be viewed as a quantification of this type. They limit the extent to which oscillations can occur in $\{p_n\}$ if the state is classical. These limits are obtained by translating properties (e), (f) of $\{q_n\}$ above into corresponding properties of $\{p_n\}$. For classical $\{q_n\}$, (e) and (f) allow only four generic patterns of behaviour: (i) q_n nondecreasing, provided a constant phase (if any) is *followed, not preceded*, by an increasing phase (if any); (ii) q_n non increasing, provided a constant phase (if any) is *preceded, not followed*, by a decreasing phase (if any); (iii) q_n constant (actually subsumed in the two previous possibilities); (iv) q_n passing through one local minimum, monotonic decreasing (increasing) before (after) the minimum. The monotonic increasing case is part of (i), the monotonic decreasing case is part of (ii). Well known classical states constitute examples of these generic behaviours of classical $\{q_n\}$, as shown in Fig.2.

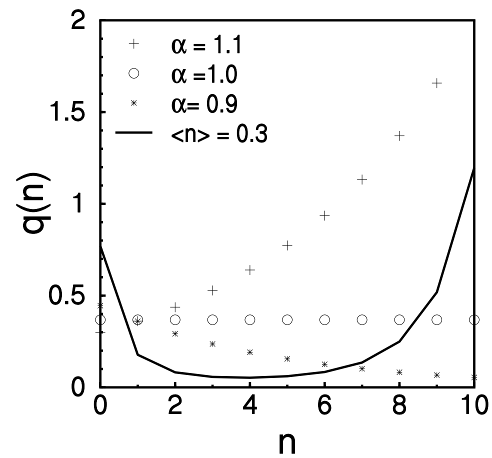

FIG. 2. Generic behaviours of the sequence $\{q_n\}$ for different classical states. First three are coherent states with different displacements and the fourth one is a thermal state.

It turns out that constant phases in $\{q_n\}$ in cases (i) and (ii) above, allowed by the minimal local conditions (3.3), but not shown in Fig.2, are indeed forbidden by the higher order local conditions as will be shown in Section IV.

All these restrictions on classical $\{q_n\}$ translate into allowed behaviours for classical $\{p_n\}$. First, local maxima, $p_n > p_{n\pm1}$, are permitted, provided

$$\frac{p_n}{p_{n-1}}\,\frac{p_n}{p_{n+1}} \le 1 + \frac{1}{n}\,. \tag{3.11}$$

Next, again because of the factor $\left(1 + \frac{1}{n}\right)$ on the right hand side in the minimal classicality condition (3.4), some amount of oscillation in $\{p_n\}$ is allowed. This is just the kind of oscillation seen in Fig.1. Note that the period of oscillation (difference of n values at two successive maxima in $\{p_n\}$) in Fig.1 is substantially greater than two as against the period two oscillations occurring in the PND of a squeezed or cat state. For period two oscillations the conditions (3.4) place substantial restrictions on the amplitude. (Indeed the next higher order local conditions to be derived in Section IV make these restrictions even more stringent). Further, as the factor $\left(1 + \frac{1}{n}\right)$ approaches unity as n becomes large, the restrictions on the amplitude of a classically allowed period two oscillation are stronger at higher values of n. To end the discussion in this paragraph we note that the *period two classical oscillation* in $\{p_n\}$, allowed by our lowest order local condition (3.3,4), indeed survives all higher order local conditions, as can be seen by the example shown in Fig.3.

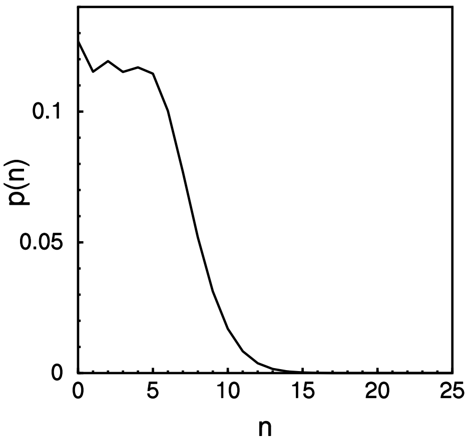

FIG. 3. Limits to PND oscillations from classicality conditions- an example of incoherent mixture of three coherent states: $\rho = \lambda_1|\alpha_1\rangle\langle\alpha_1| + \lambda_2|\alpha_2\rangle\langle\alpha_2| + \lambda_3|\alpha_3\rangle\langle\alpha_3|$ with $|\alpha_1|^2 = 0.0$, $|\alpha_2|^2 = 1.56$, $|\alpha_3|^2 = 5.42$, $\lambda_1 = 0.06$, $\lambda_2 = 0.305$ and $\lambda_3 = 0.635$. Period two oscillations though restricted in amplitude can be present for a classical state.

An instructive example illustrating many of the points made above is provided by the case of a pure state obtained as the superposition of two "opposite" coherent states:

$$|\psi(z_0,\theta)\rangle = \mathcal{N}^{-1/2}\left(|z_0\rangle + e^{i\theta}|-z_0\rangle\right) ,$$
$$\mathcal{N} = 2\left(1 + e^{-2|z_0|^2}\cos\theta\right) . \qquad (3.12)$$

Here θ is the relative phase (in the Pancharatnam sense [21]) between the two components of the superposition. As special cases we get the Yurke-Stoler states (2.12) when $\theta = \pm\pi/2$, and the cat states when $\theta = 0, \pi$. For the PND we have

$$p_n = |\langle n|\psi(z_0,\theta)\rangle|^2$$
$$= e^{-|z_0|^2}\,\frac{|z_0|^{2n}}{n!}\,\frac{(1+(-1)^n\cos\theta)}{\left(1+e^{-2|z_0|^2}\cos\theta\right)} , \qquad (3.13)$$

so that

$$q_{n-1}q_{n+1}/q_n^2 = \{(1-(-1)^n\cos\theta)/(1+(-1)^n\cos\theta)\}^2$$
$$\equiv f_n(\theta) . \qquad (3.14)$$

Clearly, $f_n(\theta) < 1$ for even n if $-\pi/2 < \theta < \pi/2$; and $f_n(\theta) < 1$ for odd n if $\pi/2 < \theta < 3\pi/2$. Thus the state $|\psi(z_0,\theta)\rangle$ violates the minimal local condition (3.3) and is nonclassical for all $\theta \neq \pm\pi/2$. For $\theta = \pm\pi/2$ these conditions are saturated at every n, and we have a Poissonian $\{p_n\}$.

It is clear that nonclassicality of the Yurke-Stoler state (2.12) can be exhibited only through phase sensitive considerations, since the PND is poissonian. It is also well known [13] that all pure states other than coherent states are nonclassical at least at the phase sensitive level. What is really interesting is the fact that for every $\theta \neq \pm\pi/2$, the nonclassicality of $|\psi(z_0,\theta)\rangle$ is coded in the phase insensitive PND, and that our minimal conditions (3.3,4) capture this nonclassicality!

We conclude this Section with one more application of the local condition. Recent years have witnessed a remarkable progress in quantum state reconstruction using techniques of optical homodyne tomography [22]. As a consequence it is now possible to 'map out' the Wigner distribution of a state using the inverse Radon transform, or reconstruct the density matrix in the Fock basis using a set of pattern functions. Schiller et al [23] report such a reconstructed density matrix $\rho_{m,n}$ for $m, n \leq 6$. The reported values of q_n for $n = 0$ to 6 are 0.44, 0.07, 0.26, 0.30, 1.44, 3.60 and 28.80. The local conditions are clearly violated: $q_1q_3 < q_2^2$, $q_3q_5 < q_4^2$. Hence the state is nonclassical. The state was, of course, known to be quadrature squeezed, and hence nonclassical. What is interesting about the present illustration is the fact that the nonclassicality of the Schiller et al state survives phase averaging, and that we are able to arrive at this definitive conclusion with just a few values of the diagonal elements of the density matrix!

IV. NECESSARY AND SUFFICIENT CONDITIONS FOR CLASSICALITY OF THE PND - THE STIELTJES MOMENT PROBLEM

We have seen in the previous Section that pointwise nonnegativity of $\mathcal{P}(I)$, hence of $\tilde{\mathcal{P}}(I)$, leads to the minimal local conditions (3.3,4) on the sequences $\{q_n\}, \{p_n\}$. As already remarked, these are necessary conditions for classicality. Central to this derivation was appreciation of the fact that the q_n are *moments* of $\tilde{\mathcal{P}}(I)$, eqn.(2.10). We have also seen that if the state is classical, then $q_0^{-1}\tilde{\mathcal{P}}(I)$ is well-defined and mathematically interpretable as a probability distribution over $[0,\infty)$. In this Section we exhibit the *necessary and sufficient* conditions on the PND $\{q_n\}$ in order that the state $\hat{\rho}$ be classical. We then use these conditions to examine the extension of the inequalities (2..23,24) to the five-term case.

The reconstruction of a probability distribution from its moment sequence constitutes the classical moment problem on which there exists an enormous amount of literature [12]. When the probability distribution is over $[0,\infty)$, one calls it the Stieltjes moment problem. The Hamburger moment problem corresponds to the case where the probability distribution is over $(-\infty,\infty)$; and the Hausdorff or the *little* moment problem corresponds to the finite interval $[0,1]$. Since the argument of $\tilde{\mathcal{P}}(I)$ is nonnegative, our problem of deriving necessary and sufficient conditions on the moment sequence $\{q_n\}$ in order that $q_0^{-1}\tilde{\mathcal{P}}(I)$ is a true probability distribution is then a

Stieltjes moment problem. In this Section we can take advantage of the fact that these moments $\{q_n\}$ of $\tilde{\mathcal{P}}(I)$ are always well-defined and finite, whatever the state $\hat{\rho}$ may be.

The solution of this classical problem is well known. To exhibit it, construct from the sequence $\{q_n\}$ two symmetric matrices $L^{(N)}, \tilde{L}^{(N)}$ of dimension $(N+1)$ defined by

$$\begin{aligned} L^{(N)}_{mn} &= q_{m+n} \, , \\ \tilde{L}^{(N)}_{mn} &= q_{m+n+1} \, , \; m,n = 0,1,2,\ldots, \; N. \end{aligned} \tag{4.1}$$

That is,

$$L^{(N)} = \begin{pmatrix} q_0 & q_1 & q_2 & \cdots & q_N \\ q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ \vdots & & & & \\ q_N & q_{N+1} & q_{N+2} & \cdots & q_{2N} \end{pmatrix} ,$$
$$\tilde{L}^{(N)} = \begin{pmatrix} q_1 & q_2 & q_3 & \cdots & q_{N+1} \\ q_2 & q_3 & q_4 & \cdots & q_{N+2} \\ \vdots & & & & \\ q_{N+1} & q_{N+2} & q_{N+3} & \cdots & q_{2N+1} \end{pmatrix} . \tag{4.2}$$

Thus, $\tilde{L}^{(N)}$ arises from $L^{(N+1)}$ by deletion of the first column and the last row in the latter. Notice also that the diagonal elements of $L^{(N)}$ are q_n for even $n \leq 2N$, while those of $\tilde{L}^{(N)}$ are q_n for odd $n \leq 2N+1$.

Now we quote the fundamental classical theorem [12].

Theorem 1:

The necessary and sufficient condition on the PND sequence $\{q_n\}$ in order that the associated distribution $q_0^{-1}\tilde{\mathcal{P}}(I)$ be a true probability distribution over $[0,\infty)$ is that the above matrices be nonnegative for all $N \geq 0$:

$$\begin{aligned} \text{Classical PND} &\Leftrightarrow \mathcal{P}(I) \geq 0 \Leftrightarrow \tilde{\mathcal{P}}(I) \geq 0 \\ &\Leftrightarrow L^{(N)}, \tilde{L}^{(N)} \geq 0, \; N = 0,1,2,\ldots . \end{aligned} \tag{4.3}$$

Proof:

We make use of the following two facts.

(i) A distribution $\tilde{\mathcal{P}}(I)$ over $[0,\infty)$ is pointwise nonnegative if and only if it leads to a nonnegative expectation value for every polynomial $f(I)$ which is itself pointwise nonnegative over $[0,\infty)$ [24]:

$$\begin{aligned} \tilde{\mathcal{P}}(I) \geq 0 \text{ for all } I \Leftrightarrow \langle f(I)\rangle_{\tilde{\mathcal{P}}} &= \int_0^\infty dI \tilde{\mathcal{P}}(I) f(I) \geq 0, \\ \text{for all polynomial } f(I) &\geq 0. \end{aligned} \tag{4.4}$$

(ii) Any polynomial $f(I)$ pointwise nonnegative over $[0,\infty)$ can be written in terms of two perfect square polynomials as [12]

$$f(I) = (f_1(I))^2 + I(f_2(I))^2 \, . \tag{4.5}$$

To prove necessity, suppose that $\tilde{\mathcal{P}}(I)$ is pointwise nonnegative. Consider the polynomial $f_1(I) = \sum_{n=0}^{N} c_n I^n$, where the c_n are arbitrary real coefficients. By (4.4) we have $\langle (f_1(I))^2\rangle_{\tilde{\mathcal{P}}} \geq 0$. That is,

$$\begin{aligned} \langle (f_1(I))^2\rangle_{\tilde{\mathcal{P}}} &= \sum_{m,n=0}^{N} c_m c_n \langle I^{m+n}\rangle_{\tilde{\mathcal{P}}} \\ &= \sum_{m,n=0}^{N} c_m c_n q_{m+n} \\ &= \sum_{m,n=0}^{N} c_m c_n L^{(N)}_{mn} \geq 0 \, . \end{aligned} \tag{4.6}$$

This means $L^{(N)} \geq 0$ for every $N \geq 0$. Similarly, writing $f_2(I) = \sum_{n=0}^{N} d_n I^n$ and evaluating the expectation value of the nonnegative polynomial $I(f_2(I))^2$ we have from (4.4):

$$\begin{aligned} \langle I(f_2(I))^2\rangle_{\tilde{\mathcal{P}}} &= \sum_{m,n=0}^{N} d_m d_n \langle I^{m+n+1}\rangle_{\tilde{\mathcal{P}}} \\ &= \sum_{m,n=0}^{N} d_m d_n q_{m+n+1} \\ &= \sum_{m,n=0}^{N} d_m d_n \tilde{L}^{(N)}_{mn} \geq 0 \, . \end{aligned} \tag{4.7}$$

This means $\tilde{L}^{(N)} \geq 0$ for every $N \geq 0$. Thus nonnegativity of $\tilde{\mathcal{P}}(I)$ implies that of every $L^{(N)}, \tilde{L}^{(N)}$.

To prove sufficiency, assume $L^{(N)}, \tilde{L}^{(N)} \geq 0$ for $N \geq 0$. Given any nonnegative polynomial $f(I)$, writing it in the form (4.5) we find that $\langle f(I)\rangle_{\tilde{\mathcal{P}}} \geq 0$. This implies, by fact (i) stated at the beginning of the proof, that $\tilde{\mathcal{P}}(I) \geq 0$. This completes the proof of the Theorem.

Now each matrix $L^{(N)}$ (respectively $\tilde{L}^{(N)}$) arises from $L^{(N-1)}$ (respectively $\tilde{L}^{(N-1)}$) by addition of one extra row and column, leaving the rest intact. Therefore the result (4.3) translates into a series of determinant conditions:

$$\begin{aligned} D_N \; = \det L^{(N)} \, , \; \tilde{D}_N &= \det \tilde{L}^{(N)} \; : \\ \text{Classical PND} &\Leftrightarrow \mathcal{P}(I), \tilde{\mathcal{P}}(I) \geq 0 \\ &\Leftrightarrow D_N \, , \; \tilde{D}_N \geq 0 \, , \; N = 0,1,2,\ldots . \end{aligned} \tag{4.8}$$

Assume now that we have a classical PND. Then either $D_N, \tilde{D}_N > 0$ for all N, or $D_N, \tilde{D}_N > 0$ for $N \leq k$ and $D_N = \tilde{D}_N = 0$ for $N > k$. In the latter case the support of $\tilde{\mathcal{P}}(I)$ (the set of values of I at which $\tilde{\mathcal{P}}(I) > 0$) is a finite set of k points [25]. In the former case it is an infinite set. This can be intuitively understood along the following lines. Suppose $\tilde{\mathcal{P}}(I) = \delta(I - I_0)$. Then $q_n = I_0^n$ so that $L^{(N)}_{mn} = I_0^m I_0^n$ and $\tilde{L}^{(N)}_{mn} = I_0 I_0^m I_0^n$. That is, $L^{(N)}$ and $\tilde{L}^{(N)}$ are (essentially) projection matrices. Thus when the support of $\tilde{\mathcal{P}}(I)$ is a finite set of points, $L^{(N)}$ (as also $\tilde{L}^{(N)}$) is the sum of k projections. Since

$I_0 = 0$ contributes a projection only to $L^{(N)}$ but not to $\tilde{L}^{(N)}$, one has the following refinement: if the support of $\tilde{\mathcal{P}}(I)$ consists of k points including the point $I = 0$, then $D_N > 0 (\tilde{D}_N > 0)$ if and only if $N \leq k (N \leq k-1)$.

It is useful to remark that the first fact (4.4) we have used in the proof of the theorem is common for all the three types of moment problems. What changes from one moment problem to another is the second fact dealing with the decomposition of nonnegative polynomials into sums of square polynomials, thus enabling us to convert (4.4) into simple matrix conditions. For instance, for the Hamburger moment problem on $(-\infty, \infty)$ we have, in place of (4.5), the statement that every polynomial nonnegative over $(-\infty, \infty)$ can be written as the sum of two square polynomials. Thus in this case we have to deal only with the matrix $L^{(N)}$, and the conditions $L^{(N)} \geq 0$ for all $N \geq 0$ are both necessary and sufficient for positivity of the underlying Hamburger distribution.

It is immediate to relate the minimal local conditions (3.3,4) to the above theorem. Nonnegativity of $L^{(N)}, \tilde{L}^{(N)}$ demands as a necessary condition nonnegativity of every 2×2 determinant obtained from 2×2 blocks along the principal diagonals of $L^{(N)}$ and $\tilde{L}^{(N)}$. This is precisely the condition (3.3). It is also clear why these minimal conditions are only necessary: positivity of the diagonal 2×2 blocks of $L^{(N)}, \tilde{L}^{(N)}$ does not capture in its entirety the positivity of $L^{(N)}$ and $\tilde{L}^{(N)}$.

We now go beyond the minimal conditions (3.3,4) and derive the next hierarchy of local conditions for a classical PND. Given $\{q_n\}$ we define

$$x_n = q_{n-1} q_{n+1} / q_n^2 \ , \ n = 1, 2, \ldots . \qquad (4.9)$$

This definition is legitimate since in any case for a classical PND each $q_n > 0$. Then the minimal conditions (3.3) simply read:

$$\text{Classical PND} \Rightarrow x_n \geq 1 \ , \ n = 1, 2 \ldots . \qquad (4.10)$$

As one may anticipate, the next order local conditions which we now derive, and which are still only necessary conditions for classicality, involve five consecutive q_n's, or equivalently three consecutive x_n's. A necessary condition for the nonnegativity of $L^{(N)}, \tilde{L}^{(N)}$ is that their diagonal 3×3 blocks be nonnegative. That is,

$$A_n = \begin{pmatrix} q_{n-2} & q_{n-1} & q_n \\ q_{n-1} & q_n & q_{n+1} \\ q_n & q_{n+1} & q_{n+2} \end{pmatrix} \geq 0 \ , \ n = 2, 3, \ldots . \qquad (4.11)$$

If we bring in x_{n-1}, x_n, x_{n+1} here and eliminate $q_{n\pm 2}$ in favour of the other variables, this reads:

$$A_n = \frac{1}{q_n} \begin{pmatrix} q_{n-1}^2 x_{n-1} & q_{n-1} q_n & q_n^2 \\ q_{n-1} q_n & q_n^2 & q_n q_{n+1} \\ q_n^2 & q_n q_{n+1} & q_{n+1}^2 x_{n+1} \end{pmatrix} \geq 0 \ ,$$
$$n = 2, 3, \ldots . \qquad (4.12)$$

This entails three conditions: $q_{n-1}^2 x_{n-1} \geq 0, \ x_{n-1} - 1 \geq 0, \ \det A_n \geq 0$. The first two are already obeyed; and after elementary algebra the third reduces to a condition expressible in terms of three consecutive x_n's:

$$(x_{n-1} - 1)(x_{n+1} - 1) \geq \left(\frac{x_n - 1}{x_n} \right)^2 \ , \ n = 2, 3, \ldots . \qquad (4.13)$$

These are our next to minimal or *second order local conditions* on the PND for classicality.

To conclude this Section, we present an interesting implication of these conditions. As already noted, if $\{p_n\}$ is Poissonian then $\{q_n\}$ is a geometric sequence, so $x_n = 1$ identically and (3.3) are saturated. We now ask whether it is possible to have a classical state for which $q_{n-1} q_{n+1} = q_n^2$ for some values of $n \geq 1$, whereas $q_{n-1} q_{n+1} > q_n^2$ for other n's. Such classical states, if they exist, can be said to be *locally Poissonian* at the former values of n, namely whenever $x_n = 1$.

Suppose a classical state is locally Poissonian at n_0, with $x_{n_0} = 1$. Then two applications of (4.13), once with $n = n_0 - 1$ and then with $n = n_0 + 1$, lead to $x_{n_0 \pm 1} = 1$. Continuing this process we end up with $x_n = 1$ for all $n \geq 1$. Thus a classical state cannot be only locally Poissonian : it is either locally Poissonian throughout, ($x_n = 1$ for all $n \geq 1$) or locally superpoissonian throughout ($x_n > 1$ for all n). Translating this to $\{p_n\}$ we can now strengthen our minimal conditions (3.4) to say:

For a classical PND, either

$$p_{n-1} p_{n+1} = \frac{n}{n+1} p_n^2 \ \text{(Poissonian)}, \ n \geq 1$$
$$\text{or}$$
$$p_{n-1} p_{n+1} > \frac{n}{n+1} p_n^2 \ \text{(Superpoissonian)}, \ n \geq 1. \qquad (4.14)$$

This is a refinement of the minimal or first order necessary local classicality conditions (3.4) in the light of the second order conditions (4.13).

It is now clear that the constant phases allowed in patterns (i) and (ii) under (g) preceding eqn.(2.31) are forbidden by the above refined condition (4.14). This renders Fig.2 generic and exhaustive.

V. ALTERNATIVE FORMULATION VIA FACTORIAL MOMENTS

In this Section we present an approach to classicality of a PND based on the normal ordered moments of $\hat{\rho}$ or factorial moments of $\{p_n\}$, namely the sequence $\{\gamma_m\}$ defined in eqn.(2.17). This approach will be along the lines of earlier work of Agarwal and Tara [6]; however our conditions for classicality will be both *necessary and sufficient*. As is amply clear from the discussion in Section II, though, we must bear in mind that for the vast majority of states $\hat{\rho}$ and PND's $\{p_n\}$, the γ_m are not all defined.

Suppose we have a PND $\{p_n\}$ whose factorial moments $\{\gamma_m\}$ are all finite and known. (This certainly places very severe restrictions on the behaviour of p_n for large n).

Our problem is to find necessary and sufficient conditions on $\{\gamma_m\}$ in order that the PND be classical. Now the γ_m are the moments of $\mathcal{P}(I)$, just as in the preceding Section the q_n were moments of $\tilde{\mathcal{P}}(I) = \mathcal{P}(I)e^{-I}$. At the risk of repetition, the finiteness of all γ_m places very strong restrictions on $\mathcal{P}(I)$.

It is now clear that we have again a Stieltjes moment problem. We want the necessary and sufficient conditions on $\{\gamma_m\}$ to ensure pointwise nonnegativity of $\mathcal{P}(I)$. Form two matrices $M^{(N)}, \tilde{M}^{(N)}$ of dimension $(N+1)$ using the γ's:

$$M^{(N)} = \begin{pmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_N \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{N+1} \\ \vdots & & & & \\ \gamma_N & \gamma_{N+1} & \gamma_{N+2} & \cdots & \gamma_{2N} \end{pmatrix} ;$$

$$\tilde{M}^{(N)} = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \cdots & \gamma_{N+1} \\ \gamma_2 & \gamma_3 & \cdots & \cdots & \gamma_{N+2} \\ \vdots & & & & \\ \gamma_{N+1} & \gamma_{N+2} & \cdots & \cdots & \gamma_{2N+1} \end{pmatrix} . \qquad (5.1)$$

Both are real symmetric; $\tilde{M}^{(N)}$ equals $M^{(N+1)}$ with first column and last row deleted; and $M^{(N)}\left(\tilde{M}^{(N)}\right)$ has γ_m for m even (odd) along the diagonal. Thus, exactly as in Section IV, we have:

Theorem 2:

The necessary and sufficient conditions for the PND $\{p_n\}$ to be classical (given the existence of all factorial moments γ_m) are

$$M^{(N)} \geq 0,\ \tilde{M}^{(N)} \geq 0,\ N = 0, 1, 2, \ldots \qquad (5.2)$$

The proof is exactly parallel to that in Section IV, with $\tilde{\mathcal{P}}(I)$, q_n, $L^{(N)}$, $\tilde{L}^{(N)}$ replaced respectively by $\mathcal{P}(I)$, γ_n, $M^{(N)}$, $\tilde{M}^{(N)}$.

This theorem completes the work initiated by Agarwal and Tara [6], by improving their necessary conditions for classicality (they had only the condition $M^{(N)} \geq 0$) into the necessary and sufficient condition (5.2). Thus the constraints on the factorial moments arising from the requirements $M^{(N)} \geq 0$ are the same as in their work: with $N = 1$ we have $\gamma_0\gamma_2 - \gamma_1^2 \geq 0$, ie., $\gamma_2 - \gamma_1^2 \geq 0$ which (see eqn.(2.18)) is the same as requiring the Mandel Q-parameter to be nonnegative. With $N = 2$ we obtain the additional condition $\det M^{(2)} \geq 0$, and so on as in [6]. However the constraints arising from $\tilde{M}^{(N)} \geq 0$ are new. For $N = 0$ we have $\gamma_1 \geq 0$; for $N = 1$ we have $\gamma_1\gamma_3 - \gamma_2^2 \geq 0$; and so on.

To conclude this Section, let us re-examine the class of pure states $|\psi(z_0, \theta) >$ defined in eqn.(2.32) within the present approach. All the γ_m do exist in this case and are easily calculated.

$$\begin{aligned} \gamma_m &= \langle\psi(z_0,\theta)|\hat{a}^{\dagger m}\hat{a}^m|\psi(z_0,\theta)\rangle \\ &= |z_0|^{2m} \frac{\left(1 + (-1)^m e^{-2|z_0|^2}\cos\theta\right)}{\left((1 + e^{-2|z_0|^2}\cos\theta\right)} . \end{aligned} \qquad (5.3)$$

This leads to the following matrices $M^{(N)}$ and $\tilde{M}^{(N)}$:

$$\begin{aligned} M^{(N)} &= A \begin{pmatrix} 1 & \sigma & 1 & \sigma & \ldots \\ \sigma & 1 & \sigma & 1 & \ldots \\ 1 & \sigma & 1 & \sigma & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} A , \\ \tilde{M}^{(N)} &= B \begin{pmatrix} \sigma & 1 & \sigma & 1 & \ldots \\ 1 & \sigma & 1 & \sigma & \ldots \\ \sigma & 1 & \sigma & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} B , \\ A &= \mathrm{diag}\left(1, |z_0|^2, |z_0|^4, |z_0|^6, \ldots\right) , \\ B &= \mathrm{diag}\left(|z_0|, |z_0|^3, |z_0|^5, \ldots\right) \\ \sigma &= \frac{\left(1 - e^{-2|z_0|^2}\cos\theta\right)}{\left(1 + e^{-2|z_0|^2}\cos\theta\right)} . \end{aligned} \qquad (5.4)$$

It is clear that both $M^{(N)}$ and $\tilde{M}^{(N)}$ are matrices of rank 2. It is further clear from their structures that $M^{(N)} \geq 0$ if and only if $\sigma \leq 1$, while $\tilde{M}^{(N)} \geq 0$ if and only if $\sigma \geq 1$. We thus get again the same results as in Section II: for $\theta \neq \pm\pi/2$, the state $|\psi(z_0, \theta)\rangle$ has a nonclassical PND.

We wish to remark once again that the fact that the conditions $\tilde{M}^{(N)} \geq 0$ in (5.2) are needed over and above $M^{(N)} \geq 0$ to form a set of necessary and sufficient conditions for classicality is ultimately due to the fact that our moment problem is a Stieltjes problem: had it been a Hamburger problem, the Agarwal-Tara conditions $M^{(N)} \geq 0$ would have been both necessary and sufficient!

VI. CONNECTION BETWEEN THE TWO APPROACHES

We have presented two approaches to the problem of phase - insensitive nonclassicality of a quantum state $\hat{\rho}$: one based on $\{q_n\}$, the moment sequence of $\tilde{\mathcal{P}}(I)$; the other dual approach based on $\{\gamma_m\}$, the moment sequence of $\mathcal{P}(I)$ when defined. In each case we obtained necessary and sufficient conditions for a PND to be classical, exploiting the fact that the underlying problem was a Stieltjes moment problem. In this Section we bring out explicitly the connection between these dual approaches and establish their equivalence, in the case (naturally) when all γ_m are defined.

The fact that $\tilde{\mathcal{P}}(I) = \mathcal{P}(I)e^{-I}$ suggests the use of Laplace transforms [25]. Let $\Phi(s), \tilde{\Phi}(s)$ be the Laplace transforms of $\mathcal{P}(I), \tilde{\mathcal{P}}(I)$ respectively:

$$\begin{aligned} \Phi(s) &= \int_0^\infty dI\ \mathcal{P}(I)\ e^{-sI} , \\ \tilde{\Phi}(s) &= \int_0^\infty dI\ \tilde{\mathcal{P}}(I)\ e^{-sI} \\ &= \Phi(s+1) . \end{aligned} \qquad (6.1)$$

We now exploit the fact that the moments of a distribution are simply related to the derivatives of its Laplace transform at the origin:

$$\gamma_n = \int_0^\infty dI\ \mathcal{P}(I)\ I^n = (-1)^n \frac{d^n\Phi(s)}{ds^n}\Big|_{s=0}\ ;$$
$$q_n = \int_0^\infty dI\ \mathcal{P}(I)\ e^{-I}\ I^n = (-1)^n \frac{d^n\tilde{\Phi}(s)}{ds^n}\Big|_{s=0}$$
$$= (-1)^n \frac{d^n\Phi(s)}{ds^n}\Big|_{s=1}\ . \qquad (6.2)$$

Making two Taylor series expansions of $\Phi(s)$, once about $s = 0$ and then about $s = 1$, equating the two expansions and using eqn.(6.2), we have:

$$\sum_{k=0}^{\infty}(-1)^k\gamma_k s^k/k! = \sum_{\ell=0}^{\infty}(-1)^\ell q_\ell (s-1)^\ell/\ell!\ . \qquad (6.3)$$

Equating derivatives of both sides first at $s = 0$ and then at $s = 1$ gives the pair of relations

$$\gamma_n = \sum_{k=0}^{\infty} q_{n+k}/k!\ ,\ n \geq 0\ ;$$
$$q_n = \sum_{k=0}^{\infty}(-1)^k\gamma_{n+k}/k!\ ,\ n \geq 0\ . \qquad (6.4)$$

Writing the two sequences $\{q_n\}, \{\gamma_n\}$ as two column vectors Q and Γ respectively we have the infinite matrix equations

$$\Gamma = S\ Q\ ,$$
$$Q = S^{-1}\ \Gamma\ ,$$
$$S = \begin{pmatrix} 1 & 1/1! & 1/2! & 1/3! & \dots \\ 0 & 1 & 1/1! & 1/2! & \dots \\ 0 & 0 & 1 & 1/1! & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}\ ,$$
$$S^{-1} = \begin{pmatrix} 1 & -1/1! & 1/2! & -1/3! & \dots \\ 0 & 1 & -1/1! & 1/2! & \dots \\ 0 & 0 & 1 & -1/1! & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}\ . \qquad (6.5)$$

We have displayed S, S^{-1} to exhibit the fact that these matrices are unchanged if the first row and first column are deleted. We shall have occasion to return to this important feature.

While Q and Γ considered as (infinite dimensional) column vectors are connected by the matrix S, the corresponding infinite dimensional matrices $L = L^{(\infty)}$ and $M = M^{(\infty)}$ are connected through a symmetric transformation using $S^{1/2}$. We have:

$$(S^{1/2})_{jk} = 2^{j-k}/(k-j)!\ ,$$
$$(S^{-1/2})_{jk} = (-2)^{j-k}/(k-j)!\ . \qquad (6.6)$$

(So both matrices are upper triangular: these matrix elements vanish for $k < j$). That $S^{1/2}$ and $S^{-1/2}$ so defined are indeed inverses of one another follows simply from the familiar properties of binomial coefficients. Using these same properties it may be verified that

$$M = S^{1/2}L(S^{1/2})^T\ ,$$
$$L = S^{-1/2}M(S^{-1/2})^T\ . \qquad (6.7)$$

This proves that $M \geq 0$ if and only if $L \geq 0$.

Let $\tilde{Q}$ be the moment sequence derived from Q by simply dropping q_0. Clearly, $\tilde{L} = \tilde{L}^{(\infty)}$ is in the same relation to $\tilde{Q}$ as L is to Q. Similarly if we form $\tilde{\Gamma}$ from Γ by simply dropping γ_0, then $\tilde{M} = \tilde{M}^{(\infty)}$ is in the same relation to $\tilde{\Gamma}$ as M is to Γ. Now recalling the fact that $S, S^{-1}, S^{1/2}, S^{-1/2}$ have the interesting property of each being unchanged upon deletion of the first row and first column we conclude:

$$\tilde{\Gamma} = S\tilde{Q}\ ,\ \tilde{Q} = S^{-1}\tilde{\Gamma}\ ;$$
$$\tilde{M} = S^{1/2}\tilde{L}(S^{1/2})^T\ ,\ \tilde{L} = S^{-1/2}\tilde{M}(S^{-1/2})^T\ . \qquad (6.8)$$

This proves that $\tilde{M} \geq 0$ if and only if $\tilde{L} \geq 0$. Combined with the statement after eqn.(6.7), we have thus established the equivalence of the two approaches. They are dual to one another, and in particular Theorem 1 is equivalent to Theorem 2.

The above analysis shows that if we drop the first k terms from the moment sequence of a bonafide Stieltjes probability distribution (in the semi-infinite interval), the result is again a bonafide Stieltjes moment sequence (of some other valid probability distribution). This is a distinguishing feature of the Stieltjes moment problem. It is not difficult to see that the corresponding statement is true for the Hamburger moment problem (on the entire real line) only if an *even* number of initial terms are dropped from the moment sequence.

VII. CONCLUDING REMARKS

Every probability distribution $\{p_n\}$ over the nonnegative integers, admissible according to the laws of classical probability theory, can appear as the PND of some quantum state $\hat{\rho}$ of the single mode radiation field. We have given explicit necessary and sufficient conditions - in two equivalent or dual forms - for a PND to be the result of a classical state in the sense appropriate to quantum optics. From the perspective of classical probability theory, the significant points are these: that we are able to view the quantities $q_n = n!p_n$ as the moments, in the Stieltjes sense, of an auxiliary function $\tilde{\mathcal{P}}(I)$, and that the properties of $\tilde{\mathcal{P}}(I)$ or equally well of $\mathcal{P}(I)$ determine whether the state is classical or not. The route from $\{p_n\}$ to $\mathcal{P}(I)$ is via the pair of eqns.(2.14,15).

We have emphasized the following points in our work: (i) that local classicality conditions on the PND are naturally available for all states $\hat{\rho}$; and (ii) all statements based on the factorial moments make sense only for a quite limited set of states $\hat{\rho}$, for which the probabilities p_n go to zero "sufficiently fast" as $n \to \infty$. It seems to us that some of these points have not been given in the past as much attention as they deserve. Above this, we stress the completeness of our results, and the more careful treatment of oscillations in the PND as a signature of nonclassicality, which becomes possible with our local approach.

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