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Folds, Bosonization and non-triviality of the classical limit of 2D string theory

Sumit R. Das *

Tata Institute of Fundamental Research

Homi Bhabha Road , Bombay 400005, INDIA.

Samir D. Mathur†

Center for Theoretical Physics, Massachussetts Institute of Technology

Cambridge, MA 02139. U.S.A.

Abstract

In the 1-dimensional matrix model one identifies the tachyon field in the asymptotic region with a nonlocal transfom of the density of fermions. But there is a problem in relating the classical tachyon field with the surface profile of the fermi fluid if a fold forms in the fermi surface. Besides the collective field additional variables $w_j(x)$ are required to desrcibe folds. In the quantum theory we show that the w_j are the quantum dispersions of the collective field. These dispersions become $O(1)$ rather than $O(\hbar)$ precisely after fold formation, thus giving additional 'classical' quantities and leading to a rather non-trivial classical limit. A coherent pulse reflecting from the potential wall turns into high energy incoherent quanta (if a fold forms), the frequency amplification being of the order of the square root of the number of quanta in the incident wave.

*E-mail: das@theory.tifr.res.in

†E-mail: me@ctpdown.mit.edu

The double scaling limit of the one dimensional matrix model provides a non-perturbative formulation of two dimensional string theory in the Liouville background [1]. Classically, in the two dimensional string theory one expects that a matter wave with sufficient energy and energy density will collapse and a black hole will be formed. The matrix model should describe this classical process. However, since the graviton and dilaton are not present as explicit fields in the matrix model, collapse would be signalled by properties of the matter alone, e.g. an infinitely long time delay for return of matter thrown into the liouville wall. In addition, the matrix model should be able to answer questions in black hole physics at the quantum level *exactly*.

However, it turns out to be surprisingly difficult to understand a potential gravitational collapse in the matrix model. The natural field which describes the matrix model scalar - the collective field - is *not* the string theory massless tachyon. Recent work has revealed that for *weak* waves of low energy the string theory tachyon may be expressed, in the asymptotic region, as a *non-local* transform of the collective field [2] which is the fourier transform of the well-known "leg-pole" factor [3]. The nonlocal transform then relates the space of eigenvalues in the matrix model with the space in string theory.

Unfortunately we cannot use this correspondence for waves of *finite* amplitude and width, in particular for the ones which we expect will form black holes. At the fundamental level the matrix model is equivalent to a system of N mutually noninteracting fermions in a background inverted harmonic oscillator potential. The classical state of the system is specified by a distribution function in phase space $u(x,p,t)$ - in fact $u(x,p,t) = \frac{1}{2\pi\hbar}$ in some region bounded by the fermi surface. We will consider only states which correspond to a single connected filled region of phase space. We will find it convenient to parametrize the fermi sea in terms of an infinite number of functions $w_n(x,t)$ $(n = 0, 1, \cdots)$ following [6]

$$\begin{aligned}
\int dp\ u(x,p,t) &= \frac{1}{2\pi\hbar}[\beta_+ - \beta_-] \\
\int dp\ p\ u(x,p,t) &= \frac{1}{2\pi\hbar}[\frac{1}{2}(\beta_+^2 - \beta_-^2) + (w_{+1} - w_{-1})] \\
\int dp\ p^2\ u(x,p,t) &= \frac{1}{2\pi\hbar}[\frac{1}{3}(\beta_+^3 - \beta_-^3) \\
&\quad +(\beta_+ w_{+1} - \beta_- w_{-1}) + (w_{+2} - w_{-2})]
\end{aligned} \tag{1}$$

and so on. Let a line of constant x in phase space cut the upper edge of the fermi sea at points $p_i(x,t)$ and the lower edges of the fermi sea at $q_i(x,t)$ where $(i = 1, 2, \cdots i_m(x,t))$. Then

$$\int dp\ p^n\ u(x,p,t) = \frac{1}{2\pi\hbar(n+1)} \sum_{i=1}^{i_m} [p_i^{n+1} - q_i^{n+1}] \tag{2}$$

$i_m(x,t)-1$ denotes the number of "folds" at the point x at time t. In the absence of folds, one can set $\beta_+(x,t) = p_1(x,t)$ and $\beta_-(x,t) = q_1(x,t)$ which then implies that all the $w_{\pm,n} = 0$. This is the standard bosonization in terms of the collective field theory. As emphasized in [4] and [5] only in the absence of folds is the state described by a scalar field $\eta(x,t)$ and its momentum conjugate $\Pi_\eta(x,t)$, which are related to $\beta_\pm$ by $\beta_\pm = \Pi_\eta \pm \partial_x \eta$. In the asymptotic region the string theory massless tachyon is a nonlocal transform of $\eta(x,t)$. In the presence of folds we do not know how to extract the string theory space-time from the matrix model, since the $w_{\pm,n} \neq 0$ and are needed to specify the classical state of the fermi fluid while in the far asymptotic region the collective field configurations alone exhaust the possible configurations of the string theory tachyon field [1].

Under time evolution folds will generically form even if we start with profiles without any fold. Nevertheless, if we start with a non-folded tachyon wave with sufficiently small amplitude (for a given wavelength) the outgoing pulse at late times does not develop a fold. This is why one can specify the incoming and outgoing states by the collective field and obtain an S- matrix by performing a perturbation expansion in the strength of the wave. However, for a pulse which is sufficiently narrow and/or tall the outgoing pulse in the asymptotic region can have a fold even if the ingoing one did not. An extreme example is a pulse which is tall enough that its tip goes over to the other side of the potential barrier - it is these pulses for which we may expect black holes to form. We wish to emphasize that fold formation is invisible if the classical collective field equation is solved *only* in a perturbation expansion (in the weakness of the field). In this sense it is a genuinely nonlinear and nonperturbative effect.

In this paper we give the physical meaning of these "folds". (We do not however address the implications of fermi fluid crossing the potential barrier.) A basic question that arises is: under the boson-fermion map what is the bose state corresponding to the classical fermi fluid with fold? (The choice of map is distinct from the choice of Hamiltonian. In particular we can ask for the bosonisation of fold states for relativistic fermions as well.)

We will show that a folded configuration correspond to a rather nontrivial classical limit. Normally the classical limit of a quantum system is obtained by considering coherent states. The quantum fluctuations of Heisenberg picture operators in such states are suppressed by powers of $\hbar$ so that their expectation values become classical dynamical variables in the $\hbar \to 0$ limit. We will argue that for fermionic systems under consideration, folded classical configurations correspond to non-coherent states which have a large $O(1)$

[1]Away from the asymptotic region higher moments of $u(x,p,t)$ are required for determining the string theory tachyon. See Dhar et. al. in [2].

dispersion.

For relativistic fermions a (Schrodinger picture) coherent state cannot develop into such a noncoherent state under time evolution. However, for nonrelativistic fermions this happens generically in a time interval of order unity. In fact the time at which this happens is exactly equal to the time at which a fold forms in the classical picture. We will demostrate this explicitly in a model closely related to the fermionic field theory of the matrix model. The derivation makes it clear that the physics is entirely similar in the matrix model. This allows a natural way to identify the space-time in the matrix model based on a map of the quantum operators.

Let us first discuss the dynamics of fold formation in the classical theory. As shown in [6] the standard classical fermion algebra is consistent with imposing the Poisson brackets

$$\begin{aligned}
[\beta_\pm(x,t),\beta_\pm(y,t)]_{PB} &= 2\pi\hbar\partial_x\delta(x-y) \\
[w_{\pm,m}(x,t),w_{\pm,n}(y,t)]_{PB} &= 2\pi\hbar[m w_{\pm,m+n-1}(x,t) \\
&\quad + n w_{\pm,m+n-1}(y,t)]\partial_x\delta(x-y) \\
[w_{\pm,n}(x,t),\beta_\pm(y,t)]_{PB} &= 0
\end{aligned} \tag{3}$$

It is well known that powers of $\beta_\pm$ satisfy a w_∞ algebra. In this construction the $w_{\pm,n}$ themselves satisfy an independent w_∞ algebra. [2]

So far our considerations were independent of the fermion dynamics. Given the hamiltonian we can now obtain the time evolution equations for the quantities $\beta_\pm, w_{\pm,n}$ using the above definitions and the Poisson brackets. Since the $\pm$ components are decoupled we can treat only one of them. In the following we will omit the $\pm$ subscript. We will also concentrate on *free* fermions. A background potential can be easily incorporated and is not essential to the main point. For relativistic fermions one has a hamiltonian $H = \int dx \int dp\ p\ u(x,p,t)$. Using (1) and (3) we easily get

$$\partial_t\beta = -\partial_x\beta \qquad \partial_t w_m = -\partial_x w_m \tag{4}$$

For nonrelativistic fermions one has $H = \int dx \int dp\ \frac{1}{2}p^2\ u(x,p,t)$ and using the Poisson brackets one gets the evolution equations

$$\partial_t\beta = -\beta\partial_x\beta - \partial_x w_1; \quad \partial_t w_m = -2w_m\ \partial_x\beta - \beta\ \partial_x w_m - \partial_x w_{m+1} \tag{5}$$

The β, w_n are expressible in terms of the (p_i, q_i) introduced in (2). We choose a parametrization associated to the upper edge of the fermi surface,

[2] In [6] it was also found that these statements are true at the quantum level as well (W_∞ algebra) and the central charge of $w_{\pm,n}$ turns out to be zero. In this paper we will not be concerned with the quantum algebra.

i.e. we will choose all the $w_{-,n} = 0$ and

$$\beta_-(x,t) = q_{i_m} \qquad \beta_+(x,t) = \sum_{i=1}^{i_m} p_i - \sum_{i=1}^{i_m - 1} q_i \tag{6}$$

This identification is natural when the fold is close to the upper edge of the fermi sea but may be used even for a fold near the lower edge. We can then use the same identification for relativistic fermions where the fermi sea has no lower edge and one has to set all $\beta_- = w_{-n} = 0$. In fact since the question of folds has nothing to do with the presence or absence of a lower edge of the fermi sea we will simply imagine that the fermi sea is bottomless even for nonrelativistic fermions. This is a good approximation for boson energy densities small enough to not reach the bottom of the fermi sea in the fermi picture. Using (6), (1) and (2) we can now easily calculate all the $w_{\pm,n}$'s. Since the p_i, q_i do not contain any $\hbar$, the w_n 's, if nonzero, are also independent of $\hbar$.

Each of the p_i, q_i satisfy an evolution equation determined by the single particle hamiltonian: $\partial_t p_i = -\partial_x p_i$ for relatvisitic fermions and $\partial_t p_i = -p_i \partial_x p_i$ for nonrelativistic fermions (and similarly for q_i's). It may be checked that these equations then imply the corresponding evolutions for the β and w_m 's.

Let the profile of the fermi surface at $t = 0$ be given by $p(x,0) = a(x)$. Then at a later time t it is easy to show that

$$\begin{aligned} p(x,t) &= a(x-t) && \text{relativistic} \\ p(x,t) &= a(x - p(x,t)\ t) && \text{nonrelativistic} \end{aligned} \tag{7}$$

If there is a fold, there must be some point where $\frac{dx}{dp} = 0$ [10]. Since the profile in a relativistic system is unchanged in time a fold cannot develop from a profile with no folds. On the other hand for a nonrelativistic system, even if we start with a single valued $a(x)$ the solution to the second equation in (7) will generically lead to $\frac{dx}{dp} = 0$ at some point on the fermi surface at some time. We give the result for the time of fold formation t_f for two initial profiles:

$$p(x,0) \;=\; b\ e^{-\frac{(x-a)^2}{c^2}} - b\ e^{-\frac{(x+a)^2}{c^2}}, \qquad t_f \approx \frac{ce^{\frac{1}{2}}}{b\sqrt{2}} \quad \text{for} \quad \frac{a}{c} >> 1 \tag{8}$$

$$p(x,0) \;=\; k\ \mathrm{Re}(C_k\ e^{ikx}), \qquad t_f = \frac{1}{(|C_k|k^2)} \tag{9}$$

These examples clearly show that fold formation can occur for pulses of arbitrarily small energy density (and total energy), provided the width is sufficiently small.

It is now clear what happens if we set $w_n = 0$ in the time evolution equations (5). The evolution of β proceeds smoothly till $t = t_f$ at which point $\partial_x \beta$ diverges. The equations cannot be evolved beyond this point and the collective field theory fails. The time evolution of $u(p, q, t)$ is of course unambiguous [5]. The point is that one needs nonzero values of w_n beyond this time, whose presence render the classical time evolution of the entire β, w_n system completely well defined. While we have illustrated this phenomenon in the simple case of free nonrelativistic fermions, it is clear that the conclusions are same for the inverted harmonic oscillator potential which appears in 2d string theory.

We now turn to the quantum theory. Consider N free fermions of mass $M = 1$ in a box of size L. Define the field operators

$$\hat{\psi}(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \hat{\psi}_n \, e^{i\frac{2\pi n}{L}x} \qquad\qquad \hat{\psi}^\dagger(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \hat{\psi}^\dagger_n \, e^{-i\frac{2\pi n}{L}x} \tag{10}$$

where $\hat{\psi}_n$, $\hat{\psi}^\dagger_n$ are annihilation and creation operators for a fermion at level n. One has the standard anticommutation relations

$$[\hat{\psi}^\dagger_n, \hat{\psi}_m]_+ \;=\; \delta_{n,m}, \quad [\hat{\psi}_n, \hat{\psi}_m]_+ \;=\; [\hat{\psi}^\dagger_n, \hat{\psi}^\dagger_m]_+ \;=\; 0. \tag{11}$$

As mentioned above for the purpose of discussing the physics of folds the presence of a lower edge of the fermi sea for nonrelativistic fermions is an unnecessary complication. Henceforth we will assume that the fermi sea is bottomless. Thus the vacuum is defined, for both relativistic and nonrelativistic fermions by [3]

$$\hat{\psi}_n|0>= 0 \;\; \text{for} \;\; n > 0; \qquad\qquad \hat{\psi}^\dagger_n|0>= 0 \;\; \text{for} \;\; n \leq 0 \tag{12}$$

Let

$$\alpha_{-n}(t) \;=\; \textstyle\sum_{m=-\infty}^{\infty} :\hat{\psi}^\dagger_{n+m}(t)\hat{\psi}_m(t): \qquad \text{or} \qquad \alpha(x,t) =: \hat{\psi}^\dagger(x,t)\hat{\psi}(x,t):$$
$$\alpha(x,t) \;=\; \tfrac{1}{L}\textstyle\sum_{n=-\infty}^{\infty} \alpha_n(t) e^{i\frac{2\pi}{L}nx} \;\equiv\; \partial_x \phi(x,t) \tag{13}$$

The modes $\alpha_n(t)$ are thus shift operators on fermion levels. (See e.g. [7].) This bosonisation is exact as long as the action of the α_n do not move a state past the bottom of the fermi sea; in particular it is exact if the sea is bottomless. The normal ordering is defined as

$$:\hat{\psi}^\dagger_m\hat{\psi}_n: \;=\; \hat{\psi}^\dagger_m\hat{\psi}_n \;\; \text{if } n > 0 \qquad\quad :\hat{\psi}^\dagger_m\hat{\psi}_n: \;=\; -\,\hat{\psi}_n\hat{\psi}^\dagger_m \;\; \text{if } n \leq 0 \tag{14}$$

[3]We have chosen phases in (10) so that $n = 0$ is the vacuum fermi level

Then it may be verified that

$$[\alpha_n(t),\alpha_m(t)] \;=\; n\delta_{n+m,0} \quad \text{or} \quad [\alpha(x,t),\alpha(x,t)] \;=\; -i\frac{1}{2\pi}\delta'(x-x') \quad (15)$$

The inverse correspondence to (13) is given as follows:

$$\begin{aligned}\hat{\psi}(x) &= \frac{1}{\sqrt{L}}e^{-i\beta_0}\; e^{-i\frac{2\pi}{L}x\alpha_0}\; e^{-\sum_{j\geq 1}\frac{\alpha_{-j}}{j}\; e^{i\frac{2\pi}{L}jx}} e^{\sum_{j\geq 1}\frac{\alpha_j}{j}\; e^{-i\frac{2\pi}{L}jx}} \\ \hat{\psi}(x)^{\dagger} &= \frac{1}{\sqrt{L}}e^{i\frac{2\pi}{L}x\alpha_0}\; e^{i\beta_0}\; e^{\sum_{j\geq 1}\frac{\alpha_{-j}}{j}e^{i\frac{2\pi}{L}jx}}\; e^{-\sum_{j\geq 1}\frac{\alpha_j}{j}e^{-i\frac{2\pi}{L}jx}} \qquad (16)\end{aligned}$$

where the zero-mode operators satisfy $[\alpha_0,\beta_0]=-i$. In the classical limit, the classical quantities in (1) are then related to expectation values of appropriate operators in some quantum state, e.g.

$$\begin{aligned}\int dp\; u &= <:\psi^{\dagger}\psi:>=<\alpha(x,t)> \\ \int dp\; p\; u &= <\frac{i\hbar}{2}[:(\partial_x\psi^{\dagger})\psi-\psi^{\dagger}(\partial_x\psi):]>=2\pi\hbar <:\frac{\alpha^2}{2}:> \\ \int dp\; p^2\; u &= <\frac{-\hbar^2}{6}:[\psi^{\dagger}(\partial_x^2\psi)+(\partial_x^2\psi^{\dagger})\psi-4(\partial_x\psi^{\dagger})(\partial_x\psi)]: \\ &= 2\pi^2\hbar^2 <:\frac{\alpha^3}{3}:> \qquad (17)\end{aligned}$$

and so on. The second equalities in (17) follow from the operator correspondence (16) and may be efficiently obtained using operator product expansions. The equations (17) demonstrate the main point of this paper : *w_n are a measure of the quantum fluctuations in the bosonic field*. Thus, e.g. comparing (1) and (17) we have

$$w_1(x,t)=(2\pi\hbar)^2[\frac{1}{2}<:\alpha^2(x,t):>-\frac{1}{2}<\alpha(x,t)>^2] \qquad (18)$$

Note that in the normalizations we are using, the fluctuations of α must be $O(\frac{1}{\hbar^2})$ for $w_n\sim O(1)$.

We now consider the evaluation of $w_1(x,t)$ in a Heisenberg picture coherent state of bosons

$$|\psi>=\prod_{n=1}^{\infty}e^{\frac{C_n}{\hbar}\alpha_{-n}(0)}|0> \qquad (19)$$

It is straightforward to check that

$$<\alpha(x,0)>\equiv\frac{<\psi|\alpha(x,0)|\psi>}{<\psi|\psi>}=\frac{2}{\hbar L}\sum_{n=1}^{\infty}\mathrm{Re}[n\; C_n\; e^{2\pi\; i\; n\; x/L}] \qquad (20)$$

Note we have chosen our state such that $< \alpha(x,0) > \sim \frac{1}{\hbar}$ which is required for $\beta \sim O(1)$ (see (17)). It may be easily shown that

$$<: \alpha^n(x,0) :>=< \alpha(x,0) >^n \tag{21}$$

Thus the w_n vanish at $t=0$, and we have a fermi surface without folds. In the following we will consider a state with only one nonzero C_n for $n=\bar{n}$. The classical fluid profile is then given by (9) This is sufficient for our purpose.

In the Heisenberg picture, the fermion operators evolve as

$$\hat{\psi}_n(t) \ = \ \hat{\psi}_n e^{-i(\frac{2\pi}{L})^2 \frac{\hbar}{2} n^2 t} \qquad \hat{\psi}^\dagger_n(t) \ = \ \hat{\psi}^\dagger_n e^{i(\frac{2\pi}{L})^2 \frac{\hbar}{2} n^2 t} \tag{22}$$

The modes of the bosonic field then evolve as

$$\alpha_n(t) = e^{i\frac{\hbar}{2}(\frac{2\pi}{L})^2 n^2 t} \sum_m : \hat{\psi}^\dagger_{-n+m} \hat{\psi}_m : \lambda^m(n) \tag{23}$$

where

$$\lambda(n) = exp[-i\hbar(\frac{2\pi}{L})^2 \ n \ t] \tag{24}$$

Using the evolution of the fermion operators we can compute the expectation value of the bose operators. The details of the calculation will be given in [8]; here we note the results. Define

$$\Phi(n,p) = \frac{4|C_n|}{\hbar} \sin\,[(\frac{2\pi}{L})^2 \ \frac{\hbar p n t}{2}] \tag{25}$$

Let $p/\bar{n}$ be an integer. Then the one point function is

$$< \alpha_p(t) > \ = \ (-i)^{p/\bar{n}} \frac{\lambda(p)}{\lambda(p)-1} (\frac{C^{\frac{1}{2}}_{\bar{n}}}{C^{\frac{1}{2}*}_{\bar{n}}})^{\frac{p}{\bar{n}}} \ J_{p/\bar{n}}(\Phi(\bar{n},p)) \tag{26}$$

($J_n(z)$ is the Bessel function.) If $p/\bar{n}$ is not an integer then $< \alpha_p(t) >= 0$. Similarily, the two point function is nonzero only if $(p+q)/\bar{n}$ an integer:

$$\begin{aligned} < \alpha_p(t)\alpha_q(t) >_c \quad &\equiv \quad < \alpha_p(t)\alpha_q(t) > - < \alpha_p(t) >< \alpha_q(t) > \\ &= \quad \sum_{s > \frac{q}{\bar{n}}} (-i)^{\frac{p+q}{\bar{n}}} (\frac{C^{\frac{1}{2}}_{\bar{n}}}{C^{\frac{1}{2}*}_{\bar{n}}})^{\frac{p+q}{\bar{n}}} \ \ J_s(\Phi(\bar{n},q)) \ J_{\frac{p+q}{\bar{n}}-s}(\Phi(\bar{n},p)) \\ &\quad\quad \lambda(p)\lambda(q) \frac{(\lambda(p)\lambda(q))^{(q-\bar{n}s)/2} - (\lambda(p)\lambda(q))^{(\bar{n}s-q)/2}}{1-\lambda(p)\lambda(q)} \end{aligned} \tag{27}$$

As a special case we note $G(p,t) \equiv < \alpha_{-p}(t)\alpha_p(t) >_c$

$$G(p,t) = \sum_{s > \frac{p}{\bar{n}}} (\bar{n}s - p) J^2_s(\Phi(\bar{n},p)) \tag{28}$$

One may verify that the result (28) reproduces at $t = 0$ the expected result that $< \alpha_q(t)\alpha_p(t) >_c = q\theta(q)\ \delta_{p+q,0}$.

We can estimate on heuristic grounds the quantity $G(p,t)$ by considering the Schrodinger picture of evolution. The state $|\psi(t) >$ describes the fermi fluid, with a fold after some time t_0. Let $|p/\bar{n}| >> 1$, $p > 0$. For the action of such $\alpha(p)$ we may consider a succesion of small x space intervals, over each of which the sections of the fermi surface are approximately constant at heights p_i, q_i. The operator α_p is a lowering operator, which destroys a fermion at some level m and creates one at a level $m - p$. If there is no fold then $G(p,t) = 0$ since fermions cannot be lowered further into the already occupied levels. If there is one fold then α_p can move fermions from the interval (p_1, q_1) to the vacant levels in (q_1, p_2). The operator α_{-p} moves the fermions back, and we find $G(p,t) \neq 0$. More generally we get a result proportional to the number of vacant 'bands' which equals the number of folds. It may be also seen that $G(p) = 0$ for p very large. This is because for very large p, α_p must lower a fermion to a level inside the main bulk of the fermi sea. Since the *momenta* in the fermion picture are all of order unity, $G(p)$ should valish rapidly for $p > p_M$ where $p_M \sim O(\frac{1}{\hbar})$.

The result (28) has exactly this feature. First, it may be seen from (25) that when

$$p > p_M = \frac{4|C_{\bar{n}}|\ \bar{n}}{\hbar} \tag{29}$$

$G(p)$ vanishes exponentially with increasing p regardless of the $\hbar \to 0$ limit. This is because in this case the index of the Bessel function is *always* greater than the argument and Bessel functions decay exponentially when the ratio of the index to the argument grows large (see (30) below).

Consider now the classical limit for $p >> \bar{n}$, but $p < p_M$. In the $\hbar \to 0$ limit one has has $\lambda(p) = -\Psi(\bar{n}, p) = 1$ and $\Phi(\bar{n}, p) = 2|C_{\bar{n}}|(\frac{2\pi}{L})^2\ \bar{n}\ p\ t$. Because of the exponential decay of the Bessel function, the dominant contribution comes from the minimum allowed value of s in the sum in (28) which is $s_m = \frac{p}{\bar{n}}$. For large p the relevant Bessel function in (28) behaves as [9]

$$J_{s_m}(\Phi(\bar{n}, p)) \sim (2\frac{p}{\bar{n}}\pi \tanh\beta)^{-(1/2)} e^{-\frac{p}{\bar{n}}(\beta - \tanh\beta)} \tag{30}$$

where we have defined

$$\cosh\beta = (2|C_{\bar{n}}|(\frac{2\pi}{L})^2 \bar{n}^2 t)^{-1} \tag{31}$$

So long as $\beta > 0$, $G(p)$ is thus exponentially suppressed for large p. The exponential suppression disappears when $\beta = 0$ or when

$$t = t_0 = (2|C_{\bar{n}}|(\frac{2\pi}{L})^2 \bar{n}^2)^{-1} \tag{32}$$

The time t_0 is *exactly* equal to the time of fold formation t_f as calculated in equation (9).

When $t > t_f$ the relevant Bessel function behaves, for large $p/\bar{n}$ as

$$J_{\frac{p}{\bar{n}}}(x) \sim (\frac{2}{\frac{p}{\bar{n}}\pi \tan\beta})^{(1/2)} \cos(\frac{p}{\bar{n}}\tan\beta - n\beta - \pi/4) \tag{33}$$

where $\cos\beta = (2|C_{\bar{n}}|(\frac{2\pi}{L})^2\bar{n}^2 t)^{-1}$. At late times it may be shown that for $p/\bar{n} >> 1$ (but $p << p_M$)

$$G(p) \sim 2|C_{\bar{n}}|(\frac{2}{L})^2\pi\bar{n}^2 pt = p\ n_{\text{fold}} \tag{34}$$

where $n_{\text{fold}} = 2|C_{\bar{n}}|(\frac{2}{L})^2\pi\bar{n}^2 t = 2|p_{max}(x)|\frac{\bar{n}t}{L}$ is the number of folds computed from the classical motion of the fermi fluid.

Finally we estimate the quantities $w_n(x,t)$ and see whether they are nonzero in the classical limit. We will consider the quantity

$$\begin{aligned} w_{1,0}(t) &\equiv \int dx w_1(x,t) = (2\pi\hbar)^2 \sum_p < \psi(t)| : \alpha_{-p}\alpha_p : |\psi(t) >_c \\ &= 2(2\pi\hbar)^2 \sum_{p>0} G(p,t) \end{aligned} \tag{35}$$

Since $G(p,t)$ decays exponentially for $p > p_M$, we can effectively put an upper bound on the sum over p at p_M.

1. For $t < t_f$, one has a $G(p)$ which decays exponentially with p at rate *independent of* $\hbar$ (see equation (30)). Thus in this situation one has $w_{1,0}(t) \sim \hbar^2$ which vanishes in the classical limit.

2. For $t > t_f$ one has $G(p) \sim p$. In this case one clearly has $w_{1,0}(t) \sim \hbar^2 p_M^2$. Using (29) one then has $w_{1,0}(t) \sim O(1)$ and survives in the classical limit.

Thus we see that the presence of folds in the classical description signifies quantum fluctuations of the bosonic field which *survive in the* $\hbar \to 0$ *limit.*

While we have demonstrated our result in a simple model, it is clear from the derivation that our main contention is valid for the matrix model described in terms of fermions in an inverted harmonic oscillator potential - though the details would be more complicated. Since at the quantum level it is sufficient to have one scalar field to describe states, we have a natural way to extract the space-time of the string theory from that of the matrix model. The idea is to express an asymptotic state in terms of the bosonic oscillators $\alpha(\omega)$ and use a quantum map of the oscillators into oscillators of the string theory tachyon, $S(\omega)$. This map is the same as given in [2]

$$\alpha(\omega) = \frac{\Gamma(i\omega)}{\Gamma(-i\omega)}\ S(\omega) \tag{36}$$

States which come out folded at $t = +\infty$ are distinguished from non-folded states by their lack of coherence.

To summarize, we have shown that the classical limit of the two dimensional string theory defined as the double scaling limit of matrix quantum mechanics is rather nontrivial and differs from that in most other theories. The number of degrees of freedom needed for a classical description is more than the number in the exact quantum theory - the extra degrees of freedom correspond to nonzero fluctuations *even in the* $\hbar \to 0$ limit. This feature could not have been guessed from the the low energy effective action which is *locally* Lorentz invariant and in which unfolded configurations can never develop into folds; thus any 'classical' field configuration retains its small ($O(\hbar)$) dispersion throughout the evolution.

The fact that a fold corresponds to incoherent quanta agrees with the spirit of [2] where a fold state was termed 'radiation'. The spectrum we find however is not thermal. In fact from the discussion above we can estimate the distribution of energies if, say, a single fold forms when an incident coherent wave reflects from the tachyon potential in 1+1 string theory. If the incident wave has $\sim N^2$ quanta of energy E each, then the state returning from the wall has, crudely speaking, quanta with energies $E,\ 2E, \ldots \sim NE$ (with total energy equalling the incident energy). Thus the mean energy of the returning quanta is higher by a factor $\sim N$ compared to the incident quanta. In the classical limit $\hbar \to 0$, $N \to \infty$, this frequency amplification factor goes to infinity.

One possibility is that we cannot trust the naive summation of the string perturbation theory for non-infinitesimal coupling, and nonperturbative effects prevent the phenomenon discussed above. The other possibility is that string theory has a nontrivial sense of classical limit; in that case one wonders if such phenomena might occur in higher dimensions as well.

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