

DEGREES OF FREEDOM IN TWO DIMENSIONAL STRING THEORY *

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Abstract

We discuss two issues regarding the question of degrees of freedom in two dimensional string theory. The first issue relates to the classical limit of quantum string theory. In the classical theory one requires an infinite number of fields in addition to the collective field to describe “folds” on the fermi surface. We argue that in the quantum theory these are not additional degrees of freedom. Rather they represent quantum dispersions of the collective field which are *not* suppressed when $\hbar \rightarrow 0$ whenever a fold is present, thus leading to a nontrivial classical limit. The second issue relates to the ultraviolet properties of the geometric entropy. We argue that the geometric entropy is finite in the ultraviolet due to *nonperturbative* effects. This indicates that the true degrees of freedom of the two dimensional string at high energies is much smaller than what one naively expects.

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1 Introduction

Recent developments seem to indicate that the true degrees of freedom in string theory are quite different from what is expected from perturbation theory. In fact we already know this from our experience with noncritical strings : the fundamental formulation of two dimensional string theory is in terms of nonrelativistic fermions via matrix models and a stringy description in terms of the massless tachyon emerges as a low energy perturbative picture [1]. For higher dimensional strings we do not have a nonperturbative formulation, but the remarkable duality properties indicate that string theory has to be formulated in terms of some yet unknown entities which are not the strings of perturbation theory.

In this talk I will summarize some results which clarify the nature of degrees of freedom of the two dimensional string. This is a good laboratory since some quantities can be calculated nonperturbatively. , even though the space-time interpretation of the theory is rather involved and not yet fully understood. The first set of results, obtained in collaboration with S. Mathur [2] indicate that the classical limit of the theory is rather nontrivial and requires more degrees of freedom than the quantum theory itself. The second set of results [3] indicate that a suitably defined geometric entropy (or entropy of entanglement), which provides a measure of the degrees of freedom, is ultraviolet finite due to essentially *nonperturbative* effects This is relevant to the question of black hole entropy in string theory.

2 Folds and the Classical limit

Two dimensional string theory is described by N mutually noninteracting nonrelativistic fermions in $1+1$ dimensions in an external inverted harmonic oscillator potential with the second quantized action in terms of fermi fields $\psi(x, t)$

$$A_f = \lambda \int dt \int dx \psi^\dagger \left[-\frac{1}{2\lambda^2} \partial_x^2 + \mu_c - \frac{1}{2} x^2 \right] \psi \quad (1)$$

and $\lambda = \frac{N}{g}$ and g is the quartic coupling in the underlying matrix model. If μ denotes the fermi level of the single particle hamiltonian in (1), the continuum string theory is described by the double scaling limit $N \rightarrow \infty$ and $\mu \rightarrow \mu_c$ with $\kappa = \lambda(\mu_c - \mu) = \text{fixed}$.

2.1 Classical Dynamics

Let us first consider the classical dynamics of the fermion system in a manner which is independent of the form of the hamiltonian. The classical

state is specified by a distribution function $u(x, p, t)$ in phase space - in fact $u(x, p, t) = \frac{1}{2\pi\hbar}$ in some region bounded by the fermi surface and zero elsewhere. The fermi sea may be conveniently parameterized in terms of two functions β_{\pm} and an infinite set of additional fields $w_{n,\pm}$ [4]

$$\begin{aligned}
\int dp u(x, p, t) &= \frac{1}{2\pi\hbar}[\beta_+ - \beta_-] \\
\int dp p u(x, p, t) &= \frac{1}{2\pi\hbar}\left[\frac{1}{2}(\beta_+^2 - \beta_-^2) \right. \\
&\quad \left. + (w_{+1} - w_{-1})\right] \\
\int dp p^2 u(x, p, t) &= \frac{1}{2\pi\hbar}\left[\frac{1}{3}(\beta_+^3 - \beta_-^3) \right. \\
&\quad \left. + (\beta_+ w_{+1} - \beta_- w_{-1}) \right. \\
&\quad \left. + (w_{+2} - w_{-2})\right]
\end{aligned} \tag{2}$$

and so on. Let a line of constant x in phase space cut the upper edge of the fermi sea at points $p_i(x, t)$ and the lower edges of the fermi sea at $q_i(x, t)$ where $(i = 1, 2, \dots, i_m(x, t))$. Then

$$\int dp p^n u(x, p, t) = \sum_{i=1}^{i_m} \frac{[p_i^{n+1} - q_i^{n+1}]}{2\pi\hbar(n+1)} \tag{3}$$

$i_m(x, t) - 1$ denotes the number of “folds” at the point x at time t . In the absence of folds, one can set $\beta_+(x, t) = p_1(x, t)$ and $\beta_-(x, t) = q_1(x, t)$ which then implies that all the $w_{\pm,n} = 0$. This is the standard bosonization in terms of the collective field theory. As emphasized in [5] only in the absence of folds is the state described by a scalar field $\eta(x, t)$ and its momentum conjugate $\Pi_{\eta}(x, t)$, which are related to β_{\pm} by $\beta_{\pm} = \Pi_{\eta} \pm \partial_x \eta$.

In the deep asymptotic region, the string theory massless tachyon is a nonlocal transform of $\eta(x, t)$ [6]. In the presence of folds we do not know how to extract the string theory space-time from the matrix model, since the $w_{\pm,n} \neq 0$ while in the far asymptotic region the collective field configurations alone exhaust the possible configurations of the string theory tachyon field ¹.

As shown in [4] the Poisson bracket algebra for β_{\pm} is that of a chiral (antichiral) boson, while the $w_{n,\pm}$ commute with the β_{\pm} and themselves satisfy a w_{∞} algebra, for the \pm components separately. In the following we will omit the \pm subscript.

We will also concentrate on *free* fermions. A background potential can be easily incorporated and is not essential to the main point.

¹Away from the asymptotic region higher moments of $u(x, p, t)$ are required for determining the string theory tachyon. See Dhar et. al. in [6].

For free relativistic right moving fermions one has a hamiltonian $H = \int dx \int dp p u(x, p, t)$. The Poisson brackets lead to the equations of motion

$$\partial_t \beta = -\partial_x \beta \quad \partial_t w_m = -\partial_x w_m \quad (4)$$

For free nonrelativistic fermions one has $H = \int dx \int dp \frac{1}{2} p^2 u(x, p, t)$ and using the Poisson brackets one gets the evolution equations

$$\begin{aligned} \partial_t \beta &= -\beta \partial_x \beta - \partial_x w_1 \\ \partial_t w_m &= -2w_m \partial_x \beta - \beta \partial_x w_m - \partial_x w_{m+1} \end{aligned} \quad (5)$$

The β, w_n are expressible in terms of the (p_i, q_i) introduced in (3). We choose a parametrization associated to the upper edge of the fermi surface, i.e. we will choose all the $w_{-,n} = 0$ and

$$\beta_-(x, t) = q_{i_m} \quad \beta_+(x, t) = \sum_{i=1}^{i_m} p_i - \sum_{i=1}^{i_m-1} q_i \quad (6)$$

For relativistic fermions the fermi sea has no lower edge and one has to set all $\beta_- = w_{-,n} = 0$. Using (6), (2) and (3) we can now easily calculate all the $w_{\pm,n}$'s. Clearly, the w_n 's are independent of \hbar .

Each of the p_i, q_i satisfy an evolution equation determined by the single particle hamiltonian: $\partial_t p_i = -\partial_x p_i$ for relativistic fermions and $\partial_t p_i = -p_i \partial_x p_i$ for nonrelativistic fermions (and similarly for q_i 's). It may be checked that these equations then imply the corresponding evolutions for the β and w_m 's.

Let the profile of the fermi surface at $t = 0$ be given by $p(x, 0) = a(x)$. Then at a later time t it is easy to show that $p(x, t) = a(x - t)$ for relativistic chiral fermions and $p(x, t) = a(x - p(x, t) t)$ for nonrelativistic fermions. If there is a fold, there must be some point where $dx/dp = 0$. Since the profile in a relativistic system is unchanged in time a fold cannot develop from a profile with no folds. On the other hand for a nonrelativistic system, even if we start with a single valued $a(x)$ one will have $dx/dp = 0$ at some point on the fermi surface at some time. We give the result for the time of fold formation t_f for two initial profiles:

$$\begin{aligned} p(x, 0) &= b e^{-\frac{(x-a)^2}{c^2}} - (a \rightarrow -a), \quad t_f \approx \frac{ce^{\frac{1}{2}}}{b\sqrt{2}} \\ p(x, 0) &= k \operatorname{Re}(C_k e^{ikx}), \quad t_f = \frac{1}{(|C_k|k^2)} \end{aligned} \quad (7)$$

where $a \gg c$. Thus fold formation can occur for pulses of arbitrarily small energy density (and total energy), provided the width is sufficiently small.

In the following we will only consider low energy pulses so that we can ignore the presence of the bottom edge of the fermi sea.

It may be noted that the notion of folds depends on a particular choice of coordinates and momenta in phase space. In fact for nonrelativistic fermions in an inverted harmonic oscillator potential, one may perform a transformation to light cone like coordinates $x_{\pm} = p \pm x$ in phase space in which the fermions appear *relativistic* and the resulting bosonic fields β_{\pm} would be *free* (see O’Loughlin in [6]). In such a parametrization a non-folded profile cannot evolve into a fold. However, in the physical scattering process incoming states are defined in terms of quanta using x_- as the “space” while for the outgoing state x_+ has to be used as “space”. The classical scattering matrix is in fact given in terms of a Bogoluibov transformation Since a nonfolded profile in the x_- parametrization generically becomes a folded profile in the x_+ configuration one cannot evade the question of fold formation.

It is now clear what happens if we set $w_n = 0$ in the time evolution equations (5). The evolution of β proceeds smoothly till $t = t_f$ at which point $\partial_x \beta$ diverges. The equations cannot be evolved beyond this point and the collective field theory fails. The time evolution of $u(p, q, t)$ is of course unambiguous. The point is that one needs nonzero values of w_n beyond this time, whose presence render the classical time evolution of the entire β, w_n system completely well defined. The same phenomenon happens in the presence of a background potential.

2.2 The quantum theory

Let us now turn to the quantum theory. Since the bottom edge of the fermi sea is irrelevant for the question of fold formation, we can completely ignore its presence. Then we have an exact operator bosonization of the system. Define Schrodinger picture field operators

$$\hat{\psi}(x) = \frac{1}{\sqrt{L}} \sum_{n=-\infty}^{\infty} \hat{\psi}_n e^{i\frac{2\pi n}{L}x} \quad (8)$$

with a similar mode decomposition for $\psi^{\dagger}(x)$ which define the modes $\psi_m^{\dagger}, \hat{\psi}_n, \hat{\psi}_n^{\dagger}$ are annihilation and creation operators for a fermion at level n obeying the standard anticommutation relations $[\hat{\psi}_n^{\dagger}, \hat{\psi}_m]_{+} = \delta_{n,m}$ and all other anticommutators being zero. The vacuum is defined by ² $\hat{\psi}_n|0\rangle = 0$ for $n > 0$ and $\hat{\psi}_n^{\dagger}|0\rangle = 0$ for $n \leq 0$.

Let

$$\alpha_{-n} = \sum_{m=-\infty}^{\infty} : \hat{\psi}_{n+m}^{\dagger} \hat{\psi}_m : \quad (9)$$

²We have chosen phases in (8) so that $n = 0$ is the vacuum fermi level

The position space field is defined by a usual fourier transform. The modes α_n are thus shift operators on fermion levels. (See e.g. [7].) The normal ordering is defined according to the vacuum defined above. Then it may be verified that

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0} \quad (10)$$

The inverse correspondence to (9) is also well known and given e.g. in [7].

In the classical limit, the classical quantities in (2) are then related to expectation values of appropriate operators in some quantum state, $|\xi, t\rangle$

$$\begin{aligned} \int dp u &= \langle : \psi^\dagger \psi : \rangle = \langle \alpha(x) \rangle \\ \int dp p u &= \langle \frac{i\hbar}{2} [: (\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) :] \rangle \\ &= 2\pi\hbar \langle : \frac{\alpha^2}{2} : \rangle \end{aligned} \quad (11)$$

Similarly, $\int dp p^n u \sim \langle : \alpha^n : \rangle$. In (11) $\langle A \rangle = \langle \xi, t | A | \xi, t \rangle$. The second equalities in (11) follow from the operator expressions of ψ in terms of α [7]. The equations (11) demonstrate the main point : *w_n are a measure of the quantum fluctuations in the bosonic field.* Thus, e.g. comparing (2) and (11) we have

$$w_1(x, t) = \frac{(2\pi\hbar)^2}{2} [\langle : \alpha^2(x) : \rangle - \langle \alpha(x) \rangle^2] \quad (12)$$

Note that in the normalizations we are using, the fluctuations of α must be $O(\frac{1}{\hbar^2})$ for $w_n \sim O(1)$.

2.3 The classical limit of the quantum theory

Normally the classical limit of a quantum system is obtained by considering coherent states. In such states, quantum dispersions of operators are suppressed by powers of \hbar . In an interacting theory, time evolution does not keep the state coherent, but once again the departures are such that the quantum dispersions are still suppressed in the $\hbar \rightarrow 0$ limit. Consequently, expectation values of operators in such coherent states act as classical dynamical variables and obey the classical equations of motion.

In our system the coherent states which represent the deformations of the fermi sea are coherent states of the boson field $\phi(x, t)$. Consider such a state at $t = 0$ given by

$$|\xi, 0\rangle = \prod_{n=1}^{\infty} e^{\frac{c_n}{\hbar} \alpha_{-n}} |0\rangle \quad (13)$$

It is straightforward to check that

$$\langle \xi, 0 | \alpha(x) | \xi, 0 \rangle = \frac{2}{\hbar L} \sum_{n=1}^{\infty} \text{Re}[n C_n e^{2\pi i n x / L}] \quad (14)$$

Note that $\langle \alpha(x) \rangle \sim \frac{1}{\hbar}$ which is required for $\beta \sim O(1)$ (see (11)). It may be easily shown that (at $t = 0$)

$$\langle : \alpha^n(x) : \rangle = \langle \alpha(x) \rangle^n \quad (15)$$

Thus the w_n vanish at $t = 0$, and we have a fermi surface without folds. In the following we will consider a state with only one nonzero C_n for $n = \bar{n}$. The classical fluid profile is then given by (7) This is sufficient for our purpose.

The time evolution of n -point functions of $\alpha(x)$ is most easily computed in the Heisenberg picture using the free equations of motion of the fermi fields. The details of the calculation are given in [2]. To illustrate our main point it is sufficient to give the result for the quantity $G(p, t) = \langle \xi, t | \alpha_{-p} \alpha_p | \xi, t \rangle$

$$G(p, t) = \sum_{s > \frac{p}{\bar{n}}} (\bar{n}s - p) J_s^2(\Phi(\bar{n}, p)) \quad (16)$$

where J_s denotes the Bessel function and

$$\Phi(n, p) = \frac{4|C_n|}{\hbar} \sin \left[\left(\frac{2\pi}{L} \right)^2 \frac{\hbar p n t}{2} \right] \quad (17)$$

It may be seen from (16) and (17) that when

$$p > p_M = \frac{4|C_{\bar{n}}| \bar{n}}{\hbar} \quad (18)$$

$G(p)$ vanishes exponentially with increasing p regardless of the $\hbar \rightarrow 0$ limit. This is because in this case the index of the Bessel function is *always* greater than the argument and Bessel functions decay exponentially when the ratio of the index to the argument grows large (see (19) below).

Consider now the classical limit for $p \gg \bar{n}$, but $p < p_M$. The argument of the Bessel function is now smaller than the index and the dominant contribution comes from the minimum allowed value of s in the sum in (16) which is $s_m = \frac{p}{\bar{n}}$. For large p the relevant Bessel function in (16) behaves as

$$J_{s_m}(\Phi(\bar{n}, p)) \sim \frac{e^{-\frac{p}{\bar{n}}(\beta - \tanh \beta)}}{(2\frac{p}{\bar{n}}\pi \tanh \beta)^{(1/2)}} \quad (19)$$

where we have defined

$$\cosh \beta = (2|C_{\bar{n}}| \left(\frac{2\pi}{L} \right)^2 \bar{n}^2 t)^{-1} \quad (20)$$

So long as $\beta > 0$, $G(p)$ is thus exponentially suppressed for large p . The exponential suppression disappears when $\beta = 0$ or when

$$t = t_0 = (2|C_{\bar{n}}|(\frac{2\pi}{L})^2\bar{n}^2)^{-1} \quad (21)$$

The time t_0 is *exactly* equal to the time of fold formation t_f as calculated in equation (7).

In fact the above results are expected in the absence of folds. For the calculation of $G(p, t)$ with $p \gg \bar{n}$ the fermi surface may be considered to be flat in a small interval in x . Since α_p is a lowering operator for $p > 0$ it is clear that $G(p) = 0$ for such large p since there is no empty level below the fermi surface. However, if there are folds, there are empty bands separating filled levels and $G(p)$ can be nonzero. However for p very large (of $O(\frac{1}{\hbar})$ in order to have an order one momentum) $G(p)$ has to be again zero since α_p would try to move a fermion into the bulk of the fermi sea.

Indeed for $t > t_f$ the relevant Bessel function behaves, for large p/\bar{n} as

$$J_{\frac{p}{\bar{n}}}(x) \sim \frac{\cos(\frac{p}{\bar{n}} \tan \beta - n\beta - \pi/4)}{(\frac{p}{\bar{n}} \pi \tan \beta)^{(1/2)}} \quad (22)$$

where $\cos \beta = (2|C_{\bar{n}}|(\frac{2\pi}{L})^2\bar{n}^2t)^{-1}$. At late times it may be shown that for $p/\bar{n} \gg 1$ (but $p \ll p_M$)

$$G(p) \sim 2|C_{\bar{n}}|(\frac{2}{L})^2\pi\bar{n}^2pt = p n_{\text{fold}} \quad (23)$$

where $n_{\text{fold}} = 2|C_{\bar{n}}|(\frac{2}{L})^2\pi\bar{n}^2t = 2|p_{\text{max}}(x)|\frac{\bar{n}t}{L}$ is the number of folds computed from the classical motion of the fermi fluid.

Finally we estimate the quantities $w_n(x, t)$ and see whether they are nonzero in the classical limit. We will consider the quantity

$$w_{1,0}(t) \equiv \int dx w_1(x, t) = 2(2\pi\hbar)^2 \sum_{p>0} G(p, t) \quad (24)$$

Since $G(p, t)$ decays exponentially for $p > p_M$, we can effectively put an upper bound on the sum over p at p_M .

1. For $t < t_f$, one has a $G(p)$ which decays exponentially with p at a rate *independent of \hbar* (see equation (19)), and the upper limit of integration is irrelevant. Thus in this situation one has $w_{1,0}(t) \sim \hbar^2$ which vanishes in the classical limit.
2. For $t > t_f$ one has $G(p) \sim p$. In this case one clearly has $w_{1,0}(t) \sim \hbar^2 p_M^2$. Using (18) one then has $w_{1,0}(t) \sim O(1)$ and survives in the classical limit.

Thus we see that the presence of folds in the classical description signifies quantum fluctuations of the bosonic field which *survive in the $\hbar \rightarrow 0$ limit*.

While we have demonstrated our result in a simple model, it is clear from the derivation that our main contention is valid for the matrix model described in terms of fermions in an inverted harmonic oscillator potential - though the details would be more complicated. Furthermore the effect of the lower edge of the fermi sea becomes relevant for states with high energy. As mentioned the question of folds is independent of this, but a complete treatment should incorporate this feature.

3 Geometric entropy in the 2d string

Let us now turn to an effect which is a direct result of the presence of a lower edge of the fermi sea of nonrelativistic fermions. The physical question relates to the notion of geometric entropy or the entropy of entanglement and is relevant in black hole physics. The entropy of black holes has a part which is *intrinsic* - the Hawking-Beckenstein entropy [8]. This is a classical contribution and is usually evaluated by computing the classical action for euclidean black holes. In addition to this, there is the quantum correction to the black hole entropy [9], which is the entropy of matter outside the black hole horizon. This, in turn, is related to the entropy of entanglement [10] (in a given quantum state of matter) between the region inside and outside the black hole.

The large mass limit of a black hole is Rindler space. In Rindler space the geometric entropy can be obtained as follows. Let x denote a spatial coordinate in *flat* space. Denote fields in the region $x > 0$ ($x < 0$) by ϕ_R (ϕ_L). In some given quantum state with a wave functional $\Psi_i[\phi_L, \phi_R]$ the reduced density matrix (unnormalized) is obtained by integrating out ϕ_L is given by

$$\rho[\phi_R, \phi'_R] = \int \mathcal{D}\phi_L \Psi_i[\phi_L, \phi_R] \Psi_i[\phi_L, \phi'_R] \quad (25)$$

Then the geometric entropy is given by

$$S_g = -\text{tr}[\rho \log \rho] \quad (26)$$

For scalar and spinor fields the geometric entropy and the quantum correction to the Rindler space entropy are exactly the same. For like gauge fields the two differ by contact terms corresponding to states exactly on the horizon [11].

The geometric entropy is in turn the same as the ordinary thermal entropy in a space with *position-dependent* temperature, the position dependence being given by the standard Tolman relation in Rindler space.

A crucial property of the geometric entropy is that it is ultraviolet divergent in usual field theories. This may be understood from the above thermodynamic interpretation. For Rindler space the position dependent temperature is given by $T(x) = \frac{1}{\beta(x)} = \frac{1}{2\pi x}$. Consider for example a massless scalar field in $d + 1$ dimensions. The geometric entropy may be calculated by integrating the entropy density in this position dependent temperature.

$$S_g = \int dx \int \frac{d^d k}{(2\pi)^d} \left[\log(1 - e^{-2\pi x|k|}) - \frac{1}{1 - e^{-2\pi x|k|}} \right] \quad (27)$$

The ultraviolet divergence in (27) may be seen by either performing the x integral first so that the resulting k integral is divergent from the high momentum limit for $d > 1$, or by integrating over k first so that the resulting x integral is divergent at the small x end. For $d = 1$ there is an additional infrared divergence coming from $k = 0$ or equivalently $x = \infty$ behaviour.

In the following we will be interested in $1 + 1$ dimensional theories. In that case there is a simple way to understand the divergence of the geometric entropy [12]. Consider dividing space x in a box of size L into two halves at $x = 0$. The minimum wave number is $k_0 = \frac{2\pi}{L}$. Now construct a set of *non-overlapping* wave packets with wave numbers in the range $2^j k_0 < k < 2^{j+1} k_0$ with integers j to be specified in a moment. For each j there is just *one* wave packet which has support *both* in the region $x < 0$ as well as $x > 0$. This one wavepacket alone contributes to the geometric entropy an amount of order one. By conformal invariance we have an equal contribution from each wave number range, i.e. for each allowed value of j . However since there is an ultraviolet cutoff Λ this allowed range is $1 < j < j_{max}$ where $j_{max} \sim \log(L\Lambda)$. The total geometric entropy is given by some number proportional to j_{max} , thus explaining the $d = 1$ answer.

The above reasoning may be applied to a situation which is more closely related to the calculation of the entanglement entropy in an evolving black hole geometry [13],[12]. This is the entropy of entanglement between the region $x_1 < x < x_2$ and the region outside this one on a one-dimensional line. The relevant part of entanglement is now due to modes of wavelengths less than the size of the region integrated over, i.e. $\lambda < (x_2 - x_1)$. Applying the above argument one gets

$$S_G(\{x_1, x_2\}) \sim \log[(x_2 - x_1)\Lambda] \quad (28)$$

There is another cutoff independent contribution to this entropy : this comes from modes which are constant in the region (x_1, x_2) , and will be unimportant for our purpose.

In an evaporating black hole geometry, the ultraviolet cutoff in (28) has to be multiplied with the scale factor of the metric at the horizon. This gives the true significance of the ultraviolet divergence : the evolution of the scale factor gives the time dependence of the geometric entropy. In this case this implies that the geometric entropy may increase indefinitely and one may dump arbitrarily large amounts of information into the black hole. Thus, as long as the semiclassical approximation is valid information will be lost.

String theories are ultraviolet finite and one may think that the geometric entropy in string theory is finite (see Susskind in [9]). However this is not so in string perturbation theory. It turns out that the genus one contribution is still divergent - though this divergence may be now interpreted as an infrared divergence related to the Hagedorn transition at the very high local temperatures at the horizon [14]. String physics beyond the Hagedorn temperature is necessarily nonperturbative in nature.

Since two dimensional string theory is the only model for which a nonperturbative formulation is known it is interesting to ask how nonperturbative effects alter the behaviour of the geometric entropy in this theory. Unfortunately despite several attempts we still do not have a description of black holes in the matrix model. However the notion of geometric entropy can be formulated in the usual vacuum of the matrix model : and this is precisely what we will do.

To understand the quantity we want to compute it is best to use the collective field description of the fermionic field theory of the two dimensional string [15]. The dynamics of fluctuations of the collective field is denoted by $\xi(\tau, t)$ and is governed by the hamiltonian

$$H_c = \int d\tau \left[\frac{1}{2} (\pi_\xi^2 + (\partial_\tau \xi)^2) + \frac{1}{6\rho_0^2} ((\partial_\tau \xi)^3 + 3\pi_\xi (\partial_\tau \xi) \pi_\xi) \right] \quad (29)$$

plus some singular terms which are responsible for a *finite* quantum ground state energy, but unimportant for our purposes. In (29) $\rho_0(x) = \sqrt{x^2 - \kappa}$ is the classical value of the collective field and we have introduced a time of flight coordinate $\tau = \int^x dx/\rho_0$. In terms of the space coordinate x the effective coupling of the theory is given by $g_{eff} = \frac{1}{x^2 - \kappa} = \frac{1}{\kappa \sinh^2 \tau}$. The fermi level κ therefore controls the strength of the coupling. The coupling is weak in the asymptotic region $\tau = \pm\infty$.

In the asymptotic region the massless field ξ may be related to the massless “tachyon” $S(\phi, t)$ of the two dimensional string by a nonlocal transform

$$S(\phi, t) = \int dk d\tau e^{-ik(\phi - \tau)} \frac{\Gamma(-ik)}{\Gamma(ik)} \xi(\tau, t) \quad (30)$$

What we mean by the geometric entropy in this model is the following. Pick some region \mathcal{I} in the space defined by the coordinate τ (or equivalently x which is *locally* related to τ) and obtain the entropy of entanglement in the ground state between this region and its complement. It is clear from (30) that this is not the same as the geometric entropy of some region in ϕ space. However, we are interested in the ultraviolet behavior of the entropy of entanglement and as we will explain later, we expect this to be similar in ϕ space.

3.1 Free nonrelativistic fermions in a box

As explained above the geometric entropy obtained by integrating out the degrees of freedom in half of the space, $x < 0$ is the same as the ordinary thermodynamic entropy in a position dependent temperature $T(x) = \frac{1}{2\pi x}$. Let us first consider the entropy *density* s for a gas of free nonrelativistic fermions in a box of size L at temperature $T = \frac{1}{\beta}$ and chemical potential μ_F . The fermi momentum is $k_F = \frac{2\pi N}{L}$. For $\beta\mu_F \gg 1$, i.e low temperatures one has

$$s = \frac{\pi}{6\beta k_F} + \frac{8\pi^3}{45\beta^3 k_F^5} + \dots \quad (31)$$

The first term is the same as that for a relativistic boson or a chiral relativistic fermion and would lead to a logarithmically divergent geometric entropy. However the divergence comes from the high temperature behavior where the expression (31) is invalid. In this regime $\beta k_F \ll 1$ and one has

$$s = \frac{k_F}{2\pi} \left[1 - \frac{1}{2} \ln \left(\frac{\beta k_F^2}{2\pi} \right) + \dots \right] \quad (32)$$

Thus at high temperatures, the entropy density increases only logarithmically - and this does not lead to any divergence from the $x = 0$ end of the integral $\int_0^\infty dx s(\beta = 2\pi x)$ for the geometric entropy. The $x = \infty$ behavior of this integral is still governed by (31) and leads to the standard infrared divergence. However there is no ultraviolet divergence.

It is clear that the answer is ultraviolet finite because the fermi sea has a finite depth. The low temperature expansion is an expansion around the relativistic limit, in which the depth of the fermi sea is infinite. In terms of the collective field theory the ultraviolet finiteness is a nonperturbative effect since for free fermions the effective coupling is $g_{eff} = \frac{1}{\mu_F}$. Perturbation theory would correspond to an expansion around $\mu_F = \infty$ and would lead to a divergence by virtue of (31)

We now outline the direct calculation of the geometric entropy from the ground state wave functional, where the above points can be clearly seen. We

will write the wave functional in a coherent state basis in terms of grassmann fields $\psi(k)$ and $\bar{\psi}(k)$ which are eigenvalues of the corresponding operators in appropriate coherent states. It is convenient to shift the origin of momentum and define $k = k_F + q$ and new fields $\chi(q) = \psi(k_F + q)$. The ground state wave functional is then given by

$$\begin{aligned} \Psi_0 = & \exp \left[-\frac{1}{2} \int_{-\infty}^{\infty} dq \bar{\chi}(q) \chi(q) \right. \\ & \left. + \int_{-2k_F}^0 dq \bar{\chi}(q) \chi(q) \right] \end{aligned} \quad (33)$$

When the terms in the exponent are converted into position space integrals, the first term is the integral over a *local* quantity. Therefore this cannot contribute to the geometric entropy. The entire contribution comes because of the second term, which is the contribution of the single particle modes in the fermi sea. For a finite box size, the number of these modes is *finite*. The perturbation expansion in the collective field theory is however an expansion around the point $k_F = \infty$. With an ultraviolet cutoff Λ , this means an expansion around $k_F = \Lambda$. In fact for $k_F = \infty$ the wave functional (33) exactly reproduces that for relativistic fermions with the fermi sea replaced by the Dirac sea. In this limit, there are an large number of modes which contribute to the geometric entropy (the largeness controlled by the ultraviolet cutoff) and one has the usual dependence on Λ . Non-perturbatively, however, the finiteness of k_F (which is always much less than the ultraviolet cutoff) renders the answer independent of Λ .

To obtain an expression for the geometric entropy we have to first expand the field in terms of modes which are localized either in the region $x < 0$ or in the region $x > 0$. A convenient way is to write

$$\begin{aligned} \chi(x) = & \frac{1}{\sqrt{|x|}} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} [\theta(x)(x/a)^{-i\omega} f_R(\omega) \\ & + \theta(-x)(-x/a)^{-i\omega} f_L(\omega)] \end{aligned} \quad (34)$$

The modes $f_R(\omega)(f_L(\omega))$ are now modes localized in the regions $x > 0(x < 0)$. One has to now express the modes $\chi(q)$ above in terms of f_L and f_R to write the wave functional as $\Psi_0[f_L, f_R]$. The density matrix for the region $x > 0$ is then simply given by a grassmann integration over f_L , from which one can obtain the geometric entropy. This calculation, described in [3] clearly shows that geometric entropy has no dependence on the ultraviolet cutoff.

3.2 Geometric Entropy in Matrix Model

The above exercise contains almost all the physics involved in understanding the geometric entropy in the matrix model. Let us first examine the high temperature behaviour of the theory. Fortunately, it is sufficient to know the singlet sector thermodynamics if we are interested in the geometric entropy in the ground state, which is a singlet. This follows from the relation between geometric entropy and thermodynamic entropy with a position dependent temperature. The singlet sector thermodynamics has been completely solved in [16] whose results will be used below.

Let g_c be the critical value of the coupling in the matrix model and define $\Delta = g_c - g$. Then the chemical potential μ is determined by the equation

$$\frac{\partial \Delta}{\partial \mu} = \text{Re} \left[\int_0^\infty dt \frac{(1/\kappa)e^{-it}}{2\pi \sinh(t/\kappa)} \frac{(\pi t/\kappa\beta)}{\sinh(\pi t/\kappa\beta)} \right] \quad (35)$$

plus an unimportant constant. The free energy is then given by $\partial F/\partial \Delta = \lambda^2(\mu - \mu_c)$ and the entropy is given by $S = \beta^2[(\partial F/\partial \beta)_\mu - \lambda^2\mu(\partial \Delta/\partial \beta)_\mu]$.

The expression (35) has an important symmetry - T duality - under $\beta \rightarrow \frac{\pi^2}{\beta}$ together with $\lambda \rightarrow \frac{\lambda\beta}{\pi}$. The transformation of λ or equivalently the string coupling $g_s = \frac{1}{\kappa}$ is the usual shift of the dilaton field required in T -duality. This is a very stringy symmetry and it is valid exactly in this model.

The standard genus expansion consists of performing a $\frac{1}{\kappa}$ expansion, which means expanding both the hyperbolic functions in (35) in power series. The genus one term in the free energy is exactly equal to the genus one Polyakov path integral calculation and may be expressed as a modular invariant integral over the moduli space of the torus [17].

Modular invariance is the underlying reason for ultraviolet finiteness of string theories. However it is clear from our result that the geometric entropy obtained by simply using this one loop answer is ultraviolet divergent as usual. Modular invariance is clearly not sufficient to make the geometric entropy finite in string theory. The issue is the behaviour at very high temperatures.

Strings in two dimensions do not have a hagedorn transition. In the matrix model the singlet sector free energy is perfectly well defined in the genus expansion. This may seem to indicate that there is nothing special about high temperatures in this caricature string theory.

The crucial result of our work is that the genus expansion breaks down at sufficiently high temperatures. The genus expansion [16] has the property that terms with higher and higher powers of the string coupling contain also higher and higher powers of the temperature - which clearly shows that while the genus expansion is consistent with a low temperature expansion, it is inconsistent with a high temperature expansion.

However we can use T -duality to obtain the high temperature behaviour from the *exact* (i.e. nonperturbative) low temperature behaviour, which can be obtained from the above formulae. The exact low temperature behaviour is required since T -duality involves a rescaling of the string coupling κ . The results are given in [3] and show that indeed there is a drastic modification of the high temperature behaviour compared to perturbative expectations.

However these results turn out to be pathological : the specific heat turns out to be negative in the high temperature limit. This is due to the fact that the model of inverted harmonic oscillator with no walls at infinity has a built-in instability in it which is, however, invisible in the genus expansion. One may work with a regulated potential by putting walls at a distance of order N from the hump and there cannot be any instability in this model. In fact at very high temperatures there are a large number of very high energy fermion excitations which perceive an almost constant potential. Using our previous results for free fermions we then expect an ultraviolet finite behaviour of the geometric entropy. The results would be however necessarily non-universal.

Finally let us turn to an estimate of the geometric entropy of the matrix model from the direct evaluation of the wave functional and the density matrix [18]. We want to compute the entropy of entanglement between some region $x_2 < x < x_1$ and its complement. This quantity is logarithmically divergent (see (28)) in 2d massless scalar field theory - which is the one loop answer in our model. In principle, this quantity may be computed using exact eigenstates of the inverted harmonic oscillator hamiltonian (parabolic cylinder functions) and transforming to a basis of modes which are localized either inside or outside this interval.

When the box $x_2 < x < x_1$ is very far away from the potential hump and has a size $(x_1 - x_2)$ much smaller than the total size, it is, however, possible to obtain an estimate of the geometric entropy. This is because one may use plane wave modes in this region to obtain an estimate. The logic is similar to the one used in deriving (28). The entropy comes from wavepackets which are made of waves of wavelength less than the size of the region, i.e. $\lambda < (x_1 - x_2)$. However for fermions the entropy of entanglement comes *only* from the modes which are in the filled fermi sea. If $x_0 = \frac{1}{2}(x_1 + x_2)$ is the central position of the box, the effective depth of the fermi sea at this point is $\sqrt{x_0^2 - \kappa}$ so that the relevant wavelengths lie in the interval $\sqrt{x_0^2 - \kappa} < \lambda < (x_1 - x_2)$. Using the expression for the effective coupling g_{eff} we get [18]

$$S_{geom} \sim \log [(x_1 - x_2)^2 g_{eff}(x_0)] \quad (36)$$

The answer depends on the logarithm of the effective coupling which now replaces the ultraviolet cutoff.

4 Conclusions

We have described two effects which throw light on the nature of true degrees of freedom in two dimensional string theory. The first demonstrates that the classical limit of the theory is rather nontrivial : one needs more variables at the classical level than in the full quantum theory. These extra dynamical variables are in fact *quantum* fluctuations which, however, survive in the $\hbar \rightarrow 0$ limit. The second effect implies that the degrees of freedom which contribute to the geometric entropy in the model are much less in number than what is expected from perturbative considerations and leads to a ultraviolet finite answer - the underlying reason is the finite depth of the fermi sea and hence nonperturbative.

At present the formalism of string theories is inadequate to determine whether there are similar effects in higher dimensions. However, the first effect we described has an uncanny resemblance with the discovery that quantum effects are not suppressed even in the $g_{st} \rightarrow 0$ limit of Type II strings moving on a Calabi Yau conifold [19] - though the mechanism appears to be rather different. Finally there are several circumstantial evidences that even higher dimensional string theories are fundamentally described by fewer degrees of freedom than what one might naively think [20]. However we dont know what these fundamental degrees of freedom (like fermions for two dimensions) are.

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