A THEOREM ON SUMS OF POWERS WITH APPLICATIONS TO THE ADDITIVE THEORY OF NUMBERS.

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1. LET us denote by N(k) the least value of s such that the equation

$$(1) \qquad \sum_{m \leqslant s} a_m^t = \sum_{n \leqslant s} b_n^t \quad (1 \leqslant t \leqslant k)$$

has a solution in integers a_m $(m \leqslant s)$ and b_n $(n \leqslant s)$ where

$$(2) \qquad a_m \ (m \leqslant s) \neq b_n \ (n \leqslant s).$$

We shall show that

Theorem.

$$(3) \qquad \mathrm{N}(k) \leqslant \frac{k^2+k}{2} + 1.$$

This is an improvement on previous results.[1]

2. In what follows *the* B'*s are real numbers which vary from place to place but are greater than a positive constant depending only on* k *and* s. $n_1, \cdots, n_s$ denote positive integers.

It follows from Dirichlet's principle ("Schubfachschluss") that there infinitely many integers n with the property that the equation

$$(4) \qquad n_1^k + \cdots + n_s^k = n$$

has $\mathrm{B}n^{\frac{s}{k}-1}$ solutions in $n_1, \cdots, n_s$.

Now (4) implies

$$(5) \qquad n_1 + \cdots + n_s \leqslant sn^{\frac{1}{k}}.$$

From (4), (5) and the "Schubfachschluss" it follows that there are infinitely many sets n, n' of integers with the property that the equations

$$(6) \qquad \left.\begin{aligned} n &= \sum_{r \leqslant s} n_r^k \\ n' &= \sum_{r \leqslant s} n_r \end{aligned}\right\}$$

have $\mathrm{B}n^{\frac{s}{k}-1-\frac{1}{k}}$ solutions in n_r $(r \leqslant s)$.

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[1] See *Proc. Ind. Acad. Sci.* (A), 1935, **1**, 528-530.

Again (6) implies

$$(7) \qquad \sum_{r \leqslant s} n_r^2 \leqslant s n^{\frac{2}{k}} .$$

Hence ("Schubfachschluss") from (6) and (7) it follows that there are infinitely many sets of integers (n, n', n'') with the property that the equations

$$(8) \qquad \begin{cases} n = \sum_{r \leqslant s} n_r^k, \\ n' = \sum_{r \leqslant s} n_r, \\ n'' = \sum_{r \leqslant s} n_r^2 \end{cases}$$

have $Bn^{\frac{s}{k} - 1 - \frac{1}{k} - \frac{2}{k}}$ solutions in n_r $(r \leqslant s)$. Proceeding in this manner we arrive at the result that there are infinitely many sets of integers n, n', n'', $\cdots, n^{(k-2)}, n^{(k-1)}$ such that the system of equations

$$(9) \qquad n = \sum_{r \leqslant s} n_r^k$$

$$(9) \qquad n^{(m)} = \sum_{r \leqslant s} n_r^m \quad (1 \leqslant m \leqslant k-1)$$

has Bn^p solutions where

$$(10) \qquad p = \frac{s}{k} - 1 - \left(\frac{1}{k} + \frac{2}{k} + \cdots + \frac{k-1}{k}\right).$$

Now

$$(11) \qquad p > 0 \text{ if } s \geqslant \frac{k^2+k}{2} + 1.$$

Hence our theorem.

3. Now (1) implies

$$(12) \qquad \sum_a (x+a)^k = \sum_b (x+b)^k.$$

Hence our theorem shows that if $s \geqslant \frac{k^2+k}{2} + 1$, we can find two different sets of integers $a_1, \cdots, a_s$ and $b_1, \cdots, b_s$ such that

$$(13) \qquad \sum_{m \leqslant s} (x + a_m)^k = \sum_{n \leqslant s} (x + b_n)^k.$$

It follows immediately that

$$(14) \qquad \beta(k) \leqslant \frac{k^2+k}{2} + 1$$

here $\beta(k)$ is the function defined by Steen and Rao[2].

[2] *Journal London Math. Soc.*, 1934, **9**, 170-71.

4. Let $v(k)$ denote the function recently introduced by Wright.[3] Then our theorem shows that for every $k > 1$ there is a $t = t(k)$ such that

$$v(t) = O(k^2)$$

for some t in $k \leqslant t \leqslant \frac{k^2+k}{2}$.

[*In particular,*

$$v(k) = O(k^2)$$

for infinitely many k.]

[3] *Journal London Math. Soc.*, 1934, **9**, 267-72. **We use $\Gamma(k) \leqslant 4k$.**