THE CLASS-NUMBER OF BINARY QUADRATIC FORMS.

By S. Chowla, Ph.D.,
Andhra University, Waltair.

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1. Let $h(\Delta)$ denote the number of primitive classes of binary quadratic forms of negative determinant $-\Delta$; $\pi(x; k, l)$ the number of primes $\equiv l \pmod{k}$ not exceeding x; $\phi(n)$, Euler's totient function. I have recently[1] shown that

(I) *If* $m > \frac{1}{2}$, $x \geqslant exp\ (k^m)$, *then*

$$(1) \quad \lim_{k \to \infty} \frac{\pi(x;\ k,\ l)}{x/\phi(k) \log x} = 1 \quad [(k, l) = 1].$$

(II) *If* (I) *is true for* $m < \frac{1}{2}$, *then*

$$(2) \quad h(\Delta) > \Delta^{\frac{1}{2}-m-\epsilon}$$

for every $\epsilon > 0$ *and every* $\Delta > \Delta_0(m, \epsilon)$.

I gave a complete proof of (I) but only indicated how (II) could be proved by a combination of my arguments with transcendental methods due to Gronwall and Landau. My object here is to give a direct proof of (II) based on elementary reasoning. In fact I prove the slightly stronger result:

(III) *Let* $0 < m < \frac{1}{2}$. *If there is a positive constant*[2] c *such that when* $(k, l) = 1$,

$$(3) \quad \pi(x;\ k,\ l) > \frac{c\,x}{\phi(k) \log x} \quad [x \geqslant \exp(k^m),\ k \geqslant k_0(m)]$$

then

$$(4) \quad h(\Delta) > \Delta^{\frac{1}{2}-m-\epsilon}$$

for every $\epsilon > 0$ *and all* $\Delta > \Delta_0(m, \epsilon)$.

2. *Notation.*

w denotes a typical prime $\equiv 1 \pmod{4\Delta}$;

t is a typical positive integer $\equiv 1 \pmod{4\Delta}$;

$y = \exp.\ (\Delta^m)$;

u, v are integers $\gtreqless 0$.

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[1] In a paper entitled "Primes in Arithmetical Progression," *Indian Physico-Mathematical Journal*, 1934, **5**, 35-43, I have used (I) to prove an asymptotic formula (conjectured by Hardy and Littlewood) for the number of representations of a positive integer as a sum of four squares and a prime. See *Zentralblatt für Mathematik*, 1934, Band **9**, 153.

[2] c is an absolute positive constant.

We further assume that Δ is a prime, but the argument is easily extended to general Δ.

Under a summation sign Σ we first indicate the variables of summation and then the conditions of summation.

Let $(a_n, b_n, c_n) \equiv a_n u^2 + 2b_n uv + c_n v^2$ be a typical reduced primitive form of negative determinant $-\Delta$, so that by giving n the values $1, 2, \ldots, h(\Delta)$ we get all the (reduced) primitive classes of negative determinant $-\Delta$. Then[3]

$$(5) \qquad \sum_{n=1}^{h(\Delta)} \quad \sum_{\substack{w \\ w=(a_n, b_n, c_n) \\ w \leqslant y}} 1 \geqslant \sum_{\substack{w \\ w \leqslant y}} 1$$

$$(6) \qquad \sum_{\substack{w \\ w=(a_n, b_n, c_n) \\ w \leqslant y}} 1 \leqslant \sum_{\substack{t \\ t=(a_n, b_n, c_n) \\ t \leqslant y}} 1$$

Now $\quad t = a_n u^2 + 2b_n uv + c_n v^2$ gives

$$a_n t = (a_n u + b_n v)^2 + \Delta v^2,$$
$$c_n t = (c_n v + b_n u)^2 + \Delta u^2,$$

which with $t \leqslant y$ imply that v can assume at most[4]

$$B\sqrt{\frac{a_n y}{\Delta}}$$

consecutive values, and that u can assume at most

$$B\sqrt{\frac{c_n y}{\Delta}}$$

consecutive values. Further, for fixed v, the congruence $a_n u^2 + 2b_n uv + c_n v^2 \equiv 1 \pmod{4\Delta}$ has B solutions (mod Δ). From these considerations it follows that

$$(7) \qquad \sum_{\substack{t \\ t=(a_n, b_n, c_n) \\ t \leqslant y}} 1 = B\sqrt{\frac{a_n c_n}{\Delta^2}} \times \frac{y}{\Delta} = \frac{By}{\Delta^{3/2}},$$

since $a_n c_n = \Delta + b_n^2 \leqslant \frac{4\Delta}{3}$. From (3), (5), (6) and (7) it follows that

$$(8) \qquad \frac{y}{(\Delta-1)\log y} = B\frac{h(\Delta)\,y}{\Delta^{3/2}}$$

If $h(\Delta) < \Delta^{\frac{1}{2}-m-\epsilon}$ then (8) is false for all $\Delta > \Delta_0(m, \epsilon)$ and hence (4) is proved.

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[3] Since w can be represented by (a_n, b_n, c_n) for some n in $1 \leqslant n \leqslant h(\Delta)$.

[4] The numbers B are less than absolute positive constants.

3. As an application of (III) we note the following derivation of a well-known result.

Titchmarsh[5] has shown that

If the 'extended Riemann hypothesis' is true then, provided (k, l) = 1,

$$(9)\quad \lim_{k \to \infty} \frac{\pi(x;\, k,\, l)}{x/\phi(k) \log x} = 1 \quad [x \geqslant k^3].$$

From (III) and (9) it follows at once that

If the 'extended Riemann hypothesis' is true then

$$h(\Delta) > \Delta^{\frac{1}{2}-\epsilon} \quad [\Delta \underset{\epsilon > 0}{>} \Delta_0(\epsilon)].$$

The latter result is a special case of results due to Gronwall, Hecke and Landau.[6]

4. Dr. Heilbronn has drawn my attention to a small error in my paper[7] " An Extension of Heilbronn's Class-Number Theorem ". On page 144, line 2 of that paper the word "exactly" should be replaced by the expression " not less than ". The rest of the proof stands unaltered.

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[5] *Palermo Rendiconti*, 1930, **54**, 414-429. The result cited is a special case of Theorem 6.

[6] See Landau, *Göttinger Nachrichten*, 1918. The best-known result in this direction is due to Littlewood, *Proc. Lond. Math. Soc.*, **2**, 1928, **27**, 358-372.

[7] *Proc. Ind. Acad. Sci.*, A, 1934, **1**, 143-144.