HEILBRONN'S CLASS-NUMBER THEOREM.

BY S. CHOWLA,
Andhra University, Waltair.

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1. LET $h(d)$ denote the number of non-equivalent forms of the type

(1) $Q(X, Y) = aX^2 + bXY + cY^2, (a, b, c) = 1, a > 0$

of the discriminant

(2) $d = b^2 - 4ac.$

Heilbronn[1] has recently proved the beautiful result (which includes a classical conjecture of Gauss) :

Theorem.—If d runs over all negative fundamental discriminants, then

$$h(d) \to \infty$$
$$\text{for} \quad d \to -\infty.$$

2. I shall now introduce the problem discussed in this paper.

There are examples of negative discriminants (*e.g.*,[2] $d = -232$ and $d = -1848$) with the following property : *all non-equivalent reduced primitive forms of negative discriminant* $-$d *have their middle coefficients equal to zero.* In such cases, we shall say, briefly, that "d (or $-d$) has the property P". It seems natural to inquire whether the number of negative discriminants d with the property P is finite or not. It will be proved here by the methods developed by Heilbronn in K that :

Theorem 1.—*There is only a finite number of negative discriminants* d, *with* $\frac{1}{4}$d *quadratfrei, having the property* P.

It is clear that if d has the property P, then

(3) $h(d) = H = 2^{t-1}$

where t is the number of different prime factors contained in $\frac{d}{4}$.

In what follows we shall assume that d has the property P, and then prove that this assumption leads to a contradiction for all large $-d$.

[1] "On the class-number in imaginary quadratic fields", hereafter referred to as K. The success of Heilbronn's argument rests essentially on—

(i) a theorem of Hecke (see Landau, *Gött. Nachr.*, 1918);

(ii) a generalization of some recent work of Deuring (*Math. Ztschr.*, 1933).

I am indebted to Dr. Heilbronn for a copy of his manuscript.

[2] The numbers 232 and 1848 are "idoneal". See Dickson's *History of the Theory of Numbers*, **I**, 361–65.

Notation.

The following notation is almost identical with that used in K.

All roman letters except o, O, s, L, Q (P and K) denote rational integers. $\chi(n)$ denotes a real character mod m ($m > 0$) such that

$$L_0(s) = L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s}$$

vanishes[3] for at least one ρ in the halfplane $\sigma > \frac{1}{2}$; m, χ and ρ are fixed throughout the paper;

(4) $\rho = \theta + i\phi \qquad (\theta > \frac{1}{2},\ \phi \gtreqless 0).$

We introduce the following Dirichlet series:

(5) $$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

(6) $$L_0(s) = \sum_{n=1}^{\infty} \chi(n)\, n^{-s},$$

(7) $$L_1(s) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right) n^{-s},$$

(8) $$L_2(s) = \sum_{n=1}^{\infty} \chi(n) \left(\frac{d}{n}\right) n^{-s},$$

where $\left(\frac{d}{n}\right)$ is the Kronecker symbol.

The constants implied in the signs O and o depend only on m, s, ρ, but they are independent of d, Q and (here we differ from K) H.

Proof of Theorem 1.

Lemma 1: If $1 \leqslant l_2 \leqslant m$ and if for $\sigma > 1$

$$\psi(s) = \psi(s, m, l_1, l_2, Q) = \sum_{\substack{Y=1 \\ Y \equiv l_2 (m)}}^{\infty} \ \sum_{\substack{X=-\infty \\ X \equiv l_1 (m)}}^{\infty} Q^{-s}(X, Y),$$

then

$\psi(s)$ is regular, and

(9) $$\psi(s) = O\left(|d|^{\frac{1}{2} - \frac{1}{2}\sigma} + |d|^{-\frac{1}{2}\sigma}\right)$$

for $d \to -\infty$ if $\sigma > \frac{1}{2}$, $s \neq 1$.

Proof: This result is contained in the proof of lemma 9 of K.

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[3] If this assumption were not true for any $m > 0$ it would follow, from a theorem of Hecke (in a paper by Landau previously cited), that

$$h(d) > \frac{c\sqrt{-d}}{\log(-d)},$$

so that (3) would be false for sufficiently large $-d$.

Lemma 2: For[4] $\sigma > \frac{1}{2}$, $s \neq 1$,

(10) $$L_0(s)\, L_2(s) = \zeta(2s) \prod_{p/m} (1-p^{-2s}) \sum_a \chi(a)\, a^{-s} + O(H\, |d|^{\frac{1}{4}-\frac{1}{2}\sigma}) + O(H\, |d|^{-\frac{1}{2}\sigma}).$$

Proof: This follows from (9) above and the proof of lemma 10 in K.

Lemma 3: If $\sigma \geqslant \frac{1}{2}$,

(11) $$\Big|\sum_a \chi(a)\, a^{-s}\Big| \geqslant \tfrac{1}{4} H^{-2} + O(H\, |d|^{-\frac{1}{2}\sigma}).$$

Proof: Since d has the property P,

(12)[5] $$d = -4ac, \quad a / \frac{d}{4}, \quad 0 < a \leqslant \tfrac{1}{2}\, |\sqrt{d}\,|.$$

Hence

(13) $$\sum_a \chi(a)\, a^{-s} = \prod_{p/\frac{1}{4}d} (1 + \chi(p)\, p^{-s}) - \sum_{\substack{r > \frac{|\sqrt{d}|}{2} \\ r/\frac{1}{4}d,\ \mu(r) \neq 0}} \chi(r)\, r^{-s}$$

As in K it is easy to see that

(14) $$\Big|\prod_{p/\frac{1}{4}d} (1 + \chi(p)\, p^{-s})\Big| \geqslant \tfrac{1}{4} H^{-2}$$

Further, from (12),

(15) $$\sum_{\substack{r > \frac{1}{2}|\sqrt{d}| \\ r/\frac{1}{4}d,\ \mu(r) \neq 0}} \chi(r)\, r^{-s} = O(2^t\, |d|^{-\frac{1}{2}\sigma}).$$

(11) follows from (13), (14), (15) and (3).

Proof of Theorem 1: We put $s = \rho = \theta + i\phi$ in lemma 2 and we let d tend to $-\infty$. Then we get

(16) $$O = \lim_{d=-\infty} \zeta(2\rho) \prod_{p/m} (1-p^{-2\rho}) \sum_a \chi(a)\, a^{-\rho} + O(H\, |d|^{\frac{1}{4}-\frac{1}{2}\theta}) + O(H\, |d|^{-\frac{1}{2}\theta}),$$

and from lemma 3,

(17) $$\Big|\sum_a \chi(a)\, a^{-\rho}\Big| \geqslant \tfrac{1}{4} H^{-2} + O(H\, |d|^{-\frac{1}{2}\theta}).$$

since $\theta > \frac{1}{2}$ we see that, if

(18) $$H < |d|^{\epsilon}$$

for every $\epsilon = \epsilon(\theta) > O$, then (16) and (17) contradict each other for $-d > d_0(\theta)$. Hence there is an $\epsilon = \epsilon(\theta) > O$ such that

(19) $$H \geqslant |d|^{\epsilon} \text{ for } -d > d_0(\theta).$$

Now (19) contradicts (3) since $2^t < |d|^{\frac{1}{2}\epsilon}$ for all $-d > d_0(\epsilon)$. Hence (3) is false for large $-d$ and our theorem is proved.

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[4] a runs through the minima of the H quadratic forms belonging to d.

[5] p/q means "p is a divisor of q".