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Quantum-ohmic resistance fluctuation in disordered conductors—An invariant imbedding approach†

N KUMAR

Department of Physics, Indian Institute of Science, Bangalore 560 012, India

Abstract. It is now well known that in the extreme quantum limit, dominated by the elastic impurity scattering and the concomitant quantum interference, the zero-temperature d.c. resistance of a strictly one-dimensional disordered system is non-additive and non-self-averaging. While these statistical fluctuations may persist in the case of a physically thin wire, they are implicitly and questionably ignored in higher dimensions. In this work, we have re-examined this question. Following an invariant imbedding formulation, we first derive a stochastic differential equation for the complex amplitude reflection coefficient and hence obtain a Fokker-Planck equation for the full probability distribution of resistance for a one-dimensional continuum with a gaussian white-noise random potential. We then employ the Migdal-Kadanoff type bond moving procedure and derive the d-dimensional generalization of the above probability distribution, or rather the associated cumulant function—'the free energy'. For $d = 3$, our analysis shows that the dispersion dominates the mobility edge phenomena in that (i) a one-parameter β-function depending on the mean conductance only does not exist, (ii) one has a line of fixed-points in the space of the first two cumulants of conductance, (iii) an approximate treatment gives a diffusion-correction involving the second cumulant. It is, however, not clear whether the fluctuations can render the transition at the mobility edge 'first-order'. We also report some analytical results for the case of the one-dimensional system in the presence of a finite electric field. We find a cross-over from the exponential to the power-law length dependence of resistance as the field increases from zero. Also, the distribution of resistance saturates asymptotically to a Poissonian form. Most of our analytical results are supported by the recent numerical simulation work reported by some authors.

Keywords. Quantum-ohmic resistance; disordered conductors; invariant imbedding; finite electric field; mobility edge.

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1. Introduction

The indefiniteness and the non-self-averaging nature of the quantum ohmic residual resistance of a disordered one-dimensional resistor has been the subject of many investigations in the recent past (Anderson *et al* 1980; Mel'nikov 1980a, b; Abrikosov 1981; Kumar 1985; Kumar and Mello 1985; Ioffe *et al* 1985). These statistical fluctuations of residual resistance over the ensemble of macroscopically identical samples have been treated exactly for a strictly one-dimensional system, i.e., for the one-dimensional 1-channel case. But the results do presumably generalize to the case of a physically thin wire that should correspond to the one-dimensional n-channel case,

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†The author felicitates Prof. D S Kothari on his eightieth birthday and dedicates this paper to him on this occasion.

with n large. It is the serial randomness of 1-space dimensionality that seems to be the determining property. However, the question whether or not these fluctuations really get harnessed by the multiple connectivity of higher space dimensions ($d > 1$) is still open, and may be crucial to the proper understanding of the physics at the mobility edge. Indeed, the results of the recent numerical analysis (Ioffe *et al* 1985) on a finite three-dimensional lattice with off-diagonal disorder show large dispersion of physical quantities such as the conductivity and the participation ratio at the mobility edge, indicating a breakdown of the *ansatz* of one-parameter scaling suggested earlier by Abrahams *et al* (1979). This has prompted us to report some of our recent analytical results based on an invariant imbedding approach (Kumar 1985; Chandrasekhar 1960; also see Bellman and Wing 1976) to this problem that lend support to this conclusion. We find that for the three-dimensional case with extremely anisotropic disorder, the variance dominates the mean value of resistance on the insulating side of the mobility edge as expected. But surprisingly enough, on the "metallic" side too the mean resistance and its dispersion remain comparable even in the infinite-sample limit. We call this the "meta-metallic" phase. The full probability distribution of resistance in this case has no non-trivial unstable fixed point. For the isotropic case, on the other hand, we have a line of fixed points in the two-parameter space of mean conductance and its variance. We also report some new results on the influence of an electric field in the one-dimensional case.

2. Invariant imbedding (d = 1)

Our starting point is the well-known Landauer (1970) expression for the resistance $\rho(L)$ given by

$$\rho(L) = \frac{R^*(L)R(L)}{1 - R^*(L)R(L)}, \tag{1}$$

where $\rho(L)$ is measured in natural units (of $\pi\hbar/e^2$) and $R(L)$ is the complex amplitude reflection coefficient for the one-dimensional conductor of length L. The one-electron motion is described by the Schrödinger equation

$$\frac{\partial^2\psi}{\partial x^2} + k^2(x)\psi = 0, \qquad k^2(x) = \frac{2m}{\hbar^2}[E - V(x)] \tag{2a}$$

conditioned by matching to the scattering states at the boundaries $x = 0, L$. Here $k(x)$ is the local wave number, assumed to be random in as much as the potential $V(x)$ is. L is the sample length. In the invariant imbedding approach, however, one addresses directly the "emergent" quantity, namely, the complex amplitude reflection coefficient $R(L)$. The latter is known to obey the Riccati equation (Bellman and Wing 1976)

$$\partial R(L)/\partial L = f_1(L) + 2if_0(L)R(L) - f_1(L)R^2(L),$$

with

$$f_1(x) = 2(\partial k/\partial x)/k(x) \text{ and } f_0(x) = k(x). \tag{2b}$$

Now, we will consider for simplicity the situation where $k^2(x) > 0$. In the description of wave propagation in random media this implies that the "refractive" index is random but stays real. Thus any exponential spatial attenuation (localization) is due entirely to interference of random phases and not due to an imaginary "refractive" index that would correspond to barrier penetration. This would be the case for an one-dimensional disordered metal with the electron Fermi energy of interest exceeding the random potential fluctuations. This restriction can, however, be easily relaxed by the use of a slightly different imbedding (Heinrichs 1986). It is then convenient to write $R = R_1 + iR_2$, with R_1, R_2 real, and separate (2) for the complex R as

$$\frac{\partial R_1(L)}{\partial L} = f_1(L) - 2f_0(L)R_2(L) - f_1(L)(R_1^2 - R_2^2), \tag{3}$$

$$\frac{\partial R_2(L)}{\partial L} = 2f_0(L)R_1(L) - 2f_1(L)R_1(L)R_2(L). \tag{4}$$

These coupled stochastic differential equations are difficult to treat. What we need, however, is the probability density $W_R(R_1 R_2; L)$ for which we now derive a Fokker-Planck equation. To this end we introduce a spread of "phase points" of density $Q(R_1, R_2 L)$ in the (R_1, R_2) "phase space" evolving in L according to (3) and (4), subject to the definite, "initial" condition that $R(L) = 0$ at $L = 0$. For $R(L) = 0$ the angle $\theta = \tan^{-1}(R_2/R_1)$ is undefined and may be taken to correspond to a circular ensemble uniform in θ space. The phase fluid will now evolve according to the stochastic Liouville equation

$$\frac{\partial Q}{\partial L} = -\frac{\partial}{\partial R_1}\left(Q\frac{\partial R_1}{\partial L}\right) - \frac{\partial}{\partial R_2}\left(Q\frac{\partial R_2}{\partial L}\right), \tag{5}$$

where the "velocities" $\mathrm{d}R_1/\mathrm{d}L$ and $\mathrm{d}R_2/\mathrm{d}L$ are given by (3) and (4). We have then the well-known result that

$$W_R(R_1, R_2; L) = \langle Q(R_1, R_2; L)\rangle_f, \tag{6}$$

where $\langle \cdots \rangle_f$ denotes averaging with respect to the basic random variables f_0 and f_1. As our interest lies in resistance, or equivalently in the reflection coefficient $r = R_1^2 + R_2^2$, it is convenient to introduce the angular (θ) average

$$W_r(r, L) = \pi\langle W_R(R_1, R_2; L)\rangle_\theta, \tag{7a}$$

with normalization

$$\int_0^1 W_r(r, L)\,\mathrm{d}r = 1. \tag{7b}$$

Assumption of circular ensemble makes $\langle Q(R_1, R_2; L)\rangle_f$ a function of $r = R_1^2 + R_2^2$ alone. This enables us to perform the θ average in the following. From (5)–(7), we get

$$\frac{\partial W_r}{\partial L} = -\pi\left\langle\frac{\partial}{\partial R_1}\langle f_1 Q\rangle_f\right\rangle_\theta + 2\pi\left\langle R_1 R_2\frac{\partial}{\partial R_2}\langle f_1 Q\rangle_f\right\rangle_\theta$$
$$+ 4\pi\langle R_1\langle f_1 Q\rangle_f\rangle_\theta + \pi\left\langle (R_1^2 - R_2^2)\frac{\partial}{\partial R_1}\langle f_1 Q\rangle_f\right\rangle_\theta, \tag{8}$$

where terms involving f_0 cancel out and only f_1, essentially the potential gradient (force), persists. Now, we must evaluate $\langle f_1 Q \rangle_f$ occurring on the right hand side of (8). For this purpose we take the derivative f_1 as our basic random variable and approximate it mathematically as Gaussian and δ-correlated in space, i.e.,

$$\langle f_1(L) f_1(L') \rangle = \frac{1}{\xi} \delta(L - L'). \tag{9}$$

Here $1/\xi$ measures the strength of scattering and essentially $\xi \simeq L_c$. In point of fact even for a realistic bounded random potential, the derivative can fluctuate very violently, even become unbounded and the above approximation makes sense as a model. This enables us to close the hierarchy of equations for the averages in virtue of the known Novikov identity for the functional Q of the Gaussian random variable f_1, namely that

$$\langle f_1(L') Q \rangle_f = \int_0^L \langle f_1(L') f_1(L'') \rangle_f \left\langle \frac{\delta Q}{\delta f_1(L'')} \right\rangle_f \mathrm{d}L'', \tag{10}$$

with $0 \leqslant L' \leqslant L$. Here $\langle \delta Q / \delta f_1 \rangle_f$ can be obtained readily from (5). Thus, closely following an earlier procedure (Soukoulis *et al* 1983) we obtain, after some tedious but straightforward algebra, the evolution equation

$$\frac{\partial W_r}{\partial l} = (1-r)^2 r \frac{\partial^2 W_r}{\partial r^2} + (1-r)(1-5r) \frac{\partial W_r}{\partial r} + 2(2r-1) W_r, \tag{11}$$

with $l = L/\xi$. Here the "initial" condition is

$$W_r(r, l) \to \delta(r) \quad \text{as} \quad l \to 0.$$

The non-self-adjoint equation (11) can be solved formally as a biorthogonal series in terms of the eigenfunctions of the associated hypergeometric equation whose domain of physical interest is bounded by the regular singularities at $r = 0$ and $r = 1$. Since, however, we are interested in the resistance (ρ) fluctuation, it is apt to change over to $\rho = r/(1-r)$ and the associated probability density $W_\rho(\rho, l)$. We then have

$$\frac{\partial W_\rho}{\partial l} = \rho(1+\rho) \frac{\partial^2 W_\rho}{\partial \rho^2} + (2\rho + 1) \frac{\partial W_\rho}{\partial \rho}. \tag{12}$$

This is our central equation. This self-adjoint equation has the remarkable property of yielding an exact two-point recursion relation for the resistance moments $\langle \rho^n \rangle = \rho_n$. Multiplying both sides by ρ^n and integrating by parts on the RHS, we get

$$\frac{\partial \rho_n}{\partial l} = n(n+1)\rho_n + n^2 \rho_{n-1}. \tag{13}$$

The boundary terms appearing on integration by parts vanish because of the conditions of normalization and of bounded moments for a given l. In particular we have

$$\rho_1^{(l)} = (\tfrac{1}{2})(e^{2l} - 1), \tag{14a}$$

$$\rho_2^{(l)} = (\tfrac{1}{12})(2e^{6l} - 6e^{2l} + 4). \tag{14b}$$

Thus the variance grows faster than the mean.

Equation (12) has the asymptotic solution ($\rho \gg 1$)

$$W_\rho(\rho, l) \sim \frac{\exp(-l/4)}{(4\pi l)^{\frac{1}{2}}} \rho^{-1/2} \cdot \exp\left[-\frac{1}{4l}\ln^2 \rho\right] \tag{15}$$

confirming that $\ln \rho$ obeys the central limit. This merely reflects the levelling property of the logarithm.

3. Finite electric field (d = 1)

It is convenient to discuss at this stage the influence of an electric field (F) on the resistance (Vijayagovindan *et al* 1986). But a technical point first. In the presence of an electric field the wavenumbers at the two ends are, of course, different and this calls for a slightly different imbedding as discussed by Heinrichs (1986). Proceeding as before, we have for the probability distribution W_ρ^F

$$\frac{\partial W_\rho^F}{\partial L} = \frac{1}{\eta\left(1+\frac{|e|FL}{E_f}\right)}\left[\frac{\partial}{\partial \rho}\left\{\rho(\rho+1)\left(\frac{\partial W_\rho^F}{\partial \rho} + \frac{|e|F\eta W_\rho^F}{E_f\left(1+\left(\frac{1}{[1+(|e|FL)/E_f]^{1/2}}\right)\right)}\right)\right\}\right], \tag{16}$$

where E_f is the Fermi-energy. We have assumed the random potential $v(x)$ to be a gaussian-white noise with $\langle v(x)\, v(x')\rangle = v_0^2\delta(x-x')$, and $\eta = E_f^2/k_f^2 v_0^2$. This gives for the mean resistance

$$\rho_1 \equiv \langle\rho\rangle \simeq \tfrac{1}{2}\left[\left(1+\frac{|e|FL}{E_f}\right)^{\frac{2E_f}{|e|F\xi}} - 1\right]$$

with $$E_f = \frac{\hbar^2 K_f^2}{2m}. \tag{17}$$

Thus, we get a cross-over from the exponential growth of resistance for weak fields ($2E_f/|e|F\xi \ll 1$) to a power-law growth for strong fields. Also, the distribution W_ρ^F saturates to a Poissonian form as $L \to \infty$. These analytic results agree with the results of numerical simulation reported by several authors (Soukoulis *et al* 1983).

4. Fluctuations at mobility edge (d > 1)

We now turn to the important question of what happens for $d > 1$. Our starting point (Kumar and Jayannavar 1986) is the evolution equation (12) for the probability distribution $W_\rho^{(1)}$, where we shall now use the subscript (1) to indicate the dimensionality. Defining the associated moment generating function $\chi_\rho^{(1)}(x, l)$ and the characteristic function (the cumulant generating function, of the "free-energy") $K_\rho^{(1)}(x, l)$ as

$$\chi_\rho^{(1)}(x, l) \equiv \int_0^\infty \exp(-x\rho)\, W_\rho^{(1)}(\rho, l)\, d\rho, \quad K_\rho^{(1)}(x, l) \equiv \ln \chi_\rho^{(1)}, \tag{18a}$$

we readily derive from (12)

$$\frac{\partial \chi_\rho^{(1)}}{\partial l} = x^2 \frac{\partial^2 \chi_\rho^{(1)}}{\partial x^2} + x(2-x)\frac{\partial \chi_\rho^{(1)}}{\partial x} - x\chi_\rho^{(1)}, \tag{18b}$$

and

$$\frac{\partial K^{(1)}}{\partial l} = x^2 \frac{\partial^2 K_\rho^{(1)}}{\partial x^2} + x^2 \left(\frac{\partial K_\rho^{(1)}}{\partial x}\right)^2 + x(2-x)\frac{\partial K_\rho^{(1)}}{\partial x} - x. \tag{18c}$$

We have in particular for the first moment $\rho_1^{(1)}$ ($\equiv$ cumulant $\kappa_1^{(1)}$)

$$\rho_1^{(1)} \equiv \langle \rho^{(1)} \rangle \equiv \kappa_1^{(1)} = \tfrac{1}{2}(\exp(2l) - 1), \tag{19}$$

where $\langle \cdots \rangle$ denotes the ensemble average.

Consider now a d-dimensional hypercubic lattice each site of which is occupied by a random 'elementary' scatterer with $2d$ incoming and $2d$ outgoing channels. Following Shapiro (1982) we partition the lattice into equal hypercubic blocks of edge b (the scale factor). The Migdal-Kadanoff procedure now involves singling out arbitrarily the direction of current flow and then cutting the bonds in the $(d-1)$ transverse directions. Thus each block now consists of $b^{(d-1)}$ chains in parallel, each chain b scatterers long. Proceeding as in Shapiro (1982) but with a difference now that for a given realization of randomness each chain has a different resistance in general, we can write for a block

$$(\rho^{(d)}(b))^{-1} = \sum_{i=1}^{b^{(d-1)}} (\rho_i^{(1)}(b))^{-1}, \tag{20a}$$

where $\rho_i^{(1)}(b)$ is the resistance of the ith one-dimensional chain of length b in the d-dimensional block and $\rho^{(d)}(b)$ is the resultant block resistance.

We will first consider the relatively simple, albeit somewhat unrealistic case of extreme anisotropy wherein the randomness evolves only along the chosen direction of current flow. For this case, we have $\rho_i^{(1)}(b) = \rho^{(1)}(b)$ (independent of the chain index i) and the law of harmonic combination in (20a) simplifies to

$$\rho^{(d)}(b) = b^{-(d-1)}\rho^{(1)}(b) \tag{20b}$$

giving

$$\chi_\rho^{(d)}(x, b) = \langle \exp[-xb^{-(d-1)}\rho^{(1)}(b)] \rangle. \tag{21}$$

To derive the recursion relation for the probability distribution in the differential form, we set the scale factor $b = 1 + d\zeta$ and let $d\zeta \to 0^{\dagger}$. We get at once

$$\frac{\partial \chi_\rho^{(d)}}{\partial \ln l} = -(d-1)x\frac{\partial \chi_\rho^{(d)}}{\partial x} + \left(\frac{\partial \chi_\rho^{(1)}}{\partial l}\right)_d \cdot \tfrac{1}{2}\ln(1 + 2\rho_1^{(d)}). \tag{22}$$

Here $(\partial \chi_\rho^{(1)}/\partial l)_d$ now signifies $(\partial \chi_\rho^{(1)}/\partial l)$ as given by (18b), but with $\chi_\rho^{(1)}$ on the right-hand side of (18b) re-interpreted as $\chi_\rho^{(d)}$ in the sense of iteration. Here, l is the actual size of the block at the present stage of length scaling. Also we have recalled the iterational relations $dl = l d\zeta$ (or $\zeta = \ln l$). Here ρ_l^d is the mean resistance of the block at the present length scale.

From (18b), (18c) and (22), we get the equation for the cumulant generating function

$$\frac{\partial K_\rho^{(d)}}{\partial \ln l} = -(d-1)\,x\frac{\partial K_\rho^{(d)}}{\partial x} + \frac{1}{2}\left[x^2\frac{\partial^2 K_\rho^{(d)}}{\partial x^2} + x^2\left(\frac{\partial K_\rho^{(d)}}{\partial x}\right)^2 + x(2-x)\frac{\partial K_\rho^{(d)}}{\partial x} - x\right]\ln(1+2\rho_1^{(d)}). \tag{23}$$

This equation contains full information on the problem in question. The fixed points of the probability distribution, if any, are obtained by setting $(\partial K_\rho^{(d)}/\partial \ln l) = 0$. The resulting second order differential equation in x as independent variable is to be solved subject to the boundary conditions $K_\rho^{(d)}(x = 0, l) = 0$ and $(-\partial K_\rho^{(d)}(x, l)/\partial x)_{x=0} = \kappa_{1\rho}^{(d)}$ ($\equiv \rho_1^{(d)}$, the mean resistance). The solution, however, also contains $\rho_1^{(d)}$ parametrically and the probability distribution so obtained must, therefore, reproduce $\rho_1^{(d)}$ self-consistently. This fixes the value $\rho_1^{(d)*}$ as the fixed point. More explicitly, we obtain from (7) the equations for the first two cumulants by series expansion of $K_\rho^{(d)}$ on both sides of (23) in powers of x and equating the coefficients of x and x^2. We get

$$\frac{\partial \kappa_{1\rho}^{(d)}}{\partial \ln l} = -(d-1)\,\kappa_{1\rho}^{(d)} + \tfrac{1}{2}(1+2\kappa_{1\rho}^{(d)})\ln(1+2\kappa_{1\rho}^{(d)}), \tag{24}$$

$$\frac{\partial \kappa_{2\rho}^{(d)}}{\partial \ln l} = -(d-1)\kappa_{2\rho}^{(d)} + (\kappa_{1\rho}^{(d)2} + \kappa_{1\rho}^{(d)} + 3\kappa_{2\rho}^{(d)})\ln(1+2\kappa_{1\rho}^{(d)}). \tag{25}$$

Equation (8) is of course the same as the one obtained by Shapiro (1982) (his equation (4)). The new information about dispersion is contained in (25). Straightforward analysis shows that for $d = 3$, $(\partial K_{1\rho}^{(d)}/\partial \ln l) = 0$ gives $\rho_1^{(d)*} = 1{\cdot}96$ which together with $(\partial K_{2\rho}^{(d)}/\partial \ln l) = 0$ gives $K_{2\rho}^{(d)*} < 0$, indicating that there is no non-trivial fixed for the full distribution. However, it is readily seen that for $\rho_1^{(d)} > \rho_1^{(d)*}$ both the average resistance $(K_{1\rho}^{(d)})$ and its variance $(\sqrt{K_{2\rho}^{(d)}})$ grow with the size l leading to the insulating phase with the variance exponentially dominating the mean value. More interestingly, for $\rho_1^{(d)} < \rho_1^{(d)*}$ both the mean and the variance flow to the "metallic" fixed-point $\kappa_{1\rho}^{(d)*} = 0 = \kappa_{2\rho}^{(d)*}$. In this regime we have $\kappa_{1\rho}^{(d)}, \kappa_{2\rho}^{(d)} \ll 1$ and (8) and (9) lead to

$$\frac{\mathrm{d}(\kappa_{1\rho}^{(d)})^2}{\mathrm{d}\ln l} \simeq -2(\kappa_{1\rho}^{(d)})^2, \tag{26}$$

$$\frac{\mathrm{d}\kappa_{2\rho}^{(d)}}{\mathrm{d}\ln l} \simeq -4\kappa_{2\rho}^{(d)} + 2(\kappa_{1\rho}^{(d)})^2. \tag{27}$$

This gives $(\partial \ln \kappa_{1\rho}^{(d)2}/\partial \ln \kappa_{2\rho}^{(d)})_{l=\infty} = 1$ showing that the mean and the variance remain comparable in the metallic regime even in the infinite sample limit. For this reason we prefer the term "meta-metallic" for this phase.

We now pass on to the physically more relevant case of isotropic disorder. Here, (4a) must be retained as such since $\rho_i^{(1)}(b)$, for $i = 1, 2, 3 \ldots b^{(d-1)}$, are no longer equal, but identically distributed independent random variables. It is convenient to re-write (4a) in terms of the conductance $(g_i^{(1)})$, rather than the resistance $(\rho_i^{(1)})$, as

$$g^{(d)}(b) = \sum_{i=1}^{b^{(d-1)}} g_i(b). \tag{28}$$

Correspondingly, we can re-write our starting equation (12) in terms of $W_g^{(1)}$ and proceed as before. There is, however, a technical problem arising from the circumstance that $W_g^{(1)}$ associated with (12) gives infinite algebraic moments $g^{(v)} = \langle g^v \rangle$ for $v > \frac{1}{2}$. While in principle one could still carry out the above programme, as the associated cumulant generating function $K_g^{(1)}(x)$ always exists, albeit now non-analytic at $x = 0$, certain intermediate mathematical operations cannot be carried out as we shall presently see. It will be quite apt to resort to a mathematical artifice of terminating the one-dimensional chains so as to include a non-zero boundary resistance. Indeed this is inherent in the treatment of Abrikosov (1981) who gets instead of (1) the equation

$$\frac{\partial W_{\rho'}^{(1)}}{\partial l} = \rho'(\rho' - 1)\frac{\partial^2 W_{\rho'}^{(1)}}{\partial \rho'^2} + (2\rho' - 1)\frac{\partial W_{\rho'}^{(1)}}{\partial \rho'} \tag{29}$$

that differs from (12) by the replacement $\rho' = \rho + 1$. With (13) one has

$$\rho_1^{(1)\prime} \equiv \langle \rho' \rangle = \tfrac{1}{2}(e^{2l} + 1) = \rho_1^{(l)} + 1. \tag{30}$$

Thus $\rho_1^{(l)}$ is the 'internal' resistance *sans* boundary. Here $\rho_l^{(1)} \to 1$ as $l \to 0$. This clearly displays the boundary resistance, and makes all moments of conductance finite. The associated probability distribution $W_{g'}^{(1)}$ of conductance g' obeys

$$\frac{\partial W_{g'}^{(1)}}{\partial l} = g'^2(1 - g')\frac{\partial^2 W_{g'}^{(1)}}{\partial g'^2} + (4g' - 5g'^2)\frac{\partial W_{g'}^{(1)}}{\partial g'} + (2 - 4g')W_{g'}^{(1)}. \tag{31}$$

Now we introduce respectively, the moment and the cumulant generating functions $\chi_{g'}^{(1)}$ and $Kg'^{(1)}$ as before and get

$$\frac{\partial K_{g'}^{(1)}}{\partial l} = x^2\left(\left(\frac{\partial K_{g'}^{(1)}}{\partial x}\right)^3 + 3\frac{\partial K_{g'}^{(1)}}{\partial x}\cdot\frac{\partial^2 K_{g'}^{(1)}}{\partial x^2} + \frac{\partial^3 K_{g'}^{(1)}}{\partial x^3}\right) + (x^2 + x)\left(\left(\frac{\partial K_{g'}^{(1)}}{\partial x}\right)^2 + \frac{\partial^2 K_{g'}^{(1)}}{\partial x^2}\right). \tag{32}$$

Steps leading to (32) from (31) involve taking the Laplace transform on both sides of (31) and interchanging the order of integration and differentiation, e.g.,

$$\int_0^\infty \exp(-xg')g'^2\frac{\partial^2 W_{g'}^{(1)}}{\partial x^2}\mathrm{d}g' = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(x^2\int_0^\infty \exp(-xg')W_{g'}^{(1)}\mathrm{d}g'\right). \tag{33}$$

It is precisely this operation that becomes inadmissible when the moments do not exist as noted above. Proceeding now as before, we get the recursion relation for $K_g^{(d)}$ in the differential form

$$\frac{\partial K_g^{(d)}}{\partial \ln l} = -(d-1)K_g^{(d)} + \tfrac{1}{2}\left[x^2\left\{\left(\frac{\partial K_g^{(d)}}{\partial x}\right)^3 + 3\frac{\partial K_g^{(d)}}{\partial x}\cdot\frac{\partial^2 K_g^{(d)}}{\partial x^2} + \frac{\partial^3 K_g^{(d)}}{\partial x^3}\right\} + (x^2 + x)\left\{\left(\frac{\partial K_g^{(d)}}{\partial x}\right)^2 + \frac{\partial^2 K_g^{(d)}}{\partial x^2}\right\}\right]\ln(1 + 2\rho_1^{(d)}). \tag{34}$$

An essential complexity of (34) for the cumulant generating function $W_g^{(d)}(x)$ of conductance g is that it involves parametrically the mean resistance $\rho_1^{(d)}$. We can still carry out an approximate analysis by writing the equations for the first two cumulants $\kappa_{1g}^{(d)}$ and $\kappa_{2g}^{(d)}$, from (34), we have

$$\frac{\partial \kappa_{1g}^{(d)}}{\partial \ln l} = (d-1)\kappa_{1g}^{(d)} - \tfrac{1}{2}\,(\kappa_{1g}^{(d)2} + \kappa_{2g}^{(d)}) \ln(1+2\rho_1^{(d)}) \tag{35a}$$

$$\frac{\partial \kappa_{2g}^{(d)}}{\partial \ln l} = (d-1)\,\kappa_{2g}^{(d)} - (\kappa_{1g}^{(d)3} + 3\kappa_{1g}^{(d)}\kappa_{2g}^{(d)} + 2\kappa_{3g}^{(d)} - \kappa_{1g}^{(d)2} - \kappa_{2g}^{(d)}) \ln(1+2\rho_1^{(d)}). \tag{35b}$$

Above we have replaced ρ_1' by $1+\rho_1$ where ρ_1 tunes the intrinsic disorder. Equations (34) and (35) constitute our main results. The form of (34) is significant. To obtain a fixed-point distribution we set the left-hand side of (34) equal to zero. This gives us a third order differential equation with x as the independent variable. We can now impose the initial (at $x=0$) condition

$$(K_g^{(d)})_0 = 0, \qquad (\partial K_g^{(d)}/\partial x)_0 = -\kappa_{1g}^{(d)} \quad \text{(mean conductance)}$$

and

$$(\partial^2 K_g^{(d)}/\partial x^2)_0 = \kappa_{2g}^{(d)} \quad \text{(dispersion)},$$

and in principle obtain the full fixed-point distribution depending parametrically on $\rho_1^{(d)}$. Now this fixed-point distribution must reproduce $\rho_1^{(d)}$ self-consistently. This gives a relation between $\kappa_{1g}^{(d)}$ and $\kappa_{2g}^{(d)}$. Thus we get a line of fixed points in the two-parameters $(\kappa_{1g}^{(d)}, \kappa_{2g}^{(d)})$ space. Also, from (35), it is clear that this line of fixed points cannot pass through $\kappa_{2g}^{(d)} = 0$, with $\kappa_{1g}^{(d)} \neq 0$. All this amounts to saying that the one-parameter β-function may not really exist. Indeed, if we were to very crudely set $\rho_1^d \simeq 1/\kappa_{1g}^{(d)}$ in (35a) on the metallic side, we get

$$\frac{\partial \ln \kappa_{1g}^{(d)}}{\partial \ln l} \simeq (d-2) - \left(\frac{\kappa_{2g}^{(d)}}{\kappa_{1g}^{(d)}} - 1\right)\frac{1}{\kappa_{1g}^{(d)}}. \tag{36}$$

Thus the diffusion correction to the Boltzmann conductivity is not quite so simple. It seems to us that the line-of-fixed points is an unstable line. Further work is necessary before any definite statement about the nature of the Anderson-Mott transition is made. But our analysis assumes significance in the light of the numerical results of Ioffe *et al* (1985) on large dispersion at the mobility edge, and the work of Efetov (1984) who obtained a non-zero $\sigma_{\min}$ for the nonlinear σ-model on the Cayley tree. The question is whether the fluctuations can drive the transition discontinuous (first-order). Further work is in progress.

Finally, we turn to the question of experimental observability of these "Sinai fluctuations. A direct approach would be to generate an ensemble of macroscopically identical (but microscopically different) samples by reversible compositional doping e.g. electrochemically. These samples should have identical resistance at elevated temperatures but multifureate to different values at ultra-low temperatures. There is however an indirect but physically more significant manifestation of "Sinai" fluctuations, namely, their time-translation as excess quantum noise at low-frequencies. For a given sample at low enough temperatures such that the sample length is less than the

Thouless length and also the electron transit time is less than the dominant phonon period at that temperature the electron senses the phonon modulation of random potential adiabatically. This will result in a low-frequency noise which is just the time-scan of the "Sinai' fluctuations – a scintillating noise.

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