Bifurcations, Chaos, Controlling and Synchronization of Certain Nonlinear Oscillators

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Abstract

In this set of lectures, we review briefly some of the recent developments in the study of the chaotic dynamics of nonlinear oscillators, particularly of damped and driven type. By taking a representative set of examples such as the Duffing, Bonhoeffer-van der Pol and MLC circuit oscillators, we briefly explain the various bifurcations and chaos phenomena associated with these systems. We use numerical and analytical as well as analogue simulation methods to study these systems. Then we point out how controlling of chaotic motions can be effected by algorithmic procedures requiring minimal perturbations. Finally we briefly discuss how synchronization of identically evolving chaotic systems can be achieved and how they can be used in secure communications.

1 Introduction

Dynamical systems are often modelled by differential equations when the flow is continuous. For an N-particle system moving under the influence of (constraint free) internal and external forces, $\vec{F}_i = 1, 2, ...N$, Newton's equations of motion,

$$m_i \frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i \left(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N, \frac{d\vec{r}_1}{dt}, \dots \frac{d\vec{r}_N}{dt}, t \right), \; i = 1, 2, \dots N, \tag{1}$$

where $\vec{r}_i \equiv \vec{r}_i(t)$ is the position vector of the ith particle in an inertial frame, subjected to $6N$ initial conditions $\vec{r}_i(0)$ and $\left(\frac{d\vec{r}_i}{dt}\right)_{t=0}$, constitute a deterministic description of the system. As long as the initial position and velocities as well as the nature of the forces are precisely prescribed, by solving the system of $6N$ coupled second order ordinary differential equations (ODEs) the future evolution can be predicted with as much accuracy as desired. Of course the Laplace dictum of complete determinism fails when statistical fluctuations and quantum effects are present. Barring the presence of these forces, one might expect that the solution of the initial value problem of (1) should lead to complete predictability. However, in recent times it has been clearly demonstrated that a new kind of indeterminism or unpredictability can arise when the forces acting on the system are nonlinear (even when statistical and quantum forces are absent) in suitable form.

When the forces $\vec{F}_i$ in (1) are linear, we only have to solve a system of linear differential equations wherein the superposition principle holds good. Any small deviation in the initial conditions or numerical errors will then grow at the worst only linearly, so the dynamics of the system can be exactly predicted. However when the forces $\vec{F}_i$ are nonlinear, one has to solve nonlinear differential equations. Then for suitable nonlinearities, one typically finds the possibility that any small deviation in the prescription of initial conditions or round-off error at any stage of the calculation can multiply quickly, leading to an exponential divergence of trajectories corresponding to nearby initial conditions. Or, in other words, the system can be extremely sensitively dependent on initial conditions leading to complex or chaotic motions. Physically this means that any small fluctuation during the evolution can lead to realizable macroscopic effects, the so-called 'butterfly effect' [1,2]. Mathematically, the phase trajectory might end up asymptotically in a strange attractor of zero volume but of fractional dimensionality in the case of dissipative systems, or it can fill the whole phase space densely in the case of conservative systems.

The notion of complexity and chaos can be realized even in low-dimensional nonlinear systems. Particularly, damped and driven oscillators are physically realizable examples exhibiting the immense varieties of bifurcations and chaos. They are often modelled by a single, second-order, nonlinear differential equation (or equivalent first order system) with periodic inhomogeneities. These include the Duffing oscillator, the Bonhoeffer-van der Pol oscillator, the Duffing-van der Pol oscillator, the Murali-Lakshmanan-Chua electronic circuit, the damped driven pendulum, the driven Morse oscillator and so on [3,4]. In this set of lectures, we will consider the extremely varied bifurcations exhibited by these systems and the various types of regular and chaotic motions admitted by them and some of their applications.

2 Linear and nonlinear oscillators

There are obvious characteristic differences between the dynamics of linear and nonlinear oscillators. We will briefly consider these aspects with simple examples.

2.1 Linear oscillators

We will point out how predictability becomes an inherent property of linear oscillator systems. We will also consider the type of attractors exhibited by these systems in this section.

2.1.1 Free linear harmonic oscillator

The ubiquitous linear harmonic oscillator of unit mass satisfying the equation of motion

$$\frac{d^2x}{dt^2} + \omega_0^2 x = 0, \;\; \omega_0^2 > 0 \tag{2}$$

and the initial conditions $x(0) = A$, $(dx/dt)_{t=0} = 0$ admits the solution

$$x(t) = A\cos\omega_0 t \tag{3}$$

such that

$$E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega_0^2 x^2 = \frac{1}{2}\omega_0^2 A^2 \quad (. = \frac{d}{dt}) \tag{4}$$

is a constant. The corresponding phase trajectories in the phase space ((x-$\dot{x}$)-space) are concentric ellipses with the origin $(0, 0)$ being a centre type (elliptic) fixed point (Fig. 1a).

2.1.2 Damped linear harmonic oscillator

When viscous damping force is present, the harmonic oscillator equation becomes

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega_0^2 x = 0, \tag{5}$$

where α (> 0) is the damping parameter. For the same initial conditions, $x(0) = A$, $\dot{x}(0) = 0$, the solution for the physically interesting underdamped case ($\alpha < 2\omega_0$) becomes

$$x(t) = A\frac{\sqrt{\alpha^2/4 + C^2}}{C} \exp\left(-\frac{\alpha t}{2}\right) \cos(Ct - \delta), \tag{5a}$$

where

$$C = \sqrt{\omega_0^2 - \frac{\alpha^2}{4}}, \ \ \delta = \tan^{-1}\frac{\alpha}{2C}. \tag{5b}$$

Solution (5) represents a damped oscillatory motion and energy is dissipated. The corresponding phase trajectory is a spiral asymptotically approaching the fixed point at the origin which is now a stable focus (Fig. 1b).

2.1.3 Damped and driven linear oscillator

When an external sinusoidal force is added to the system, we have

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega_0^2 x = f \sin\omega t. \tag{6}$$

Again for the same set of initial conditions as above, we have the solution

$$x(t) = A_t\frac{\sqrt{\alpha^2/4 + C^2}}{C} \exp\left(\frac{-\alpha t}{2}\right) \cos(Ct - \delta) + A_p\cos(\omega t - \gamma), \tag{7a}$$

where

$$A_p = \frac{f}{\sqrt{(\omega_0^2 - \omega^2)^2 + \alpha^2\omega^2}}, \ \ \gamma = \tan^{-1}\left(\frac{\alpha\omega}{(\omega_0^2 - \omega^2)}\right), \tag{7b}$$

and A_t is chosen so as to satisfy the initial condition $x(0) = A$, $\dot{x}(0) = 0$. The first term in (7) is obviously a transient which decays for large t. Thus asymptotically the system oscillates with a frequency equal to that of the external periodic force. At the resonance value $\omega = \omega_0$, the amplitude of oscillation A_p takes the maximum value. The associated phase trajectory corresponds asymptotically to an attractor which is now a limit cycle as depicted in Fig. 1(c). In all the above cases, the initial value problem is exactly solvable, the motion is completely regular and the future is fully predictable, given the initial state. Trajectories corresponding to nearby initial conditions continue to be nearby, leading to complete predictability. The attractors are only fixed points or closed curves corresponding to regular motions.

2.2 Nonlinear oscillators

As a prelude to understanding the typical behaviour of damped and driven nonlinear oscillators, we ask how the dynamics of the previously studied linear oscillators are modified when a nonlinear spring force of cubic form is added so that the equation of motion becomes

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \omega_0^2 x + \beta x^3 = f \sin \omega t. \tag{8}$$

2.2.1 Forced oscillations - Resonances ($f \neq 0$)

One immediately observes that the behaviour of nonlinear oscillators of the form (8) can be qualitatively quite different from that of the linear case, even for very small β. Near the primary resonance $\omega \approx \omega_0$, $\beta \ll 1$, one can write the lowest order approximate solution in the form

$$x(t) = A\cos(\omega t + \delta), \tag{9}$$

where δ is a constant to be fixed. Using (9) in (8) one obtains the resonance curve

$$\left[(\omega_0^2 - \omega^2)A + \frac{\beta}{4}\beta A^3\right]^2 + (\alpha A\omega)^2 = f^2. \tag{10}$$

A typical form of this curve is shown in Fig. 2 for $\beta > 0$, which clearly shows the associated hysteresis phenomenon. In addition one also observes secondary resonances (subharmonic and superharmonic) at frequencies $\omega \approx n\omega_0$, and $\omega = \frac{\omega_0}{n}$, $n = 1, 2, 3, \ldots$

2.2.2 Nonlinear oscillations and bifurcations [1-7]

However, to obtain a complete picture of the dynamics, one has to have recourse to detailed numerical analysis, using for example the fourth order Runge-Kutta integration method. As a result one obtains a rich variety of bifurcation phenomena, namely successive qualitative (and quantitative) changes in the nature of oscillations as the nonlinearity parameter (or another control parameter say f or ω or α for fixed nonlinearity parameter) is varied. A mind-boggling multitude of bifurcations leading to the onset of chaos and further bifurcations occurs in the parameter space. The details are further described in Sec. 4. The above type of numerical analysis can be used in different ways to extract information about the dynamics. Some of them are the following.

1. Trajectory plot ((x-t)-plot)
2. Phase portrait: (x-$\dot{x}$)-projection of the trajectories from three-dimensional $(x, \dot{x}, t)$-space.
3. Poincaré map: Stroboscopic (snapshot) portrait of the trajectory at every period $T(= \frac{2\pi}{\omega})$.

4. Power spectrum: $P(\hat{\omega}) = \left| \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} x(t) \exp(-i\hat{\omega}t) \right|^2$.
5. Bifurcation diagram: Plot of Poincaré points of the solution $x(t)$ vs control parameter.
6. Lyapunov spectrum: Plot of maximal Lyapunov exponent vs control parameter.

Besides these, one often uses correlation functions, basins of attractors and so on.

What are the kinds of behaviours a dissipative nonlinear system of driven type typically exhibits and how do they occur? The common types of behaviours are the following:

1. Fixed points
2. Periodic response of period ω of the input
3. Subharmonic response ($\Omega = \frac{\omega}{n}$, n integer)
4. Superharmonic response ($\Omega = n\omega$)
5. Almost periodic (quasiperiodic) response (consisting of periodic components whose frequencies are incommensurable)
6. Chaotic response (nonperiodic and sensitively dependent on initial conditions)

Some typical attractors are illustrated in Figs. 3. There are typical routes through which periodic and chaotic responses can arise as the control parameter varies. Three most predominant routes are the following:

1. Feigenbaum's period doubling scenario.
2. Ruelle-Takens-Newhouse scenario of a quasiperiodic route.
3. Pomeau-Manneville scenario of an intermittency route.

Besides these three prominent routes, there are many other less common routes, such as period adding sequence, equal-periodic bifurcations and so on, which occur occasionally in dissipative systems.

2.3 Linear and nonlinear electronic circuits [6,8]

Besides numerical and analytic studies of nonlinear oscillators there is another useful way of investigating them, namely by analog simulation through suitable electronic circuits. Such studies help in a dramatic way to scan the parameter space quickly and to avoid long transients as in the case of numerical studies. The construction of such circuits is fairly easy in typical cases, giving rise to quick responses as the control parameter is varied.

From another point of view, a variety of nonlinear electronic circuits consisting of nonlinear physical devices such as nonlinear diodes, capacitors, inductors and so on or devices constructed with piecewise-linear circuit elements have been utilized in recent times as model nonlinear systems in their own right to explore different aspects of chaotic dynamics. They further give rise to interesting technological applications.

2.3.1 Linear circuits

Linear circuits consist of only linear elements such as linear resistors, capacitors or inductors besides the source. In fact, the ubiquitous LCR circuit (Fig. 4) can be considered as a prototype analog simulation of the damped and driven linear oscillator (8). The LCR circuit of Fig. 4 consists of a linear resistor, a linear inductor, a linear capacitor and a time-dependent voltage source, $f(t) = F_s \sin \omega_s t$. Applying Kirchhoff's voltage law to the circuit of Fig. 4, we obtain

$$L\frac{di_L}{dt} + Ri_L + v = F_s \sin \omega_s t, \tag{11}$$

with $i_L(0) = i_0$, $v(0) = v_0$, where i_L is the current along the inductor, v is the voltage across the capacitor, L is the linear inductance and R is the linear resistor.

Now substituting $i_L = C\frac{dv}{dt}$ into Eq. (11) and rearranging one obtains the linear second-order inhomogeneous ODE with constant coefficients,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{F_s}{LC}\sin \omega_s t. \tag{12}$$

Obviously Eq. (12) is analogous to (6) with appropriate identifications, and its solution can be immediately written down in the form (7) corresponding to limit cycle oscillation with frequency $\omega = \omega_s$. The actual experimental result for the wave form $v(t)$ and the phase portrait in the (v-i_L)-plane is shown in Fig. 5, which is seen to mimic the solution of (6) given in Fig. 1(c).

2.3.2 Nonlinear circuits [9]

a) analog simulation

As mentioned above, nonlinear oscillator equations can be analog simulated by the use of suitable operational amplifier modules and multipliers. Normally op-amps are used as integrators, differentiators, sign-changers, adders and so on. The mathematical operations are generally obtained either by using an op-amp individually with resistor feedback, or an op-amp with capacitor feedback. Similarly, product and division operations can be performed by multiplier and divider chips such as AD532 or AD534. One such analog simulation circuit constructed with adding integrators, scale changers and multipliers allows us to study the chaotic dynamics of the cubic Duffing oscillator equation (8). Its schematic diagram is given in Fig. 6.

b) Piecewise-linear circuits: MLC circuit [10]

Considering the familiar LCR circuit given in Fig. 4, let us add a nonlinear resistor, the so called Chua's diode characterized by a three segment negative resistance characteristic curve as given in Fig. 7. The modified circuit is given in Fig. 8. This simple non-autonomous circuit was first reported by Murali, Lakshmanan and Chua [10]. The equations of motion governing this circuit are

$$C\frac{dv}{dt} = i_L - g(v) \tag{13a}$$

$$L\frac{di_L}{dt} = -v - Ri_L + F_s \sin \omega_s t, \tag{13b}$$

where $g(v) = G_b v + \frac{1}{2}(G_a - G_b)[|v+B_p| - |v-B_p|]$, and G_a, G_b and B_p are circuit parameters. It can also be considered as a prototype of the chaotic nonlinear oscillators.

3 Some ubiquitous damped and driven nonlinear oscillators [6]

3.1 Duffing oscillator

One of the widely studied damped and driven nonlinear oscillators is the Duffing oscillator given by Eq. (8). It can be considered to represent a particle (of unit mass) moving in a potential well

$$V = \frac{1}{2}\omega_0^2 x^2 + \frac{\beta}{4}x^4, \tag{15}$$

subject to viscous damping and external periodic force. Here one may distinguish three cases.

1. Double-well (twin-well) potential $\omega_0^2 < 0, \beta > 0$
2. Single-well potential $\omega_0^2 > 0, \beta > 0$
3. Double-hump potential: $\omega_0^2 > 0, \beta < 0$.

Each one of the above three cases is an important model for many physical problems. The double-well potential represents the oscillations of a buckled beam under a periodic force, plasma oscillations and so on. Similarly, the single-well oscillator is a standard model of a particle oscillating under viscous damping and a periodic force, while a double-hump potential well is a model for weakly pinned charge density waves in anisotropic solids and superionic conductors.

Now Eq. (8) has an obvious sealing property which allows one to fix two of the five parameters α, ω_0^2, β, f and ω. From an experimental point of view it is more convenient to fix the spring forces ω_0^2 and β and vary the parameters in the (α-ω-f)-space. From a practical point of view one normally studies behaviour in the f-space or (f-ω)-space by fixing the remaining parameters as well. In the next section (Sec. 4), we will briefly discuss the chaotic dynamics of all the three potential wells of Eq. (8).

3.2 BVP and DVP oscillators

The nerve impulse propagation along the axon of a neuron can be studied by combining the equations for an excitable membrane with the differential equations for an electrical core conductor cable, assuming the axon to be an infinitely long cylinder. A well known approximation of FitzHugh and Nagumo to describe the propagation of voltage pulses $V(x,t)$ along the membranes of nerve cells is the set of coupled PDEs

$$V_{xx} - V_t = F(V) + R - I, \tag{16a}$$

$$R_t = c(V + a - bR), \tag{16b}$$

where $R(x,t)$ is the recovery variable, I the external stimulus and a, b, c are related to the membrane radius, specific resistivity of the fluid inside the membrane and temperature factor respectively. Here suffixes stand for partial differentiation.

When the spatial variation of V, namely V_{xx}, is negligible, Eq. (16) reduces to a set of two coupled first-order ODEs known as the Bonhoeffer-van der Pol oscillator system,

$$\dot{V} = V - \frac{V^3}{3} - R + I \tag{17a}$$

$$\dot{R} = c(V + a - bR), \quad \left(. = \frac{d}{dt}\right) \tag{17b}$$

with $F(V) = -V + \frac{V^3}{3}$. Normally the constants in Eq. (16) satisfy the inequalities $b < 1$ and $3a + 2b > 3$, though from a purely mathematical point of view this need not be insisted upon. Then with a periodic (ac) applied membrane current $A_1 \cos\omega t$ and a (dc) bias A_0, the BVP oscillator equation becomes

$$\dot{V} = V - \frac{V^3}{3} - R + A_0 + A_1 \cos\omega t, \tag{18a}$$

$$\dot{R} = c(V + a - bR). \tag{18b}$$

Again system (18) shows a very rich variey of bifurcations and chaos phenomena as discussed in Sec. 5 below.

Further, Eqs. (18) can be rewritten as a single second-order differential equation by differentiating (18a) with respect to time and using (18b) for R,

$$\ddot{V} - (1-bc)\left\{1 - \frac{V^2}{1-bc}\right\}\dot{V} - c(b-1)V + \frac{bc}{3}V^3 = c(A_0 b - a) + A_1 \cos(\omega t + \phi), \tag{19}$$

where $\phi = \tan^{-1}\frac{\omega}{bc}$. Using the transformation $x = (1-bc)^{-(1/2)}V$, $t \longrightarrow t' = t + \frac{\phi}{\omega}$, Eq. (19) can be rewritten as

$$\ddot{x} + p(x^2 - 1)\dot{x} + \omega_0^2 x + \beta x^3 = f_0 + f_1 \cos\omega t, \tag{20}$$

where

$$p = (1-bc), \ \ \omega_0^2 = c(1-b),$$
$$\beta = bc\frac{(1-bc)}{3}, \ \ f_0 = c\frac{(A_0 b - a)}{\sqrt{1-bc}},$$
$$f_1 = \frac{A_1}{\sqrt{1-bc}}. \tag{21}$$

Eq. (20) or its rescaled form

$$\ddot{x} + p(kx^2 + g)\dot{x} + \omega_0^2 x + \beta x^3 = f_0 + f_1 \cos\omega t \tag{22}$$

is the Duffing-van der Pol equation. In the limit k=0, we have the Duffing equation discussed above (with $f_0 = 0$), and in the case $\beta = 0$ ($g = -1$, $k = 1$) we have the driven van der Pol equation. Eq. (22) again exhibits a very rich variety of bifurcations and chaos phenomena, including quasiperiodicity, phase lockings and so on, depending on whether the potential $V = \frac{1}{2}\omega_0^2 x^2 + \frac{\beta x^4}{4}$ is i) a double well, ii) a single well or iii) a double hump. We will discuss some of the details in Sec. 5.

4 Duffing oscillator: Bifurcations and chaos

As noted earlier, the Duffing oscillator (Eq. (8)) dynamics can be described by a phase diagram in a three-parameter (α-f-ω)-space. In the following, we will concentrate on the f- or ω- or (f-ω)-parameter space because of practical considerations.

4.1 The double-well oscillator ($\omega_0^2 < 0, \beta > 0$)

Considering a particle in a double-well potential $V = -1/2|\omega_0^2|x^2 + \beta x^4$ with a base vibrating periodically, we notice that there are three equlibrium points in the force-free case ($f = 0$), corresponding to $\omega_0^2 x + \beta x^3 = 0$. They are the stable fixed points (centres)

$$x_{1,2}^s = \pm\sqrt{\frac{|\omega_0^2|}{\beta}} \tag{23}$$

and an unstable fixed point which is a saddle,

$$x_0^u = 0. \tag{24}$$

Then the oscillations about the two stable equilibrium points are given (after a redefinition of x) by

$$\ddot{x} + \alpha\dot{x} + 2|\omega_0^2|x \mp 3\sqrt{|\omega_0^2|\beta}\; x^2 + \beta x^3 = f \sin\omega t. \tag{25}$$

4.1.1 Period doubling scenario in f-space

Choosing the initial condition such that the system approaches the left equilibrium point ($x_1 = -1.0$) asymptotically as example, and choosing the parameters as $\omega_0^2 = -1.0$, $\beta = 1.0$, $\alpha = 0.5$, $\omega = 1.0$, we can solve Eq. (25) numerically and obtain the following picture. To start with we realize from the theory of nonlinear oscillations that the system can exhibit two types of periodic motions:

1. Small orbits (SOs): Oscillations around the stable fixed point x_1.
2. Large orbits (LOs): Large amplitude oscillations that encircle all the three equlibrium points x_1, x_2 and x_0.

Then the following picture arises.

A. Small orbit oscillations

As f is increased slightly from zero, a stable period T limit cycle about the left equilibrium point $x_1 = -1.0$ occurs, which persists up to $f = 0.34$. Then a period doubling scenerio starts at $f = 0.35$ with a $2T$ periodic cycle, followed by a $4T$ periodic cycle at $f = 0.358$, etc. This cascade of bifurcations accumulates at $f = f_c = 0.361$, the sequence converging geometrically at the rate known as the Feigenbaum ratio. Beyond $f = f_c$, one finds chaotic orbits occurring where the motion is highly dependent on the initial conditions. This can be checked by the various characterizations discussed previously. These motions are illustrated in Figs. (9a-c).

B. Hopping Oscillations

When $f = 0.37$, the onset of a crisis occurs, in which the chaotic motion that had been confined to the left well suddenly expands to cover the right well also, in a hopping cross-well chaotic motion.

C. Large orbit oscillations

For still larger values of f, the chaotic state suddenly disappears and is replaced by a stable period-mT oscillation, overcoming the central barrier, extending over both the valleys and encompassing all the equilibrium points. (Fig. 9f, g).

Fuller details are given in Table 1 and the bifurcation diagram in Fig. 10.

D. Antimonotonicity

In addition to the periodic window resulting from the crisis, there is also another possibility due to antimonotonicity. Here one finds that periodic orbits are not only created through period doubling cascades but are also destroyed through reverse period doubling when a control parameter is monotonically increased in any neighborhood of a homoclinic tangency value. Two types of such antimonotonicity are shown in Fig. 11. Fig. 11(a) represents a period-3 bubble in the range $0.53 \leq f \leq 0.62$, while Fig. 11(b) shows a reverse-period bubble in the region $0.43 \leq f \leq 0.48$.

4.1.2 Phase-diagram in (f-ω)-space

To appreciate the rich variety of bifurcations, let us look at the (f-ω)-phase diagram (Fig. 12) given by Szemplinska-Stupnicka and Rudowski [11] in the frequency range $0.25 < \omega < 1.1$, fixing $\alpha = 0.1$, $|\omega_0|^2 = 0.5$ and $\beta = 0.5$. One observes that SO, symmetric LO, unsymmetric LO and cross-well chaotic or stable attractors coexist. The SO motions occur within the whole (f-ω)-plane except for two V-shaped regions, one with a cusp at $\omega = 0.8$ and $f = F_2$ at the principal resonance region, the other cusp at $\omega = 0.4$, and $f = 0.14$, that is the superharmonic zone. Inside the two V-shaped regions the system can exhibit cross-well chaotic (or regular) motion. Also, in some regions, two different attractors coexist while, in the others, a single steady state (globally stable) motion can be observed. Typical orbits: the form of the various single and coexisting states denoted in Fig. 12 by numbers $1, 2, \ldots, 20$ are shown in Fig. 13. Regular attractors are illustrated by their phase portraits and chaotic attractors by Poincaré maps. They include all the types of motions discussed above.

4.2 Single well and double hump potentials

Considering the single-well potential, one observes hysteresis in the primary resonance region, harmonic resonances and symmetry breaking orbits for a range of low values of f. For sufficiently high values of f, chaos occurs through a period-doubling route. Similarly, in the double-hump case, one finds that one symmetric attractor bifurcates into two mutually symmetric attractors. Both attractors then undergo period-doubling cascades to chaos, where windows of both even and odd periods occur. Then both chaotic attractors broaden and merge to form a single attractor, ultimately leading to a boundary crisis. For further details see Ref. [6].

4.3 analog simulation and experimental verification

The Duffing oscillator (8) can be easily analog-simulated using op-amps and four-quadrant multipliers. Replacing $\dot{x}(t) = \int \ddot{x} dt$, $x(t) = \int \dot{x} dt$, the equivalent schematic simulation circuit is given in Fig. 6. Here blocks IC_1 and IC_2 represent two integrators, IC_3 is an inverter (sign-changer), and M_1 and M_2 are four-quadrant multipliers. The experimental results for the double-well case are given in Figs. 14, and they agree with the theoretical numerical analysis. For further details see [6] again.

5 Dynamics of the BVP and DVP oscillators [6] and MLC Circuit

Let us now quickly review the chaotic dynamics and bifurcations of the BVP and DVP oscillators discussed in Sec. 3.2. Apart from the period-doubling route to chaos, one also observes the presence of phase-lockings and devil's staircase structures, quasiperiodic and intermittency routes to chaos predominantly. The nonlinear dynamics of the MLC circuit are also discussed. We shall therefore give only a very brief account of these structures.

5.1 BVP oscillator

Considering Eq. (18), the equilibrium points in the absence of external stimuli ($A_0 = A_1 = 0$) are the roots of the equations

$$V^3 - qV - p = 0, \tag{26a}$$

$$V + a - bR = 0, \tag{26b}$$

where $p = \frac{-3a}{b}$ and $q = 3\frac{(b-1)}{b}$. Eq. (26a) has three real roots for $27p^2 > q^3$ and one real root for $27p^2 < q^3$. The linear stability analysis of Eqs. (26) shows that for a fixed point (V_0, R_0) to be stable, the necessary and sufficient condition is

$$(V_0^2 + bc - 1) > 0$$

and

$$c(1 + bV_0^2 - b) > 0. \tag{27a}$$

Thus for suitable choices of parameters one can have one stable focus or one saddle and two stable foci.

Proceeding along the same lines, with the inclusion of constant bias ($A_1 = 0$, $A_0 > 0$), one can find fixed points by simply modifying p to $p = 3(A_0 - a/b)$ in Eqs. (26). It is of particular interest to check whether the fixed points undergo a Hopf bifurcation. This can be easily verified by checking the nature of changes in the stability property of the fixed points and the parametric choices for which the eigenvalues cross the imaginary axis. For the present problem, one can easily check that a Hopf bifurcation occurs at

$$A_{0\pm} = \frac{V_{0\pm}^3}{3} - \frac{b-1}{b}V_{0\pm} + \frac{a}{b} \tag{27b}$$

One checks that for $A_0 < A_{0-}$ and $A_0 > A_{0+}$ the fixed points are asymptotically stable, being either a node or a spiral point. For $A_0 \in (A_{0-}, A_{0+})$, the eigenvalues have positive real parts so that the fixed point is unstable and one may expect a stable limit cycle in this range.

Similarly when only a periodic input current ($A_1 \neq 0, A_0 = 0$) is present, the system, as in the case of the Duffing oscillator, undergoes period-doubling bifurcations to chaos, followed by windows, intermittency and so on. The bifurcation diagram given in Fig. 15 describes briefly the various bifurcations admitted by this system for the parametric choice $a = 0.7$, $b = 0.8$, and $c = 0.1$.

Now if we consider the situation in which both external d.c. and periodic stimuli are present, the system enters into an interesting phenomenon called phase-locking or mode-locking. Here the natural frequency of the oscillator (ν_n) and the frequency of the driving force get locked. When one calculates the winding number $W = \frac{\nu_d}{\nu_n}$ as a function of the d.c. current A_0 (which will give rise to a different ν_n), the corresponding schematic diagram has a devil's staircase structure. A typical example for the choice $A_1 = 0.74$ is shown in Fig. 16.

5.2 DVP oscillator

One can consider three different cases for the DVP oscillator (22), as in the case of the Duffing oscillator, namely i) double well, (ii) single well and (iii) double hump. They exhibit many complicated bifurcation phenomena including period-doubling, hysteresis, quasiperiodicity, phase-locking and so on. Some interesting features of the double-hump DVO oscillator include

i) the existence of a Farey sequence,

ii) double hysteresis loops,

iii) a swallow-tailed bifurcation structure of subharmonic locking,

and so on. For more details see Ref. [6].

5.3 Chaotic dynamics of the MLC circuit [10]

The simple second-order nonautonomous dissipative nonlinear circuit consisting of Chua's diode as its only nonlinear element, suggested by Murali, Lakshmanan and Chua (MLC), can exhibit a rich variety of bifurcations and chaos phenomena. The circuit diagram and its state equations have already been introduced in Sec. 2.3.2. Rescaling Eq. (13) with $v = xB_p$, $i_L = G.y.B_p$, $G = 1/R$, $\omega = \frac{\Omega C}{G}$ and $t = TCG$, and then redefining T as t, we have the following set of normalized equations,

$$\dot{x} = y - h(x), \;\; \dot{y} = -\beta y - \nu\beta y - \beta x + f \sin \omega t, \tag{28a}$$

where $\beta = \frac{C}{LG^2}$, $\nu = GR$, and $f = \frac{Fs}{\beta_p}$. Here $h(x)$ stands for

$$h(x) = \begin{cases} bx + a - b, & x \geq 1 \\ ax, & |x| \leq 1 \\ bx - a + b, & x \leq -1. \end{cases} \tag{28b}$$

Here $a = \frac{Ga}{G}$, $b = G_b G$. Now the dynamics of Eq. (28) depend on the parameters ν, β, a, b, ω and f. The experimental circuit parameters used in the actual circuit correspond to $\beta = 1$, $\nu = 0.015$, $a = -1.02$, $b = -0.55$ and $\omega = 0.75$. It is straightforward to establish that there exists a unique equilibrium (x_0, y_0) for Eq. (28) in each of the following three regions,

$$\begin{aligned} D_1 = \{(x,y) |\; x \geq 1\} : &\quad P^+ = (-k_1, -k_2) \\ D_0 = \{(x,y) |\; |x| \leq 1\} : &\quad O = (0,0) \\ D_{-1} = \{(x,y) |\; x \leq -1\} : &\quad P^- = (k_1, k_2) \end{aligned}$$

where $k_1 = [(\sigma(a-b))/(p+\sigma b)]$, $k_2 = [\beta(b-a)/(\beta+\sigma b)]$ and $\sigma = \beta(1+\nu)$. In each of the regions D_1, D_0, D_{-1}, Eq. (28) is linear. One can easily check by a stability analysis that $O \in D_0$ is a hyperbolic (saddle) fixed point, while P^+ and P^- are stable spiral fixed points. Depending on the initial conditions, any of them can be observed. As the forcing signal f is increased ($f > 0$), the stable fixed points undergo a Hopf bifurcation. A further increase in f leads to period doubling bifurcations and finally to chaos. The corresponding numerical and experimental results are given in Fig. 17.

6 Dictionary of Bifurcation

The various oscillator systems discussed in the previous sections exhibited a variety of oscillatory behaviours including fixed points, resonant periodic oscillations, period-doubled oscillations, quasiperiodic and chaotic motions. In particular, the various bifurcation diagrams clearly show that all these systems exhibit qualitative (and quantitative) changes at critical values of the control parameters as they are varied smoothly. These correspond to the different bifurcations exhibited by the systems. The question is how to understand the existence of such bifurcations and the mechanism by which they arise. One possible approach is to look for the local stability properties of solutions in the neighbourhood of the critical parameter values at which bifurcations occur. However, since no exact solutions are available for the above systems, one can prepare a dictionary of bifurcations [12] that occur in simple and interesting low dimensional nonlinear systems, and then use them as reference systems for the identification of the bifurcations in the various nonlinear oscillators. In Tables 2 and 3 we give their salient features which are self-explanatory.

7 Analytic approaches

The various types of bifurcations exhibited by a typical nonlinear oscillator can be classified by using the dictionary of bifurcation phenomena like the ones discussed in the earlier section. However, some approximate analysis can be performed analytically by constructing perturbative solutions and studying their stability properties. Moreover the analytic structure of solutions in the complex t-plane can throw much light on the integrability and nonintegrability and chaotic behaviour of dynamical systems. We will discuss some features of these aspects in this section.

7.1 Theoretical analysis of large orbit T-periodic solution in the double-well Duffing oscillator

The numerical analysis presented in Sec. 4 for the double-well Duffing oscillator conclusively showed that it admits a large orbit (LO) T-periodic solution for a wide range of drive frequencies. One might then expect a first-order approximate solution whose stability analysis might give a good estimation of the parameter domain in which it occurs. So we shall look for a T-periodic solution of the oscillator equation [11],

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} - |\omega_0^2|x + \beta x^3 = f\cos\omega t, \tag{29}$$

which is close to a harmonic function of time as

$$x(0) = A(0)\cos(\omega t + \phi), \tag{30}$$

where $A(0)$ and ϕ are to be determined. For this purpose we transform (29) into the form

$$\ddot{x} + \omega^2 x + \varepsilon f(x, \dot{x}, t) = 0 \tag{31}$$

as

$$\ddot{x} + \omega^2 x + \mu(\bar{\alpha}\dot{x} + x\sigma - \Omega^2 x - |\bar{\omega}_0|^2 x + \bar{\beta}x^3 - \bar{f}\cos\omega t) = 0, \tag{32}$$

where

$$\mu\bar{\alpha} = \alpha,\ \ \mu\sigma = \mu\Omega^2 - \omega^2,\ \ \mu\bar{f} = f,\ \ \mu|\bar{\omega}_0|^2 = |\omega_0|^2, \mu\bar{\beta} = \beta. \tag{33}$$

Here the quantity Ω has been introduced from the form of the harmonic solution $x = a\cos\Omega t$, $\Omega^2 = -|\omega_0^2| + 3/4\beta a^2 > 0$. Now one can look for a periodic solution of (32) in the form

$$x(t, \mu) = x_0(T_0, T_1, \ldots) + \mu x_1(T_0, T_1, \ldots), \tag{34}$$

where the multiple scales $T_0 = t$, $T_1 = \mu t$, $\ldots$ and $\frac{d}{dt} = \sum\limits_{n=0}^{\infty} \mu^n D_n$, $D_n = \frac{\partial}{\partial T_n}$ so that we have the system of equations

$$D_0^2 x_0 + \omega^2 x_0 = 0, \tag{35a}$$

$$D_0^2 x_1 + \omega^2 x_1 = -2D_0 D_1 x_0 - \bar{\alpha} D_0 x_0 - (\sigma - \Omega^2 - |\bar{\omega}_0)|^2)x_0 - \bar{\beta}x_0^3 + \bar{f}\cos\omega t, \tag{35b}$$

etc. The general solution of (35a) is

$$x_0 = A_1(T_1)\exp i\omega T_0 + c.c., \tag{36}$$

where A_1 is, in general, complex. Using (36) in (35b), in order to eliminate the secular terms in the solution of x_1, one requires that

$$-[2i\omega(D_1 A_1 + \bar{\alpha}/2A_1) + (\sigma - \Omega^2 - |\bar{\omega}_0|^2)A_1 + 3\beta A_1^2 A_1^*] + \bar{f}/2 = 0. \tag{37}$$

Making the substitution

$$A_1(T_1) = (1/2)a(T_1)\exp i\phi(T_1) \tag{38}$$

and equating real and imaginary parts for steady state conditions with $\Omega^2 \approx -|\omega_0^2| + 3/4\beta a^2$, we obtain

$$a = \frac{f}{\sqrt{(\Omega^2 - \omega^2)^2 + \alpha^2\omega^2}},\ \ \tan\phi = \frac{-\alpha\omega}{(\Omega^2(a) - \omega^2)}. \tag{39}$$

Finally, the first-order correction becomes

$$\mu x_1(t) = A_3\cos 3(\omega t + \phi),\ \ A_3 = \frac{\mu\bar{\beta}a^3}{32\omega^2} = \frac{\beta a^3}{32\omega^2}, \tag{40}$$

so that the *LO* solution becomes

$$x(t) = a\cos(\omega t + \phi) + \frac{a^3}{32\omega^2}\cos 3(\omega t + \phi) \equiv x(t + T). \tag{41}$$

7.1.1 Stability of LO solution,

A. Soft mode instability

Considering small disturbances to the amplitude and phase of the solution (41) leading to

$$\tilde{x}(t) = (a + \delta a)\cos(\omega t + \phi + \delta\phi) + \frac{(a + \delta a)^3}{64\omega^2}\cos 3(\omega t + \phi + \delta\phi), \qquad (42)$$

one obtains the linear eigenvalue problem

$$\begin{pmatrix} \delta\dot{a} \\ \delta\dot{\phi} \end{pmatrix} = \begin{pmatrix} -\frac{\alpha}{2} & -\frac{\alpha}{2\omega}(\Omega^2 - \omega^2) \\ \frac{1}{2\omega a}(\Omega^2 - \omega^2) & -\frac{\alpha}{2} \end{pmatrix} \begin{pmatrix} \delta a \\ \delta\phi \end{pmatrix}. \qquad (43)$$

Then the linear stability of the solution (41) depends upon the (2×2) matrix in the right hand side of (43) from which a necessary criterion for the stability of the solution can be obtained (see Fig. 20).

B. Symmetry-breaking instability

Considering more general instabilities, $\tilde{x}(t) = x(t) + \delta x(t)$, one can deduce a Hill type eigenvalue problem,

$$\delta\ddot{x} + \alpha\delta\dot{x} + (\lambda_0 + \sum_{n=2,4,6} \lambda_n \cos n\omega t)\delta x = 0, \qquad (44)$$

where λ_0, λ_2, λ_4, λ_6 are appropriate coefficients. One can check that while there is no period-doubling instability, symmetry-breaking instability can occur here. This may be verified by assuming a solution of the form

$$\delta x(t) = e^{\epsilon_1 t}(b_0 + b_{21}\cos 2\omega t + b_{22}\sin 2\omega t), \qquad (45)$$

and finding the values of ϵ_1. The corresponding curve $\omega_{SB}(a)$ is given in Fig. 20. One finds that the curves $\omega_B(a)$ and $\omega_{SB}(a)$ determine the region of existence of LO to a reasonable approximation.

7.2 Stability analysis of SO motion

The numerical analysis of the double-well Duffing oscillator discussed earlier clearly shows the presence of SO periodic motion confined to one of the wells. One can therefore again look for T- periodic solution in the right well for example and apply multiple scale perturbation theory. Comparing with Eq. (29), we consider the equation

$$\ddot{x} + \omega^2 x + \mu\bar{\omega}_1 x^2 + \mu^2(\bar{\alpha}\dot{x} + \bar{\omega}_3 x^3 + \sigma x - \bar{f}\cos\omega t) = 0,$$

where

$$\mu\bar{\omega}_1 = 3/2, \mu^2\bar{\omega}_3 = 1/2, \mu^2\bar{\alpha} = \alpha, \sigma\mu^2 = 1 - \omega^2,$$
$$\mu^2\bar{f} = f, \mu \ll 1. \quad (46)$$

Proceeding as in the previous case one obtains the second-order approximate solution

$$x(t) = a\cos(\omega t + \phi) - 3/4a^2 + 1/4a^2\cos 2(\omega t + \phi) + a^3/16\cos 3(\omega t + \phi). \quad (47)$$

Considering, as in the case of *LO* periodic solutions, the stability of the *SO* periodic solution (47), one can analyse the first-order instability as well as period-doubling instability. As a result one can conclude that the approximate T-periodic solution is unstable within a region of the driving frequency $\omega_A < \omega < \omega_{PD}$ and that no other *SO* stable solution is possible. The above result is found to give rise to a surprisingly good estimation of the system's critical parameter values for which cross-well chaos really occurs. For more details, see Ref. [11].

7.3 Analytic structure of the Duffing oscillator

An altogether different approach to investigating the nonintegrability aspects of the Duffing oscillator and other dynamical systems is the study of the analytic structure of singularities of the solutions in the complex time plane, known in the literature as the Painlevé singularity structure analysis [see for example Refs. 13-15]. Originally this method was applied successfully to isolate the integrable cases of the rigid body motion in a gravitational field. Its recent renaissance started with the discovery of soliton systems (see the lectures of Prof. Ablowitz in this School). One essentially tries to isolate the parameters of the nonlinear differential equations underlying the dynamical system for which the solution is free from movable critical points (movable branch points and essential singularities) so that the solution is meromorphic in the complex time plane. For nonintegrable systems, the singularities are often located in a complicated pattern, giving an indication that a sufficient number of analytic integrals is lacking. Consider for example the unforced, undamped (free) Duffing oscillator,

$$\ddot{x} + \omega_0^2 x + \beta x^3 = 0. \quad (48)$$

The solution to (48) can be given in terms of Jacobian elliptic functions,

$$x(t) = Acn(\Omega t + \delta), \Omega = \sqrt{\omega^2 + \beta A^2}, \quad (49)$$

where the modulus $k = \sqrt{\beta A^2/2(\omega_0^2 + \beta A^2)}$. One knows that the elliptic functions are meromorphic and doubly periodic. The only singularities they possess in the complex t-plane are poles of order 1 at the lattice points,

$$(\Omega t + \delta) = 2nK + (2m+1)iK', \ n, m \in Z. \quad (50)$$

We also note that despite the existence of these poles, the system possesses the analytic integral, $E = \frac{1}{2}\omega_0^2 A^2 + \frac{1}{4}\beta A^4$, in the complex t-plane and the system is integrable. A typical pattern of the lattice of singularities of the solution of (48) is given in Fig. 21a. The situation becomes more complicated when damping as well as forcing are present. Typical singularity structures are shown for the case when damping alone is present in Fig. 21b, and when forcing is also present in Fig. 22. One observes a progressive clustering of singularities off the real axis as the forcing strength increases. Actually, one can identify much more intricate structures through local analysis or numerical analysis. Around each singular point there is an infinite-sheeted, four-armed structure of singularities. For more details, see Refs. [13-15]. Similarly, infinite-sheeted structures are present in the case of the DVP oscillator [16].

8 Controlling Chaos

Eventhough the presence of chaotic behaviour is generic and robust for suitable nonlinearities, ranges of parameters and external forces, there are practical situations where one wishes to avoid or control chaos so as to improve the performance of the dynamical system. Also, eventhough chaos is sometimes useful as in a mixing process or in heat transfer, it is often unwanted or undesirable. For example, increased drag in flow systems, erratic fibrillations of heart beats, extreme weather patterns and complicated circuit oscillations are situations where chaos is harmful. Clearly, the ability to control chaos, that is to convert chaotic oscillations into desired regular ones with a periodic time dependence would be beneficial in working with a particular system. The possibility of purposeful selection and stabilization of particular orbits in a normally chaotic system, using minimal, predetermined efforts, provides a unique opportunity to maximize the output of a dynamical system. It is thus of great practical importance to develop suitable control methods and to analyze their efficacy. Very recently much interest has been focussed on this type of problems [6,17-19].

8.1 Algorithms for controlling chaos: Feedback and non-feedback methods

Let us consider a general n-dimensional nonlinear dynamical system,

$$\dot{X} = \frac{dX}{dt} = F(X\ ;\ p\ ;\ t), \tag{51}$$

where $X = (x_1, x_2, x_3, ..., x_n)$ represents the n state variables and p is a control or external parameter. Let $X(t)$ be a chaotic solution of Eq. (51). Different control algorithms are essentially based on the fact that one would like to effect the most minimal changes to the original system so that it will not be grossly deformed. From this point of view, controlling methods or algorithms can be broadly classified into two categories:

(i) feedback methods, and

(ii) non-feedback algorithms.

Feedback methods essentially make use of the intrinsic properties of chaotic systems, including their sensitivity to initial conditions, to stabilize orbits already existing in the systems. Some of the prominent methods are the following.

1. Adaptive control algorithm
2. Ott-Grebogi-Yorke (OGY) method of stabilizing unstable periodic orbits
3. Singer *et al.*'s method of stabilizing unstable periodic orbits
4. Various control engineering approaches.

In contrast to feedback control techniques, non-feedback methods make use of a small perturbing external force such as a small driving force, a small noise term, a small constant bias or a weak modulation to some system parameter. These methods modify the underlying chaotic dynamical system weakly so that stable solutions appear. Some of the important controlling methods of this type are the following.

1. Parametric perturbation method
2. Addition of a weak periodic signal, constant bias or noise
3. Entrainment-open loop control
4. Oscillator absorber method.

For more complete details of the various controlling methods, see for example Refs. [6,17,18].

Here is a typical example of adaptive control algorithm. We can control the chaotic orbit $X_s = (x_s, y_s)$ of the BVP oscillator by introducing the following dynamics on the parameter A_1:

$$\dot{x} = x - \frac{x^3}{3} - y + A_0 + A_1 \cos \omega t, \tag{52a}$$

$$\dot{y} = c(x + a - by), \tag{52b}$$

$$\dot{A}_1 = -\epsilon[(x - x_s) - (y - y_s)], \ \epsilon << 1 \tag{52c}$$

The actual conversion of the chaotic motion into period-T and $2T$ limit cycles in a typical case is shown in Fig. 23. All the other methods can be discussed in similar fashion.

9 Synchronized Chaotic Systems and Secure Communication

Finally, we will discuss very briefly another fascinating concept, namely, synchronization of chaos, which is attracting considerable interest among chaos researchers in recent times. The possibility of two or more chaotic systems oscillating in a coherent and synchronized way is not an obvious one. One of the main features often associated with the definition of chaotic behaviour is the sensitive dependence on initial conditions. Then one may conclude that synchronization is not feasible in chaotic systems because it is not possible in real systems either to reproduce exactly identical initial conditions or exact specification of system parameters of two similar systems. We may be able to build "nearly" identical systems, but there is always an inevitable technological mismatch and noise, impeding the exact reproduction of all the parameters and initial conditions. Thus, even an infinitesimal deviation in any one of the parameters will eventually result in the divergence of nearby starting orbits.

In this connection, the recent suggestion of Pecora and Carrol [20] that it is possible to synchronize even chaotic systems by introducing appropriate coupling between them has revolutionized our understanding. They have shown that if a reproduced part (response or subsystem) of a chaotic system responds to a chaotic signal from the full (drive) system, under some conditions the signals in the response part would converge to the corresponding signals in the drive system. Such a possibility is known as the synchronization of chaotic systems. This idea has been further generalized by cascading [20] the reproduced parts or subsystems. Here the considered chaotic system may be divided into two subsystems, each of which will synchronize with the full system when driven by the appropriate chaotic signal. By arranging one of these synchronized subsystems to drive the other subsystem, it is possible to produce synchronized signals in the subsystems for every signal in the full drive system.

Another innovative method which has been developed recently for chaos synchronization is the method of one-way coupling between two identical chaotic systems [21,22]. In this configuration the response chaotic system variables follow identically the drive chaotic system variables for appropriate one-way coupling strength. Moreover, the behaviour of the response system is dependent on the behaviour of drive system, but the drive system is not influenced by the response system's behaviour.

Thus by reproducing all the signals at the receiver under the influence of a single chaotic signal from the drive, the synchronized chaotic systems provide a rich mechanism for signal design and generation, with potential applications to communications and signal processing. Further, one can use the pair of cascading driven subsystems or the identical response system with one-way coupling term as attractor recognition devices for chaotic attractors. Hence, a single input signal will allow recognition and reconstruction of the driving system and/or rejection of non-matching device systems. Another interesting area of application would be cryptography. For example, the non-driven part of the chaotic system produces signals, which are typically broad-band, noise-like and difficult to predict, and so they can be ideally utilized in various contexts for masking information bearing waveforms. They can also further be used as modulating waveforms in spread spectrum techniques.

9.1 Pecora and Carroll method

In this method, Pecora and Carroll have considered an n-dimensional system governed by the state equation of the form

$$\frac{dX}{dt} = f[X(t)], \; X = (x_1, x_2, \cdots, x_n)^T. \tag{53}$$

They divide the system into two parts in an arbitrary way as $X = (x_D, x_R)^T$. The D part is referred to as the driving subsystem variables and the R part as the response subsystem variables. Then Eq. (53) can be rewritten as

$$\dot{x}_D = g(x_D, x_R), \quad (m{-}dimensional) \tag{54a}$$

$$\dot{x}_R = h(x_D, x_R), \quad (k{-}dimensional) \tag{54b}$$

where $x_D = (x_1, x_2, \cdots, x_m)^T$, $x_R = (x_{m+1}, x_{m+2}, \cdots x_n)^T$, $g = [f_1(X), \cdots, f_m(X)]^T$, $h = [f_{m+1}(X), \cdots, f_n(X)]^T$, and $m + k = n$.

Pecora and Carroll suggested building an identical copy of the response subsystem with variables x'_R and drive it with the x_D variables coming from the original system. In such a model, we have the following compound system of equations:

$$\dot{x}_D = g(x_D, x_R), \quad (m{-}dimensional){-}drive \tag{55a}$$

$$\dot{x}_R = h(x_D, x_R), \quad (k{-}dimensional){-}drive \tag{55b}$$

$$\dot{x}'_R = h(x_D, x'_R), \quad (k{-}dimensional){-}response. \tag{55c}$$

Under the right conditions, as time elapses the x'_R variables will converge asymptotically to the x_R variables and continue to remain in step with the instantaneous values of $x_R(t)$. Here the drive or master system controls the response or slave system through the x_D component. The other component x'_R is allowed to have different initial conditions from that of x_R. The generalized schematic set-up for this type of synchronization is shown in Fig. 24.

One can easily see that if all the Lyapunov exponents of the response system consisting of Eq. (55c) are less than zero, then after the decay of the initial transients, $x'_R(t)$ will be exactly in step with $x_R(t)$. The Lyapunov exponents for Eq. (55c) are called the conditional Lyapunov exponents (CLE's). Also the response system with negative conditional Lyapunov exponents is called a stable subsystem or a stable response system. If at least one of the conditional Lyapunov exponents is positive, then synchronization will not occur between the x'_R and x_R variables. If the x'_R and x_R subsystems are under perfect synchronization, then the difference between the x'_R and x_R variables, $x^*_R = x'_R - x_R$, will tend to zero asymptotically, that is, $\lim_{t\to\infty} x^*_R(t) \to 0$. Then we have

$$\dot{x}^*_R(t) = \dot{x}'_R(t) - \dot{x}_R(t) = h(x_D, x'_R) - h(x_D, x_R). \tag{56}$$

If the subsystem is linear, we have

$$\dot{x}^*_R(t) = Ax^*_R(t), \tag{57}$$

where A is a $(k\times k)$ constant matrix. Let the eigenvalues of A be $(\lambda_1, \lambda_2, \cdots, \lambda_k)$. The real parts of these eigenvalues are, by definition, the CLE's we seek.

If all of the CLE's are negative, then $\lim_{t\to\infty} x^*_R(t) = 0$, and the subsystems will synchronize; if there is a positive CLE, the subsystems will grow further apart as $t \to \infty$ and they will never synchronize. Interestingly, an intermediate case occurs if one or more of the CLE's are zero but none is positive; in this case, as $t \to \infty$, the trajectories of the subsystems will be separated by a fixed distance depending upon the initial conditions.

If the response system is nonlinear in nature, then the conditional Lyapunov exponents are not so easily determined and we must resort to numerical procedures to calculate them. But if the response system is linear (for example, a linear circuit with passive elements), it is trivial to calculate the CLE in certain cases. The above methodology has been successfully applied to obtain chaos synchronization in many important nonlinear systems including the Lorenz system, the Rössler system, the hysteretic circuit, Chua circuit, the driven Chua circuit, the DVP oscillator, phase-locked loops, and so on. For applications to various systems and applications to secure communications we again refer to [6].

10 Conclusions

In this set of lectures, we have tried to show how even seemingly simple looking nonlinear oscillators can give rise to a very rich and baffling variety of complex dynamics involving periodic and chaotic motions. Combined analytical, numerical and analog simulation studies can help to unravel such complex dynamics in a systematic manner. Recent developments also show clearly that chaotic dynamical behaviour can be controlled algorithmically by minimal efforts so as to avoid any harmful effects of chaos on the structure of the system. Furthermore, identically evolving chaotic systems can be synchronized in a surprising way through appropriate coupling and the concept can be profitably used for spread spectrum communications. Thus the study of even such simple nonlinear systems can bring out an enormous amount of new knowledge on dynamical systems.

Naturally, it will be of great interest to consider assemblies and arrays of such oscillators with suitable couplings. Such studies can lead to interesting spatio-temporal patterns and chaos as well as to new structures of practical importance. The associated mathematical analysis and physical realizations will be areas of intense future investigations.

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Figure Captions

Fig. 1(a) (i) Solution curve $x(t)$ of Eq. (2) for $\omega_0^2 = 1.0$ (ii) Phase portrait in the $(x - \dot{x})$ plane corresponding to (i).

Fig. 1(b) (i) Solution curve $x(t)$ of Eq. (5) for $\alpha = 0.5$, $\omega_0^2 = 1.0$. (ii) Phase Portrait of (i).

Fig. 1(c) (i) Solution curve $x(t)$ of Eq.(6) for $\alpha = 0.5$, $f = 1.0$, $\omega_0^2 = 1.0$, and $\omega = 1.0$. (ii) Phase portrait of (i) (transients excluded).

Fig. 2 Resonance curve of the nonlinear oscillator (8) for $\beta > 0$.

Fig.3 Some typical attractors of a nonlinear oscillator. (a) Trajectories ending at a stable focus. (b) Stable limit - cycle attractor of period T and angular frequency $\omega = 2\pi/T$. (c) A quasiperiodic attractor of the Duffing-van der Pol oscillator. (d) Chaotic attractor of the Murali-Lakshmanan-Chua circuit

Fig.4 A forced LCR circuit (8). Here $f(t) = F_s \sin \omega_s t$.

Fig.5 (a) Wave form v of circuit of Fig. 4. (b) Trajectory plot in the $(v - i_L)$ plane. (c) Trajectory plot in the $(v - i_L)$ plane of the experimental circuit of Fig. 4.

Fig.6 Analog simulation circuit of the driven Duffing oscillator (double-well potential).

Fig.7 A typical three-segment piecewise-linear characteristic curve of a nonlinear resistor (N).

Fig.8 Circuit diagram of the simplest dissipative nonautonomous circuit.

Fig.9 Various types of steady-state attractors of the Duffing oscillator as a function of the forcing strength f.

Fig.10 Bifurcation diagrams of the Duffing oscillator in the $(f - x)$ plane computed through numerical simulation.

Fig.11 (a): (i) Schematic diagram of a reverse bubble. (b): (i) Schematic diagram of a bubble. (a): (ii) Bifurcation diagram in the range $0.43 - 0.48$ of the parameter f. (b): (ii) Bifurcation diagram in the range $0.53 \leq f \leq 0.62$

Fig.12 The $(f - \omega)$ phase diagram indicating regions of different steady states exhibited by the double-well potential oscillator for $\alpha = 0.1$, $\omega_0^2 = -0.5$, $\beta = 0.5$ in Eq.(25) with $f(t) = f \cos \omega t$. (vertical lines): LO symmetric; (right slanting lines): LO unsymmetric; shaded area: cross-well chaotic motion; cross-thatched V areas: SO occurs outside V-shaped regions. Adapted from Ref.[11].

Fig.13 Various types of steady state attractors corresponding to Fig. 12.
Fig.14 Experimental results of the Duffing oscillator circuit (Fig. 6). (i) Phase portrait in the $(x-\dot{x})$ plane (ii) Lower trace: solution curve $x(t)$, upper trace: forcing.

Fig.15 Bifurcation diagram of the BVP Eq.(17) in the ranges (a) $A_1 \in (0, 1.8)$, (b) $A_1 \in (0.6, 0.8)$, and (c) $A_1 \in (1.0, 1.3)$.

Fig.16 The complete devil's staircase for Eq.(17) at $A_1 = 0.88$ with $a = 0.8$, $b = 0.7$, $c = 0.09$, and $\omega = 1.0$

Fig.17 MLC circuit (28): (a) Numerical simulation results: Phase portraits in $(x-y)$ plane: (i) $f = 0.065$, Period-1 limit cycle; (ii) $f = 0.08$, Period-2 limit cycle; (iii) $f = 0.091$, Period-4 limit cycle; (iv) $f = 0.1$, One-band chaos; (v) $f = 0.15$, Doubleband chaotic attractor. (b) Experimental results: Phase portrait v versus $v_s (= R_s i_L)$: (i) $F = 0.0365V_{rms}$, Period-1 limit cycle; (ii) $F = 0.0549V_{rms}$, Period-2 limit cycle; (iii) $F = 0.064V_{rms}$, Period-4 limit cycle; (iv) $F = 0.0723V_{rms}$, One-band chaos; (v) $F = 0.0107V_{rms}$, Doubleband chaotic attractor.

Fig.18 Codimension one bifurcations (see Table 2).

Fig.19 Codimension two bifurcations (see Table 3).

Fig.20 Regions of LO attractor: computer simulation and theoretical stability limits: $\alpha = 0.1$ (Ref.[11]).

Fig.21 (a) Square lattice of poles found in typical solution of free undamped Duffing oscillator ($\alpha = 0, f = 0$). (b) Singularity distribution in the complex t-domain of a solution to the damped oscillator.

Fig.22 Typical singularity distribution in complex t-domain for a solution of the forced Duffing oscillator. Accentuated poles (heavy dots) indicate commonly observed 'tunnel' structures (adapted from Ref.[13]).

Fig.23 Evolution of the variable x of the BVP oscillator. (a) Limit cycle motion without control; (b) Attraction to the fixed point $(x_s, y_s) = (-1.001, -0.376)$ under the control for $\epsilon = 0.1$. Conversion of (c) chaotic motion to (d) a period -T limit cycle and (e) period-$2T$ limit cycle. A_{1f} is the value of A_1 associated with the desired solution.

Fig.24 Schematic representation of cascading synchronization. Drive and response #1 (left slanting lines): Variables used to drive the response #1. (right slanting lines): The replica part of drive system used in response. (blank): Drive variables which are fed directly into the response #1. Response #2. Variables which are fed directly into the response #2 from response #1. (left slanting lines): The replica part of the drive system.

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Table 1: Summary of bifurcation phenomena of Eq. (25) with $\alpha = 0.5$, $\omega_0^2 = -1.0$, $\beta = 1.0$ and $\omega = 1.0$.

Value of f	Nature of solution	Attractor in the phase space in the $(x, \dot{x})$ plane (simulation results) Numerical Fig. 9 analog Fig. 14
$f = 0$	Damped oscillation to the stable focus at $x < 0$	(a)
$0 < f < 0.34$	Period-T oscillation	(b)
$0.34 < f < 0.358$	Period-$2T$ oscillation	(c)
$0.358 < f < 0.362$	Period-$4T$ oscillation	(d)
$0.362 < f < 0.37$	Further period-doubling	
$0.37 < f < 0.4$	One-band chaos	(e)
$0.4 < f = 0.42$	Double-band chaos	(f)
$0.42 < f$	Chaos, windows, reverse period-doublings, chaos, boundary crisis, etc. (See bifurcation diagram of Fig. 10)	(g), (h)

Table 2: Codimension one (one-parameter family of) bifurcations

Type of bifurcation	Model equation	Nature of bifurcation
Flows		
1. Saddle node	(i) $\dot{x} = \mu - x^3$ (ii) $\dot{x} = \mu + x^3$ $x \in \mathcal{R}^1,\ \mu \in \mathcal{R}^1$	Fig. 18a Fig. 18b
2. Transcritical	$\dot{x} = \mu - x^2$	Fig. 18c
3. Pitchfork		
(i) Supercritical	$\dot{x} = \mu x - x^3$	Fig. 18d
(ii) Subcritical	$\dot{x} = \mu x + x^3$	Fig. 18e
3. Hopf		
(i) Supercritical	$\dot{r} = r(\mu - r^2)$ $\dot{\theta} = 1$ $r^2 = x^2 + y^2$	Fig. 18f
(ii) Subcritical	$\dot{r} = r(\mu + r^2)$ $\dot{\theta} = 1$	Fig. 18g
Maps		
4. Flip		
(period doubling)	$x_{n+1} = \mu x_n(1 - x_n)$ $x \in [0, 1),\ 0 \leq \mu \leq 4$	Fig. 18h

Table 3: Codimension two (two-parameter family of) bifurcations

Type of bifurcation	Model equation	Nature of bifurcation
1. Cusp catastrophe	$\dot{x} = \lambda_1 + \lambda_2 x - x^3$ $x \in \mathcal{R}^1,\ (\lambda_1, \lambda_2) \in \mathcal{R}^2$	Fig. 19a
2. Degenerate Hopf	$\dot{r} = r(\lambda_1 + \lambda_2 r^2 + \alpha r^4)$ $\dot{\theta} = \omega + \mathbf{O}(r^2)$	Fig. 19b
3. Takens -Bogdanov	$\dot{x} = y$ $\dot{y} = \lambda_1 x + \lambda_2 y - x^3 - x^2 y$	Fig. 19c