Dynamics of a Completely Integrable N-Coupled Liénard Type Nonlinear Oscillator

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Abstract. We present a system of N-coupled Liénard type nonlinear oscillators which is completely integrable and possesses explicit N time-independent and N time-dependent integrals. In a special case, it becomes maximally superintegrable and admits $(2N-1)$ time-independent integrals. The results are illustrated for the $N=2$ and arbitrary number cases. General explicit periodic (with frequency independent of amplitude) and quasiperiodic solutions as well as decaying type/frontlike solutions are presented, depending on the signs and magnitudes of the system parameters. Though the system is of a nonlinear damped type, our investigations show that it possesses a Hamiltonian structure and that under a contact transformation it is transformable to a system of uncoupled harmonic oscillators.

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1. Introduction

In a recent paper we have shown that the modified Emden type equation (MEE) with additional linear forcing,

$$\ddot{x} + 3kx\dot{x} + k^2x^3 + \lambda x = 0, \tag{1}$$

where over dot denotes differentiation with respect to t and k and λ are arbitrary parameters, exhibits certain unusual nonlinear dynamical properties [1]. Equation (1) is essentially of Liénard type. For a particular sign of the control parameter, namely $\lambda > 0$, the frequency of oscillations of the nonlinear oscillator (1) is completely independent of the amplitude and remains the same as that of the linear harmonic oscillator, thereby showing that the *amplitude dependence of frequency is not necessarily a fundamental property of nonlinear dynamical phenomena.* In this case ($\lambda > 0$) the system admits the explicit sinusoidal periodic solution

$$x(t) = \frac{A\sin(\omega t+\delta)}{1-\left(\frac{k}{\omega}\right)A\cos(\omega t+\delta)}, \quad 0 \le A < \frac{\omega}{k}, \quad \omega = \sqrt{\lambda}, \tag{2}$$

where A and δ are arbitrary constants. In figure 1a we depict the harmonic periodic

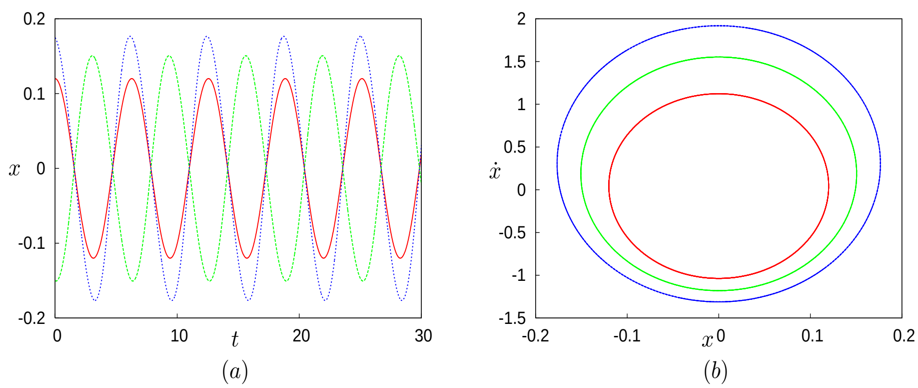

Figure 1. (Color online) Solution and phase space plots of equation (1) for the case $\lambda < 0$ (a) Periodic oscillations (b) Phase space portrait

oscillations of the MEE (1) for three different initial conditions, showing the amplitude independence of the period or frequency. The phase space plot, figure 1b, resembles that of the harmonic oscillator which again confirms that the system has a unique period of oscillations for $\lambda > 0$.

For $\lambda < 0$, equation (1) admits the following form of solution [1],

$$x(t) = \left(\frac{\sqrt{|\lambda|}(I_1 e^{2\sqrt{|\lambda|}t} - 1)}{kI_1I_2e^{\sqrt{|\lambda|}t} + k(1 + I_1e^{2\sqrt{|\lambda|}t})}\right), \tag{3}$$

where I_1 and I_2 are constants. Depending on the initial condition, the solution (3) is either of decaying type or of aperiodic frontlike type, see figure 2a. The time of decay or approach to asymptotic value is independent of the amplitude/initial value, which is

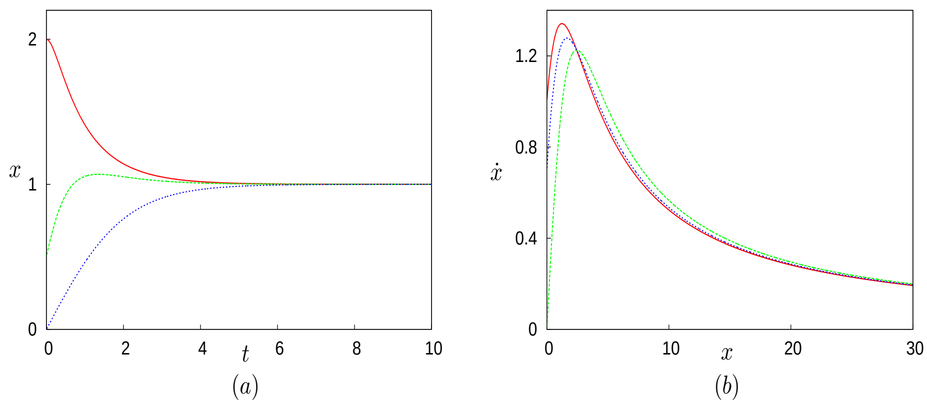

Figure 2. (Color online) (a) Decaying and frontlike solutions of (1) for the parametric choice $\lambda < 0$, (b) Solution plot of (1) with $\lambda = 0$

once again an unusual feature for a nonlinear dynamical system [1]. Finally, for $\lambda = 0$ equation (1) is nothing but the MEE [3] which has the exact general solution [4]

$$x(t) = \frac{t + I_1}{2kt^2 + I_1 kt + I_2}, \tag{4}$$

where I_1 and I_2 are the two integrals of motion (figure 2b).

A natural question which now arises is whether there exist higher dimensional coupled analogues of (1) which are integrable and exhibit interesting oscillatory properties. In this paper we first report a system of two-coupled MEEs (with additional linear forcing) which is completely integrable, possesses two time-independent integrals and two time-dependent integrals and whose general solution can be obtained explicitly. Depending on the signs of the linear term, the system admits periodic (with amplitude independent frequency) or quasiperiodic solutions or bounded aperiodic solutions (decaying or frontlike). The results are then extended to N-coupled MEEs and we prove the complete integrability of them also in the same way. From the nature of the explicit solutions we identify a suitable contact transformation which maps the coupled system onto a system of coupled canonical equations corresponding to a system of N-uncoupled harmonic oscillators, thereby proving the Hamiltonian nature of the N-coupled MEEs. We also prove that the system becomes a maximally superintegrable one [5] for any value of $N(> 1)$ when the coefficients of the linear force terms are equal.

We organize our results as follows. In Section 2 we propose a two-coupled version of the MEE (1) and construct the integrals of motion through the recently proposed modified Prelle-Singer procedure (a brief description of this procedure is given in Appendix A). In Section 3, by appropriately choosing the coefficients of the linear forcing term, we construct different forms of general solutions including periodic, quasiperiodic and frontlike solutions and discuss the dynamics in each of the cases in some detail. In Section 4, we briefly analyse the symmetry and Painlevé singularity structure properties of the proposed two-coupled version of the MEE. We also investigate the dynamics of this equation under perturbation by numerical analysis. We consider a N- coupled generalization of equation (1) and discuss the general solution/dynamics in Section 5. In Section 6, we identify a Hamiltonian structure for the N-coupled MEE by mapping it onto a system of N-uncoupled harmonic oscillators through a contact type transformation which is obtained from the general solution of the coupled MEE. Finally in Section 7, we summarize our results. In Appendix A, we briefly discuss the generalized modified Prelle-Singer procedure which has been used to derive the results discussed in Section 2.

2. Two-dimensional generalizations of the MEE

It is of considerable interest to study the dynamics of higher dimensional versions of the MEE (1). Recently Cariñena and Ranada studied the uncoupled two-dimensional version of equation (1) [6],

$$\begin{aligned} &\ddot{x} + 3k_1 x\dot{x} + k_1^2 x^3 + \lambda_1 x = 0, \\ &\ddot{y} + 3k_2 y\dot{y} + k_2^2 y^3 + \lambda_2 y = 0, \end{aligned} \tag{5}$$

where k_1, k_2, λ_1 and λ_2 are arbitrary parameters, and analyzed the geometrical properties and proved that the above system is superintegrable [6]. On the other hand, Ali et al. [7] have analysed a system of two-coupled differential equations, which is a complex version of (1) with $x = y + iz$ and $\lambda = 0$:

$$\begin{aligned} &\ddot{y} = -3(y\dot{y} - z\dot{z}) - (y^3 - 3yz^2), \\ &\ddot{z} = -3(z\dot{y} + y\dot{z}) - (3y^2 z - z^3). \end{aligned} \tag{6}$$

The above system of equations is shown to be linearizable by complex point transformation and from the solution of the linearized equation, a general solution for the equation (6) has been constructed [7].

In [2] we have pointed out that equation (1), and its generalization as well as the N^{th} order version can be transformed to linear differential equations through appropriate nonlocal transformations. In particular equation (1) under the nonlocal transformation $U = x(t)e^{\int_0^t kx(\tau)d\tau}$ gets transformed to the linear harmonic oscillator equation $\ddot{U} + \lambda U = 0$, where x and U are also related through the Riccati equation $\dot{x} = \frac{\dot{U}}{U}x - kx^2$. Substituting the expressions for U and $\dot{U}$ and solving the resultant Riccati equation one can obtain the solution (2). Now searching for possible extensions to higher dimensions by considering a generalized nonlocal transformation of the form $U = xe^{\int_0^t f(x(\tau),y(\tau))d\tau}$, $V = ye^{\int_0^t g(x(\tau),y(\tau))d\tau}$, where U and V satisfy the uncoupled linear harmonic oscillator equations, $\ddot{U} + \lambda_1 U = 0$ and $\ddot{V} + \lambda_2 V = 0$, we try to identify the forms $f(x, y)$ and $g(x, y)$ so that the transformation can be written as a system of parametrically driven Lotka-Volterra type equations or time-dependent coupled Riccati equations. In particular, with the choice $f = g = k_1 x + k_2 y$, the transformation becomes a set of coupled time dependent Riccati equations, $\dot{x} = \left(\frac{\dot{U}}{U}x - k_1 x^2 - k_2 xy\right)$, $\dot{y} = \left(\frac{\dot{V}}{V}y - k_1 xy - k_2 y^2\right)$. Consequently one obtains a system of two coupled MEEs with additional linear forcing,

$$\ddot{x} = -2(k_1x + k_2y)\dot{x} - (k_1\dot{x} + k_2\dot{y})x - (k_1x + k_2y)^2x - \lambda_1 x \equiv \phi_1,$$
$$\ddot{y} = -2(k_1x + k_2y)\dot{y} - (k_1\dot{x} + k_2\dot{y})y - (k_1x + k_2y)^2y - \lambda_2 y \equiv \phi_2, \qquad (7)$$

where k_i's and λ_i's, $i = 1, 2$, are arbitrary parameters. When either one of the parameters k_1 or k_2 is taken as zero, then one of the two equations in (7) reduces to the MEE defined by (1) while the other reduces to a linear ordinary differential equation (ODE) in the other variable or vice versa. On the other hand when one of the variables (x or y) is zero, equation (7) reduces to a MEE in the other variable. A characteristic feature of this form (7) is that it can be straightforwardly extended to higher dimensions as we see in the following sections besides admitting unusual nonlinear dynamical properties.

To obtain the solutions of the above system of nonlinear ODEs one can solve the above coupled Riccati equations. However, to obtain the integrals of motion as well as the solutions we find it more convenient to solve (7) by the generalized modified Prelle-Singer (PS) procedure introduced recently [12]. We indicate this procedure applicable to (7) briefly in Appendix A. The resultant independent integrals of motion can be written as

$$I_1 = \frac{(\dot{x} + (k_1x + k_2y)x)^2 + \lambda_1 x^2}{\left[\frac{k_1}{\lambda_1}(\dot{x} + (k_1x + k_2y)x) + \frac{k_2}{\lambda_2}(\dot{y} + (k_1x + k_2y)y) + 1\right]^2}, \qquad (8)$$

$$I_2 = \frac{(\dot{y} + (k_1x + k_2y)y)^2 + \lambda_2 y^2}{\left[\frac{k_1}{\lambda_1}(\dot{x} + (k_1x + k_2y)x) + \frac{k_2}{\lambda_2}(\dot{y} + (k_1x + k_2y)y) + 1\right]^2}, \qquad (9)$$

$$I_3 = \begin{cases} \tan^{-1}\left[\frac{\sqrt{\lambda_1}x}{\dot{x}+(k_1x+k_2y)x}\right] - \sqrt{\lambda_1}\,t, & \lambda_1 > 0 \\ \frac{e^{2\sqrt{|\lambda_1|}t}(\dot{x}+(k_1x+k_2y)x-\sqrt{|\lambda_1|}x)}{\dot{x}+(k_1x+k_2y)x+\sqrt{|\lambda_1|}x}, & \lambda_1 < 0, \end{cases} \qquad (10)$$

$$I_4 = \begin{cases} \tan^{-1}\left[\frac{\sqrt{\lambda_2}y}{\dot{y}+(k_1x+k_2y)y}\right] - \sqrt{\lambda_2}\,t, & \lambda_2 > 0 \\ \frac{e^{2\sqrt{|\lambda_2|}t}(\dot{y}+(k_1x+k_2y)x-\sqrt{|\lambda_2|}y)}{\dot{y}+(k_1x+k_2y)y+\sqrt{|\lambda_2|}y}, & \lambda_2 < 0. \end{cases} \qquad (11)$$

We note here that the forms of the time-dependent integrals of motion depend upon the signs of the parameters λ_i, $i = 1, 2$. Using the above integrals one can obtain periodic and aperiodic but bounded solutions as per (i) λ_1, $\lambda_2 > 0$ (ii) $\lambda_1, \lambda_2 < 0$ and (iii) $\lambda_1 < 0$, $\lambda_2 > 0$ (or vice versa). The case $\lambda_1 = \lambda_2 = 0$ is dealt with separately below in Section 3.3. In the following we discuss the nature of the solutions.

3. The Dynamics

3.1. Periodic and quasi periodic oscillations ($\lambda_1, \lambda_2 > 0$)

By restricting λ_1, $\lambda_2 > 0$ in the integrals (8)-(11) and solving them algebraically we obtain the following general solution,

$$x(t) = \frac{A\sin(\omega_1 t + \delta_1)}{1 - \frac{Ak_1}{\omega_1}\cos(\omega_1 t + \delta_1) - \frac{Bk_2}{\omega_2}\cos(\omega_2 t + \delta_2)},$$

$$y(t) = \frac{B\sin(\omega_2 t + \delta_2)}{1 - \frac{Ak_1}{\omega_1}\cos(\omega_1 t + \delta_1) - \frac{Bk_2}{\omega_2}\cos(\omega_2 t + \delta_2)}, \qquad \Big|\frac{Ak_1}{\omega_1} + \frac{Bk_2}{\omega_2}\Big| < 1, \qquad (12)$$

where $\omega_j = \sqrt{\lambda_j},\ j = 1, 2,\ A = \sqrt{I_1}/\omega_1,\ B = \sqrt{I_2}/\omega_2,\ \delta_1 = I_3,\ \delta_2 = I_4$. Two types of oscillatory motion can arise depending on whether the ratio ω_1/ω_2 is rational or irrational leading to $m : n$ periodic or quasiperiodic motion, respectively. One may note that the frequency of oscillations is again independent of the amplitude in the present two-coupled generalization also. The conservative nature of the above oscillatory solution can be seen from the phase space plot.

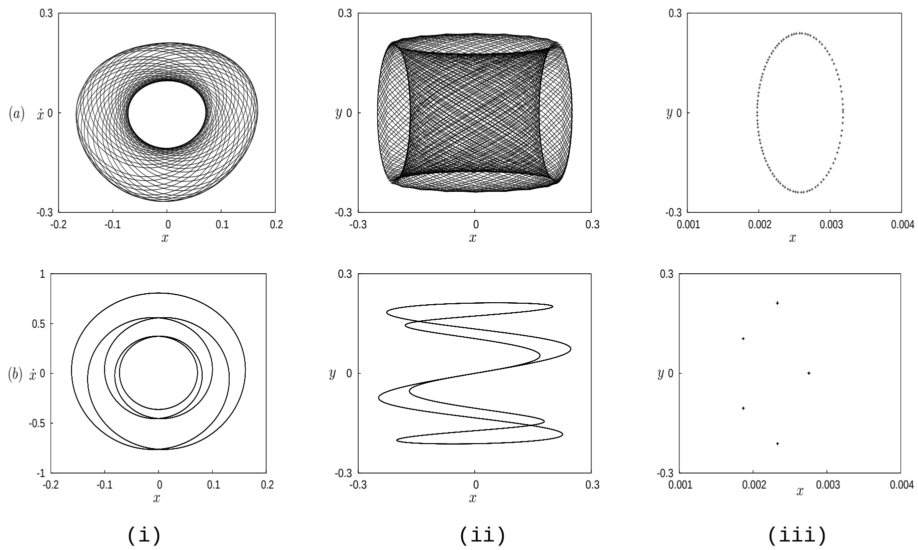

Figure 3. (a) Quasi-periodic oscillations with $\omega_1 = 1$ and $\omega_2 = \sqrt{2}$ (i) phase space plot (ii) Configuration space plot and (iii) Poincaré SOS (b) periodic oscillations with $\omega_1 = 1$ and $\omega_2 = 5$ (i) phase space plot (ii) Configuration space plot and (iii) Poincaré SOS, see equation (12).

To visualize the dynamics we plot the solutions $x(t)$ and $y(t)$ both in phase space and in configuration space for two different sets of frequencies. First we consider the case in which the ratio of frequencies is an irrational number. For illustration, we take $\omega_1 = \omega_2/\sqrt{2} = 1$. The $(x, \dot{x})$ phase space plot is depicted in figure 3a(i). The configuration space (x, y) plot is given in figure 3a(ii). We also confirm the quasi-periodic nature of the oscillations by plotting the Poincaré surface of section (SOS) [13] in figure 3a(iii). The solution is clearly not a closed orbit and is quasiperiodic or almost periodic in the sense that trajectory returns arbitrarily close to its starting point infinitely often, which is confirmed by a closed curve in the Poincaré SOS. The system parameters are chosen as $k_1 = -1, k_2 = 1$ and the values of the arbitrary constants are taken as $A = -1$ and $B = -0.4$.

Next, when the ratio of frequencies is a rational number one has periodic solutions. In figure 3b we show a 1:5 periodic solution by choosing $\omega_1 = 1$ and $\omega_2 = 5$.

3.2. Aperiodic solutions $(\lambda_1, \lambda_2 < 0)$

We next consider the case $\lambda_1 < 0$, $\lambda_2 < 0$. In this case one obtains aperiodic but bounded frontlike or decaying type solutions. To see this, we use the two time-independent integrals (I_1, I_2) and time-dependent integrals (I_3, I_4) vide equations (8),(9),(10) and (11), and obtain the following general solution for equation (7) when $\lambda_1, \lambda_2 < 0$,

$$x(t) = \frac{\lambda_2\sqrt{|\lambda_1|}(\mathfrak{I}_1 e^{\sqrt{|\lambda_1|}t} - \mathfrak{I}_3 e^{-\sqrt{|\lambda_1|}t})}{k_1\lambda_2(\mathfrak{I}_1 e^{\sqrt{|\lambda_1|}t} + \mathfrak{I}_3 e^{-\sqrt{|\lambda_1|}t}) + k_2\lambda_1(\mathfrak{I}_2 e^{\sqrt{|\lambda_2|}t} + \mathfrak{I}_4 e^{-\sqrt{|\lambda_2|}t}) - 2},$$
$$y(t) = \frac{\lambda_1\sqrt{|\lambda_2|}(\mathfrak{I}_2 e^{\sqrt{|\lambda_2|}t} - \mathfrak{I}_4 e^{-\sqrt{|\lambda_2|}t})}{k_1\lambda_2(\mathfrak{I}_1 e^{\sqrt{|\lambda_1|}t} + \mathfrak{I}_3 e^{-\sqrt{|\lambda_1|}t}) + k_2\lambda_1(\mathfrak{I}_2 e^{\sqrt{|\lambda_2|}t} + \mathfrak{I}_4 e^{-\sqrt{|\lambda_2|}t}) - 2}, \qquad (13)$$

where $\mathfrak{I}_1 = \sqrt{I_1}/(\sqrt{I_3}\lambda_1\lambda_2)$, $\mathfrak{I}_2 = \sqrt{I_2}/(\sqrt{I_4}\lambda_1\lambda_2)$, $\mathfrak{I}_3 = \sqrt{I_1 I_3}/(\lambda_1\lambda_2)$, $\mathfrak{I}_4 = \sqrt{I_2 I_4}/(\lambda_1\lambda_2)$. Here also we observe that the solution is of the same form as in the one dimensional case. The decaying type solution is plotted in figure 4a and the frontlike solution is shown in figure 4b.

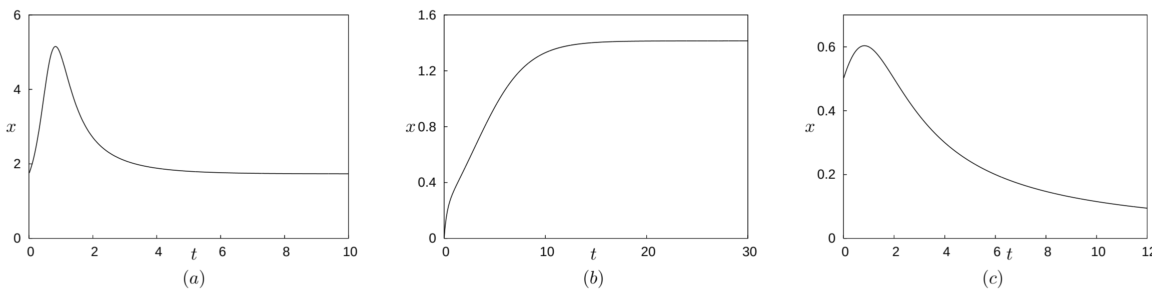

Figure 4. (a) Decaying type solution of (7) for λ_1, $\lambda_2 < 0$ (b) Frontlike solution of (7) for $\lambda_1, \lambda_2 < 0$ (c) Decaying type solution of (7) for $\lambda_1 = \lambda_2 = 0$.

3.3. Decaying type solution for $\lambda_1 = \lambda_2 = 0$

Restricting the values of the parameters to $\lambda_1 = \lambda_2 = 0$ in (7) one can get the two-dimensional generalization of the MEE (1) . In this case we find that the system (7) admits the following integrals of motion, namely,

$$I_1 = \frac{(\dot{x} + (k_1 x + k_2 y)x)}{\dot{y} + (k_1 x + k_2 y)y}, \qquad I_2 = -t + \frac{x}{\dot{x} + (k_1 x + k_2 y)x},$$
$$I_3 = -t + \frac{y}{\dot{y} + (k_1 x + k_2 y)y}, \quad I_4 = \frac{t^2}{2} + \frac{1 - t(k_1 x + k_2 y)}{(k_1\dot{x} + k_2\dot{y} + (k_1 x + k_2 y)^2)^2}. \qquad (14)$$

Then the general solution for equation (7) (with $\lambda_1 = \lambda_2 = 0$) can be written in the form

$$x(t) = \frac{2I_1(I_2 + t)}{k_1 I_1(2I_4 + (2I_2 + t)t) + k_2(2I_4 + (2I_3 + t)t)},$$
$$y(t) = \frac{2(I_3 + t)}{k_1 I_1(2I_4 + (2I_2 + t)t) + k_2(2I_4 + (2I_3 + t)t)}. \qquad (15)$$

Again the solution turns out to be a rational function in t in which the denominator is a quadratic function of t as in the one dimensional case. Choosing I_1, I_2, I_3 and I_4 suitably, one can obtain decaying type solutions as shown in figure 4c.

3.4. Cases with mixed signs of parameters λ_1 and λ_2

In the mixed case, for example with $\lambda_1 > 0$, $\lambda_2 < 0$, the solution can be derived from (13) as

$$x(t) = \frac{-\omega_1 A\sin(\omega_1 t+\delta_1)}{\frac{k_2\omega_1^2}{2}(\mathfrak{I}_2 e^{\sqrt{|\lambda_2|}t} + \mathfrak{I}_4 e^{-\sqrt{|\lambda_2|}t}) + Ak_1\cos(\omega_1 t+\delta_1) - 1}, \quad \omega_1 = \sqrt{\lambda_1},$$
$$y(t) = \frac{\omega_1^2\sqrt{|\lambda_2|}(\mathfrak{I}_2 e^{\sqrt{|\lambda_2|}t} - \mathfrak{I}_4 e^{-\sqrt{|\lambda_2|}t})}{2[\frac{k_2\omega_1^2}{2}(\mathfrak{I}_2 e^{\sqrt{|\lambda_2|}t} + \mathfrak{I}_4 e^{-\sqrt{|\lambda_2|}t}) + Ak_1\cos(\omega_1 t+\delta_1) - 1]}, \tag{16}$$

where $A = \sqrt{I_1}/\omega_1^2$.

The motion now turns out to be a mixed oscillatory-bounded frontlike one. We depict the solution (16) in figure 5 in which we have fixed $k_1 = k_2 = 1$, $\omega_1 = 2$, and $\lambda_2 = -1$.

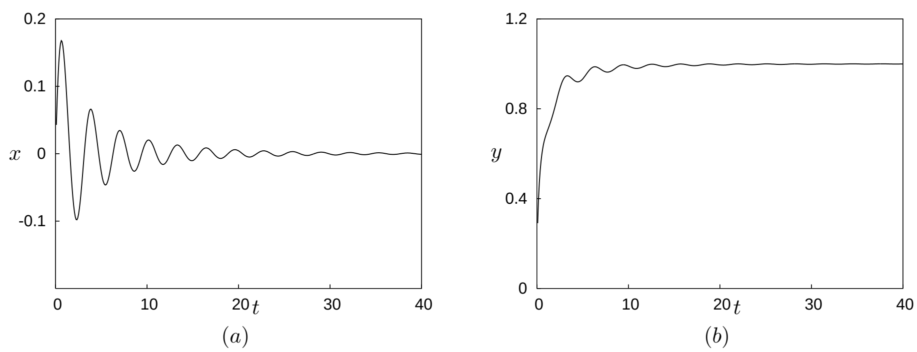

Figure 5. Solution plots of equation (7) for the mixed case $\lambda_1 > 0$, $\lambda_2 < 0$ (a) $x(t)$, (b) $y(t)$

3.5. A superintegrable case

Finally we investigate the dynamics in the limit $\lambda_1 = \lambda_2 = \lambda \neq 0$. In this case one is able to find an additional time-independent integral of motion given by

$$I_5 = \frac{x\dot{y} - y\dot{x}}{(k_1\dot{x} + k_2\dot{y} + (k_1 x + k_2 y)^2 + \lambda)^2}. \tag{17}$$

As a result one has three time-independent integrals, I_1, I_2, and I_5, besides one time-dependent integral (any one of the two time-dependent integrals I_3 or I_4), in a two degrees of freedom system which in turn confirms that the system under consideration is a superintegrable one. Obviously it is also maximally superintegrable [5].

4. Symmetry and Singularity structure analysis

In order to understand why the system (7) is integrable and whether perturbations of them lead to chaos, we analyse the symmetry properties, in particular the point symmetries and Painlevé singularity structure associated with equation (7) and investigate numerically the perturbed equation.

4.1. Lie point symmetry analysis

Considering the invariance of equation (7) under a one parameter continuous Lie point symmetry group [8], $x \to X = x + \epsilon\eta_1(t,x,y) + O(\epsilon^2)$, $y \to Y = y + \epsilon\eta_2(t,x,y) + O(\epsilon^2)$, $t \to T = x + \epsilon\xi(t,x,y) + O(\epsilon^2)$, $\epsilon << 1$, and performing the Lie point symmetry analysis on the equation (7) using the package MULIE [9], we find that the system admits only the following single Lie point symmetry vector,

$$\Gamma_1 = \frac{1}{2\lambda_1}\frac{\partial}{\partial t}, \tag{18}$$

corresponding to the invariance of (7) under time translation for $\lambda_1 \neq \lambda_2$. For the specific choice $\lambda_1 = \lambda_2$, which we have earlier proved to be superintegrable, we obtain the following two additional point symmetries,

$$\Gamma_2 = -y\frac{k_1}{k_2}\frac{\partial}{\partial y} + y\frac{\partial}{\partial x},$$
$$\Gamma_3 = -x\frac{k_2}{k_1}\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \tag{19}$$

So we conclude at this stage that since the system (7) is completely integrable and does not possess enough number of Lie point symmetries, it has to admit more general symmetries, namely nonlocal and contact symmetries. The study requires separate analysis and we do not pursue them here. Note that on the other hand, the one dimensional MEE (1) admits eight Lie point symmetries and is linearizable through point transformation which is not the case for the coupled MEE (7).

4.2. Painlevé singularity structure analysis

We now perform the standard Painlevé singularity structure analysis [10, 11] to the equation (7). Looking for the leading order behaviour of the Laurent series solution in the neighbourhood of a movable singular point t_0, we substitute $x = a_0\tau^p$ and $y = b_0\tau^q$, $\tau = (t - t_0) \to 0$, and obtain

$$a_0p(p-1)\tau^{p-2} + a_0^3k_1^2\tau^{3p} + 3a_0^2k_1p\tau^{2p-1} + 2a_0b_0k_2p\tau^{p+q-1} + a_0b_0k_2q\tau^{p+q-1}$$
$$+ 2a_0^2b_0k_1k_2\tau^{2p+q} + a_0b_0^2k_2^2\tau^{p+2q} = 0, \tag{20}$$
$$b_0q(q-1)\tau^{q-2} + b_0^3k_2^2\tau^{3q} + a_0b_0k_1p\tau^{p+q-1} + 2a_0b_0k_1q\tau^{p+q-1} + a_0^2b_0k_1^2\tau^{2p+q}$$
$$+ 3b_0^2k_2q\tau^{2q-1} + 2a_0b_0^2k_1k_2\tau^{p+2q} = 0. \tag{21}$$

Comparing the exponents of τ, we find $p = -1$ and $q = -1$. Substituting this and simplifying we get

$$(2a_0 - 3a_0^2k_1 + a_0^3k_1^2 - 3a_0b_0k_2 + 2a_0^2b_0k_1k_2 + a_0b_0^2k_2^2)\tau^{-3} = 0,$$
$$(2b_0 - 3a_0b_0k_1 + a_0^2b_0k_1^2 - 3b_0^2k_2 + 2a_0b_0^2k_1k_2 + b_0^3k_2^2)\tau^{-3} = 0.$$

Solving the above system of equation we find two possibilities for the leading order coefficients as $a_0 = (1 - b_0k_2)/k_1$ and $a_0 = (2 - b_0k_2)/k_1$, while b_0 is arbitrary. For the choice $a_0 = (1 - b_0k_2)/k_1$ we identify that the resonances (that is powers at which arbitrary constants can enter) occur at $r = -1$, $r = 0$, $r = 1$ and $r = 1$. By proceeding with the full Laurent series one can show that in addition to t_0 and b_0 being arbitrary (corresponding to $r = -1$ and $r = 0$), a_1 and b_1 are also arbitrary corresponding to $r = 1, 1$, while all higher order coefficients in the Laurent series can be determined in terms of the earlier ones. Similarly we find that the Laurent series corresponding to the second value of $a_0 = (2 - b_0k_2)/k_1$ also does not admit any movable critical singular point. We find that the equation (7) passes the Painlevé test as expected.

4.3. Perturbed system : Numerical analysis

In order to understand the dynamics of the system in the neighbourhood of the integrable parametric regime, we perturb the system leading to the form

$$\ddot{x} + 2(k_1x + k_2y)\dot{x} + (k_1\dot{x} + k_2\dot{y})x + (k_1x + k_2y)^2x + \lambda_1x + \rho_1xy = 0,$$
$$\ddot{y} + 2(k_1x + k_2y)\dot{y} + (k_1\dot{x} + k_2\dot{y})y + (k_1x + k_2y)^2y + \lambda_2y + \rho_2xy = 0, \quad (22)$$

where ρ_1, ρ_2 are the strength of the perturbation. We now numerically solve equation (22) for three different values of ρ_1 and ρ_2 using the fourth order Runge-Kutta method. Of these three parametric choices, the first one ($\rho_1 = \rho_2 = 0$) corresponds to the completely integrable equation (7) and the other two are the perturbed form of (7), namely, $\rho_1 = \rho_2 = 0.5$ and $\rho_1 = \rho_2 = 0.7$. In figures 6a we show the phase space plots of equation (22) for these three different parametric choices. The corresponding Poincaré surface of sections are plotted in figures 6b by identifying the peaks of $y(t)$ from the time series data. We find from the Poincaré surface of section of the torus that for the integrable parametric choice a closed curve is obtained while as the strength of the perturbation is increased the curve starts to break up and ends up into scattered points, confirming the onset of chaos.

5. N-coupled MEE

Proceeding further, we find that equation (1) can also be generalized to arbitrary number, N, of coupled oscillators. In order to generalize the results of two-coupled MEE (7) to N-coupled MEE, we have first investigated the three-coupled MEEs. Then from results of two and three-coupled MEE, we have generalized the results to N-coupled MEE. For brevity, we are not presenting here the results of the $N = 3$ case.

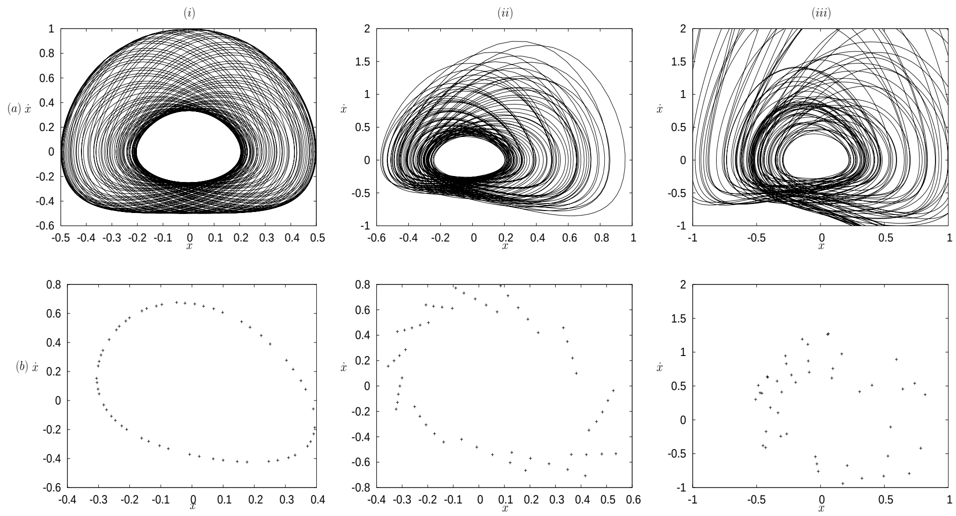

Figure 6. Figures (a) (i), (ii) and (iii) describe phase space trajectories of equation (22) with $\lambda_1 = 1$ and $\lambda_2 = 2$ for three values of ρ_1, ρ_2 corresponding to integrable case (i) $\rho_1 = \rho_2 = 0$ and perturbed cases : (ii) $\rho_1 = \rho_2 = 0.5$ and (iii) $\rho_1 = \rho_2 = 0.7$. Figures (b) (i), (ii) and (iii) describe the Poincaré surface of sections of the above phase space plots.

The N-coupled modified Emden type equation is given as

$$\ddot{x}_i + 2(\sum_{j=1}^{N} k_j x_j)\dot{x}_i + (\sum_{j=1}^{N} k_j \dot{x}_j)x_i + (\sum_{j=1}^{N} k_j x_j)^2 x_i + \lambda_i x_i = 0, \quad i = 1, 2, ..., N. \quad (23)$$

By generalizing the results of two and three-coupled cases one can now inductively construct the integrals of motion for the equation (23) which turn out to be

$$I_{1i} = \frac{(\dot{x}_i + \sum_{j=1}^{N}(k_j x_j)x_i)^2 + \lambda_i x_i^2}{\left[\sum_{j=1}^{N}\left[\frac{k_j}{\lambda_j}(\dot{x}_j + \sum_{n=1}^{N}(k_n x_n)x_j)\right] + 1\right]^2}, \quad (24)$$

$$I_{2i} = \tan^{-1}\left[\frac{\sqrt{\lambda_i}x_i}{\dot{x}_i + \sum_{j=1}^{N}(k_j x_j)x_i}\right] - \sqrt{\lambda_i}t, \;\; \lambda_i > 0, \quad i = 1, 2, ..., N. \quad (25)$$

In other words one has N time-independent integrals and N time-dependent integrals. From these integrals one can derive the general solution of equation (23) for the parametric choice $\lambda_1, \lambda_2, ..., \lambda_N > 0$ as

$$x_i(t) = \frac{A_i \sin(\omega_i t + \delta_i)}{1 - \sum_{j=1}^{N}\frac{A_j k_j}{\omega_j}\cos(\omega_j t + \delta_j)}, \;\; i = 1, ..., N, \;\; \Big|\sum_{j=1}^{N}\frac{A_j k_j}{\omega_j}\Big| < 1, \quad (26)$$

with $\omega_i = \sqrt{\lambda_i}$.

In the second case, $\lambda_i < 0$, $i = 1, 2, ..., N$, to derive the general solution we construct the time-dependent integrals in the form

$$I_{3i} = \frac{e^{2\sqrt{|\lambda_i|}t}(\dot{x}_i + \sum_{j=1}^{N}(k_j x_j)x_i - \sqrt{|\lambda_i|}x_i)}{\dot{x}_i + \sum_{j=1}^{N}(k_j x_j)x_i + \sqrt{|\lambda_i|}x_i}, \;\; i = 1, 2, ..., N. \quad (27)$$

Using the above integrals one can construct the general solution of (23) in the form

$$x_i(t) = \frac{\prod_{s=1}^{N} \lambda_s \sqrt{|\lambda_i|}(\mathfrak{I}_{1i} e^{\sqrt{|\lambda_i|}t} - \mathfrak{I}_{3i} e^{-\sqrt{|\lambda_i|}t})}{\sum_{j=1}^{N} \prod_{m=1}^{N} \lambda_m k_j (\mathfrak{I}_{1j} e^{\sqrt{|\lambda_j|}t} + \mathfrak{I}_{3j} e^{-\sqrt{|\lambda_j|}t}) - 2}, \quad i = 1, 2, ..., N, \tag{28}$$

where $j \neq m$ and $s \neq i$, $\mathfrak{I}_{1i} = \sqrt{I_{1i}}/(\sqrt{I_{3i}} \prod_{n=1}^{N} \lambda_n)$, $\mathfrak{I}_{3i} = \sqrt{I_{1i} I_{3i}}/\prod_{n=1}^{N} \lambda_n$.

In the third case we consider the mixed sign case of the parameters λ_i, $i = 1, 2, ..., N$. Let us consider that λ_r, $r = 1, 2, ..., l$, are positive constants and λ_j, $j = N - l, ..., N$, are negative constants. The solution for this case can be written as

$$\begin{aligned} x_r &= -\omega_r A_r \sin(\omega_r t + \delta_r) h^{-1}, \\ x_j &= \prod_{i=1}^{l} \omega_i^2 \prod_{m=N-l}^{N} \lambda_m \sqrt{|\lambda_j|}(\mathfrak{I}_{1j} e^{\sqrt{|\lambda_j|}t} + \mathfrak{I}_{3j} e^{\sqrt{|\lambda_j|}t})(2h)^{-1}, \end{aligned} \tag{29}$$

where

$$h = \sum_{r=1}^{l} k_r A_r \cos(\omega_r t + \delta_r) + \sum_{m=N-l}^{N} k_m \prod_{r=1}^{l} \omega_r^2 \prod_{q=N-l}^{N} \lambda_q (\mathfrak{I}_{1m} e^{\sqrt{|\lambda_m|}t} + \mathfrak{I}_{3m} e^{-\sqrt{|\lambda_m|}t}) - 1,$$

$m \neq q$ and $m \neq j$.

In the fourth case we have the N-dimensional generalization of "classical MEE". It is evident from the lower dimensional cases that the integrals of motion includes (i) $(N-1)$ time-independent ones (ii) N time-dependent integrals which are linear in 't' and (iii) one time-dependent integral which is quadratic in 't', that is

$$I_{1i} = \frac{\dot{x}_i + x_i \sum_{j=1}^{N} k_j x_j}{\dot{x}_N + x_N \sum_{j=1}^{N} k_j x_j}, \qquad i = 1, 2, \ldots, N-1, \tag{30}$$

$$I_{2i} = -t + \frac{x_i}{\dot{x}_i + x_i \sum_{j=1}^{N} k_j x_j}, \quad i = 1, 2, \ldots, N, \tag{31}$$

$$I_3 = \frac{t^2}{2} + \frac{1 - t\sum_{j=1}^{N} k_j x_j}{(\sum_{j=1}^{N} k_j \dot{x}_j + (k_j x_j)^2)}. \tag{32}$$

From these integrals, again one can deduce the general solution of (23) for $\lambda_i = 0$ in the form

$$x_i(t) = \frac{2 I_{1i}(I_{2i} + t)}{\sum_{j=1}^{N} k_j I_{1j}(2I_3 + (2I_{2j} + t)t)}, \quad i = 1, 2, \ldots, N-1, \tag{33}$$

$$x_N(t) = \frac{2(I_{2N} + t)}{\sum_{j=1}^{N} k_j I_{1j}(2I_3 + (2I_{2j} + t)t)}, \tag{34}$$

where $I_{1N} = 1$.

In the fifth case, one can consider the parameters λ_i's are all equal but nonzero, that is $\lambda_1 = \lambda_2 = ... = \lambda_N = \lambda \neq 0$. In this case, one can construct $(N-1)$ additional time-independent integrals by eliminating the variable 't' in equation (25),

$$I_{3i} = \frac{(x_i \dot{x}_{i+1} - x_{i+1}\dot{x}_i)}{\sum_j (k_j \dot{x}_j + (k_j x_j)^2 + \lambda)}, \quad i = 1, 2, ...N-1. \tag{35}$$

Again the existence of $(2N-1)$ time-independent integrals of motion (vide equations (24) and (35)) confirms that the system under consideration, namely (23) with $\lambda_i = 0$, $i = 1, 2, ..., N$, is a maximally superintegrable one.

6. Connection to uncoupled harmonic oscillators

The solution of the coupled MEE equation (7) given by equation (12) and (13) can be rewritten as

$$x = \frac{U}{1 - \frac{k_1}{\omega_1^2}\dot{U} - \frac{k_2}{\omega_2^2}\dot{V}}, \;\; y = \frac{V}{1 - \frac{k_1}{\omega_1^2}\dot{U} - \frac{k_2}{\omega_2^2}\dot{V}}, \tag{36}$$

where $U = A\sin(\omega_1 t + \delta_1)$ and $V = B\sin(\omega_2 t + \delta_2)$. Here U and V can also be interpreted as the solutions of the following uncoupled harmonic oscillator equations,

$$\ddot{U} + \omega_1^2 U = 0, \qquad \ddot{V} + \omega_2^2 V = 0. \tag{37}$$

Equation (36) gives a transformation connecting equation (7) and the harmonic oscillator equations (37). In order to invert this transformation, we need $\dot{U}$ and $\dot{V}$ for which we differentiate equation (36) once with respect to time and replace $\ddot{U}, \ddot{V}$ with $-\omega_1^2 U$ and $-\omega_2^2 V$ respectively. Thus we get the following equations

$$\dot{x} = \frac{\dot{U}(1 - \frac{k_1\dot{U}}{\omega_1^2} - \frac{k_2\dot{V}}{\omega_2^2}) - U(k_1 U + k_2 V)}{\left(\frac{k_1\dot{U}}{\omega_1^2} + \frac{k_2\dot{V}}{\omega_2^2} - 1\right)^2}, \;\; \dot{y} = \frac{\dot{V}(1 - \frac{k_1\dot{U}}{\omega_1^2} - \frac{k_2\dot{V}}{\omega_2^2}) - V(k_1 U + k_2 V)}{\left(\frac{k_1\dot{U}}{\omega_1^2} + \frac{k_2\dot{V}}{\omega_2^2} - 1\right)^2} \tag{38}$$

Solving the relations (36) and (38) one obtains the inverse transformation of (36) and (38) as

$$U = \frac{x}{(\frac{k_1}{\omega_1^2}(\dot{x} + (k_1 x + k_2 y)x) + \frac{k_2}{\omega_2^2}(\dot{y} + (k_1 x + k_2 y)y) + 1)}, \tag{39}$$

$$V = \frac{y}{(\frac{k_1}{\omega_1^2}(\dot{x} + (k_1 x + k_2 y)x) + \frac{k_2}{\omega_2^2}(\dot{y} + (k_1 x + k_2 y)y) + 1)}, \tag{40}$$

$$\dot{U} = \frac{(\dot{x} + k_1 x^2 + k_2 xy)}{(\frac{k_1}{\omega_1^2}(\dot{x} + (k_1 x + k_2 y)x) + \frac{k_2}{\omega_2^2}(\dot{y} + (k_1 x + k_2 y)y) + 1)}, \tag{41}$$

$$\dot{V} = \frac{(\dot{y} + k_1 xy + k_2 y^2)}{(\frac{k_1}{\omega_1^2}(\dot{x} + (k_1 x + k_2 y)x) + \frac{k_2}{\omega_2^2}(\dot{y} + (k_1 x + k_2 y)y) + 1)}. \tag{42}$$

This is indeed a contact transformation between the old and new variables, which can also be interpreted as a linearizing transformation to equation (7). The form of the Hamiltonian for the system of two uncoupled harmonic oscillators (37) obviously is

$$H = \frac{1}{2}\left[P_1^2 + P_2^2 + \lambda_1 U^2 + \lambda_2 V^2\right], \tag{43}$$

where $P_1 = \dot{U}$, $P_2 = \dot{V}$, $\lambda_1 = \omega_1^2$ and $\lambda_2 = \omega_2^2$. Substituting for P_1, P_2, U, and V from (39)-(42) we get,

$$H = \frac{(\dot{x} + k_1 x^2 + k_2 xy)^2 + (\dot{y} + k_1 xy + k_2 y^2)^2 + \lambda_1 x^2 + \lambda_2 y^2}{2[\frac{k_1}{\lambda_1}(\dot{x} + (k_1 x + k_2 y)x) + \frac{k_2}{\lambda_2}(\dot{y} + (k_1 x + k_2 y)y) + 1]^2}. \tag{44}$$

Here we note that $H = \frac{1}{2}(I_1 + I_2)$, where I_1 and I_2 are the time independent integrals of motion of equation (7) (vide (8) and (9)).

Further, U and V satisfy the canonical equations

$$\dot{U} = \frac{\partial H}{\partial P_1} = P_1, \quad \dot{P}_1 = -\frac{\partial H}{\partial U} = -\lambda_1 U, \tag{45}$$

$$\dot{V} = \frac{\partial H}{\partial P_2} = P_2, \quad \dot{P}_2 = -\frac{\partial H}{\partial V} = -\lambda_2 V. \tag{46}$$

The results obviously confirm the existence of a conservative Hamiltonian for equation (7).

The above results can also be extended to N dimensions by using the contact transformation derivable from the solution (26),

$$U_i = \frac{x_i}{\sum_{j=1}^{N}\left[\frac{k_j}{\lambda_j}(\dot{x}_j + \sum_{n=1}^{N}(k_n x_n)x_j)\right] + 1},$$

$$P_i = \dot{U}_i = \frac{\dot{x}_i + (\sum_{j=1}^{N} k_j x_j)x_i}{\sum_{j=1}^{N}\left[\frac{k_j}{\lambda_j}(\dot{x}_j + \sum_{n=1}^{N}(k_n x_n)x_j)\right] + 1}, \tag{47}$$

$i = 1, ..., N$. Consequently the Hamiltonian for equation (23) can be rewritten as a system of N uncoupled harmonic oscillators specified by

$$H = \frac{1}{2}\sum_{i=1}^{N}\left(P_i^2 + \lambda_i U_i^2\right). \tag{48}$$

In terms of the original coordinates this becomes

$$H = \frac{(\dot{x}_i + (\sum_{j=1}^{N} k_j x_j)x_i)^2 + \lambda_i x_i^2}{2\left[\sum_{j=1}^{N}\left[\frac{k_j}{\lambda_j}(\dot{x}_j + \sum_{n=1}^{N}(k_n x_n)x_j)\right] + 1\right]^2}, \tag{49}$$

which can be associated with the integrals of motion (24) as $H = \frac{1}{2}\sum_{i=1}^{N} I_{1i}$. The canonical equation of motion are given as

$$\dot{U}_i = \frac{\partial H}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial U_i}, \quad i = 1, 2, ..., N, \tag{50}$$

thus confirming the Hamiltonian nature of the system (23). However, we have not yet succeeded to obtain an explicit Lagrangian form, and so canonically conjugate momenta, to re-express (44) or (49) in terms of canonical coordinates. This is being pursued at present.

7. Conclusion

In this paper we have presented a system of completely integrable N-coupled Liénard type (modified Emden type) nonlinear oscillators. The system admits in general N time-independent and N time-dependent integrals whose explicit forms can also be found. For special parametric choices, the system also becomes maximally superintegrable. Using these integrals general solution of periodic, quasiperiodic, frontlike and decaying type or oscillatory type are obtained depending on the signs and magnitudes of the linear forcing terms. We have also pointed out that the system possesses a nonstandard Hamiltonian structure and is transformable to a system of uncoupled harmonic oscillators. Further analysis of the Hamiltonian structure can be expected to yield interesting information on the nonstandard Hamiltonian structure of coupled nonlinear oscillators of dissipative type. It is also of interest to investigate whether there exist other couplings of MEEs and its generalizations which are also integrable : For example, one can show [2] that under the general transformation $U(t,x) = x^n e^{\int_0^t f(x(t'))dt'}$, $f(x(t)) = \beta(t)x^m + \gamma(t)$, the one dimensional generalized MEE, $\ddot{x}+(n-1)\frac{\dot{x}^2}{x}+\frac{\beta^2}{n}x^{2m+1}+b_1(t,x)\dot{x}+b_2(t)x^{m+1}+b_3(t)x=0$, where $b_1(t,x) = \frac{1}{n}(2n\gamma+n\lambda+(m+2n)\beta x^m)$, $b_2(t) = \frac{1}{n}\left(\dot{\beta}+2\gamma\beta+\lambda\beta\right)$, $b_3(t) = \frac{1}{n}(\dot{\gamma}+\gamma^2+\lambda\gamma)$, can be reduced to the linear harmonic oscillator equation. One can expect higher dimensional generalization of such a transformation can give rise to more general higher dimensional integrable equations. These questions are being pursued currently.

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Appendix A. Generalized modified Prelle-Singer procedure

To solve the system of two-coupled second order nonlinear ODEs we apply the generalized modified Prelle-Singer (PS) approach introduced recently [12]. Let the system

$$\ddot{x} = -2(k_1x+k_2y)\dot{x} - (k_1\dot{x}+k_2\dot{y})x - (k_1x+k_2y)^2x - \lambda_1 x \equiv \phi_1,$$
$$\ddot{y} = -2(k_1x+k_2y)\dot{y} - (k_1\dot{x}+k_2\dot{y})y - (k_1x+k_2y)^2y - \lambda_2 y \equiv \phi_2,$$

admits a first integral of the form $I(t,x,y,\dot{x},\dot{y}) = C$ with C constant on the solutions so that the total differential gives

$$dI = I_t dt + I_x dx + I_y dy + I_{\dot{x}} d\dot{x} + I_{\dot{y}} d\dot{y} = 0. \qquad \text{(A.1)}$$

Equation (7) can be rewritten as the equivalent 1-forms

$$\phi_1 dt - d\dot{x} = 0, \qquad \phi_2 dt - d\dot{y} = 0. \qquad \text{(A.2)}$$

Adding null terms $s_1(t,x,y,\dot{x},\dot{y})\dot{x}dt - s_1(t,x,y,\dot{x},\dot{y})dx$ and $s_2(t,x,y,\dot{x},\dot{y})\dot{y}dt - s_2(t,x,y,\dot{x},\dot{y})dy$ with the first equation in (A.2), and $u_1(t,x,y,\dot{x},\dot{y})\dot{x}dt - u_1(t,x,y,\dot{x},\dot{y})dx$ and $u_2(t,x,y,\dot{x},\dot{y})\dot{y}dt - u_2(t,x,y,\dot{x},\dot{y})dy$ with the second equation in (A.2), respectively, we obtain that, on the solutions, the 1-forms

$$(\phi_1 + s_1\dot{x} + s_2\dot{y})dt - s_1 dx - s_2 dy - d\dot{x} = 0, \qquad \text{(A.3)}$$
$$(\phi_2 + u_1\dot{x} + u_2\dot{y})dt - u_1 dx - u_2 dy - d\dot{y} = 0. \qquad \text{(A.4)}$$

Hence, on the solutions, the 1-forms (A.1) and (A.3)-(A.4) must be proportional. Multiplying (A.3) by the function $R(t,x,y,\dot{x},\dot{y})$ and (A.4) by the function $K(t,x,y,\dot{x},\dot{y})$, which act as the integrating factors for (A.3) and (A.4), respectively, we have on the solutions that

$$dI = R(\phi_1 + S\dot{x})dt + K(\phi_2 + U\dot{y})dt - RSdx - KUdy - Rd\dot{x} - Kd\dot{y} = 0, \quad \text{(A.5)}$$

where $S = (Rs_1 + Ku_1)/R$ and $U = (Rs_2 + Ku_2)/K$. Comparing equations (A.5) and (A.1) we have, on the solutions, the relations

$$I_t = R(\phi_1 + S\dot{x}) + K(\phi_2 + U\dot{y}),\ I_x = -RS,\ I_y = -KU,\ I_{\dot{x}} = -R,\ I_{\dot{y}} = -K. \quad \text{(A.6)}$$

The compatibility conditions between the different equations in (A.6) provide us the ten relations

$$\begin{aligned}
D[S] &= -\phi_{1x} - \frac{K}{R}\phi_{2x} + \frac{K}{R}S\phi_{2\dot{x}} + S\phi_{1\dot{x}} + S^2, && \text{(A.7)}\\
D[U] &= -\phi_{2y} - \frac{R}{K}\phi_{1y} + \frac{R}{K}U\phi_{1\dot{y}} + U\phi_{2\dot{y}} + U^2, && \text{(A.8)}\\
D[R] &= -(R\phi_{1\dot{x}} + K\phi_{2\dot{x}} + RS), && \text{(A.9)}\\
D[K] &= -(K\phi_{2\dot{y}} + R\phi_{1\dot{y}} + KU), && \text{(A.10)}\\
SR_y &= -RS_y + UK_x + KU_x, \quad R_x = SR_{\dot{x}} + RS_{\dot{x}}, && \text{(A.11)}\\
R_y &= UK_{\dot{x}} + KU_{\dot{x}}, \quad K_x = SR_{\dot{y}} + RS_{\dot{y}}, && \text{(A.12)}\\
K_y &= UK_{\dot{y}} + KU_{\dot{y}}, \quad R_{\dot{y}} = K_{\dot{x}}. && \text{(A.13)}
\end{aligned}$$

Here the total differential operator, D, is defined by $D = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \phi_1\frac{\partial}{\partial \dot{x}} + \phi_2\frac{\partial}{\partial \dot{y}}$.

Integrating equations (A.6), we obtain the integral of motion,

$$I = r_1 + r_2 + r_3 + r_4 - \int \left[K + \frac{d}{d\dot{y}}\Big(r_1 + r_2 + r_3 + r_4\Big)\right]d\dot{y}, \quad \text{(A.14)}$$

where

$$r_1 = \int \Big(R(\phi_1 + S\dot{x}) + K(\phi_2 + U\dot{y})\Big)dt, \quad r_2 = -\int \Big(RS + \frac{d}{dx}(r_1)\Big)dx,$$
$$r_3 = -\int \Big(KU + \frac{d}{dy}(r_1 + r_2)\Big)dy, \quad r_4 = -\int \left[R + \frac{d}{d\dot{x}}\Big(r_1 + r_2 + r_3\Big)\right]d\dot{x}.$$

Solving equations (A.7)-(A.13) one can obtain S, U, R and K. Substituting these forms into (A.14) and evaluating the resulting integrals one can get the associated integrals of motion. Once sufficient number of integrals of motion are found (four in the present problem) then the general solution can be derived from these integrals by just algebraic manipulations. One can refer to [12] for details of the method of solving the determining equations (A.7)-(A.13).

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