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ON A QUERY OF ADRIAN WADSWORTH*

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Let v_0 be a valuation of a field K_0 with residue field k_0 of characteristic $\neq 2$. Suppose that K is a function field of a conic over K_0 and v is a prolongation of v_0 to K, whose residue field k is not algebraic over k_0. It is well known (cf. [Residue fields of valued function fields of conics, *Proc. Edinb. math. Soc.* **36** (1993)], 469-78]) that either k is a simple transcendental extension of a finite extension of k_0,or k is a regular function field of a conic over k_0. The paper contains the answer to a natural question posed by Wadsworth which states that if v_1, v_2 are any two prolongations of v_0 to K with residue fields k_1, k_2 non-algebraic over k_0 such that neither k_1 or k_2 is a simple transcendental extension of a finite extension of k_0, is it true that k_1 is k_0-isomorphic to k_2.

Key Words : Valued Fields; Valuations and Their Generalizations

0. INTRODUCTION

Let K be a function field of a conic over a field K_0. Let v_0 be a valuation of K_0 with residue field k_0 of characteristic $\neq 2$ and v be an extension of v_0 to K having residue field k which is not an algebraic extension of k_0. In 1992, we proved that either k is a simple transcendental extension of a finite extension of k_0 or it is a regular function field of a conic over k_0 (cf. [2, Theorem 1.1, 1.2]). This led Wadsworth to ask the following natural question.

Let K_0, k, v_0, k_0 be as above and v_1, v_2 be any two prolongations of v_0 to K with residue fields k_1 and k_2 not algebraic over k_0. If neither k_1 nor k_2 is a simple transcendental extention of a finite extension of k_0, does it follow that k_1 is k_0-isomorphic to k_2?

In the present paper, it is shown that the answer to the above question is "yes". Indeed we prove that there exists at most one prolongation v of v_0 to K whose residue field is not a simple transcendental extension of any finite extension of k_0 but it is transcendental over k_0. We have also given a set of necessary and sufficient conditions for the existence of such a prolongation.

1. STATEMENTS OF RESULTS

We first introduce some notation and a definition. Let (K, v) be a valued field. For any ξ in the valuation ring of v, we shall denote by ξ^* its v-residue, i.e., the image of ξ under the canonical homomorphism from the valuation ring of v onto its residue field.

We use "tr". to abbreviate transcendental, "char" to abbreviate characteristic and X, Y for indeterminates.

An extension $(K, v)/(K_0, v_0)$ of valued fields is said to be residually tr. if the residue field of v is a non-algebraic extension of the residue field of v_0.

In what follows, K is a function field of a conic over a field K_0 of char $\neq 2$ i.e. K is the quotient field of a domain $K_0[X, Y]/(f)$, where f is an irreducible polynomial of degree 2 in $K_0[X, Y]$. As in [2, Lemma 2.1], it can be easily shown that there exist explicitly constructible elements $c, d \in K_0$ such that $K = K_0(x, y)$ where (x, y) satisfies the K_0-irreducible polynomial $X^2 - cY^2 - d$, which is referred to as a defining polynomial for K/K_0. Observe that if c_1, d_1 in K_0 are suchthat $c = c_1 a^2$, $d = d_1 b^2$ for some $a, b \in K_0^\times$, then $X^2 - c_1 Y^2 - d_1$ is also a defining polynomial for K/K_0, for we can write $K = K_0(x_1, y_1)$, where $x_1 = x/b$, $y_1 = ay/b$ satisfy $x_1^2 = c_1 y_1^2 + d_1$.

We shall prove :

***Theorem* 1.1** — *Let v_0 be a valuation of a field K_0 with residue field k_0 of characteristic $\neq 2$ and value group G_0. Let $K = K_0(x, y)$ be a function field of a conic over K_0 with (x, y) satisfying $x^2 - cy^2 - d = 0$ for some c, d in K_0. There exists a residually transcendental extension v of v_0 to K whose residue field isnot a simple transcendental extension of a finite extension of k_0, if and only if, the following three conditions are satisfied :*

(i) $cd \neq 0$.

(ii) $v_0(c) \in 2G_0$, $v_0(d) \in 2G_0$.

(iii) *The equation $x^2 = c_1^* Y^2 + d_1^*$ has no solution over k_0, where $c_1, d_1 \in K_0$ are chosen such that c/c_1, d/d_1 are in K_0^2 and $v_0(c_1) = v_0(d_1) = 0$.*

Theorem* 1.2** — ***Let K, K_0, v_0, G_0, k_0 be as in the above theorem. Then there exists at most one residually transcendental extension v of v_0 to K, whose residue field k is not a simple transcendental extension of a finite extension of k_0. If such a prolongation v exists, then k can be *expressed as $k_0(t, u)$, with t transcendental over k_0 andt, u satisfying a relation $t^2 = c_1^* u^2 + d_1^*$, where c_1, d_1 belonging respectively to cK_0^2 and dK_0^2 satisfy $v_0(c_1) = v_0(d_1) = 0$.*

2. Proof of Theorems 1.1, 1.2

We retain the notation introduced in the first section. As in [2, §2 Notation] for a residually tr. extension v of v_0 to $K = K_0(x, y)$, we shall denote by v' its restriction to the simple tr. extension $K_0(y)$ of K_0 and by $E = E(v'/v_0)$ the number defined by $E = \min \{[K_0(y) : K_0(\xi)] \mid \xi \in K_0(y), v'(\xi) \geq 0, \xi^*$ is tr. over $k_0\}$.

It can be easily seen that when $E = 1$, then there exist $a, b \in K_0$ such that the v'-residue of $z = (y - a)/b$ is tr. over k_0; in this case for any $f(y) = \sum_{i \geq 0} c_i z^i$ in $K_0[y], c_i \in K_0$, we have

$$v'(f(y)) = \min_i v_0(c_i); \qquad \dots (1)$$

and the residue field of v' is the simple tr. extension $k_0(z^*)$ of k_0.

For proving the theorems, assume first that there exists a residually tr. prolongation v of v_0 to K having value group G whose residue field is not a simple tr. extension of a finite extension of k_0. As remarked in the proof of [2, Theorem 1.2], this assumption implies that $E(v'/v_0) = 1$, k_0 is algebraically closed in k and that $G = G_0$. Let $a, b \in K_0$ be such that the v'-residue z^* of $z = (y - a)/b$ is tr. over k_0. Observe that $x \neq 0$, for otherwise K would be the simple tr. extension $K_0(y)$ of K_0 andthe residue field of v would be $k_0(z^*)$, contrary to the assumption. Choose $0 \neq R \in K_0$ such that $v(x) = v_0(R)$ and set

$$\eta = (cy^2 + d)/R^2 = x^2/R^2.$$

It has been shown in the last paragraph of the proof of [2, Theorem 1.1] that in the present situation the residue field k of v is $k_0(z^* \sqrt{\eta^*})$. On expressing $cy^2 + d$ as a polynomial in $z = (y - a)/b$, we see that

$$\eta = \left(\frac{b^2 c}{R^2}\right) z^2 + \frac{2abc}{R^2} z + \frac{a^2 c + d}{R^2}. \qquad \dots (2)$$

In view of formula (1),

$$0 = v(\eta) = \min \left\{ v_0\left(\frac{b^2 c}{R^2}\right), v_0\left(\frac{2abc}{R^2}\right), v_0\left(\frac{a^2 c + d}{R^2}\right) \right\} \qquad \dots (3)$$

Therefore, (2) shows that η^* is a polynomial of degree ≤ 2 in z^* over k_0. The assumption that k is not a simple tr. extension of k_0 together with the fact that k_0 is algebraically closed in k implies that η^* must be a polynomial of degree 2 in z^* over k_0. So by virtue of (2),

$$v_0(b^2 c/R^2) = 0. \qquad \dots (4)$$

It is immediate from (3) and (4) that

$$v_0(2abc) \geq v_0(R^2) = v_0(b^2c)$$

Keeping in view that $v_0(2) = 0$, i.e., that char $k_0 \neq 2$, the above inequality gives $v_0(a) \geq v_0(b)$. Thus we see that $\left(\frac{y}{b}\right)^* = \left(\frac{y-a}{b}\right)^* + \left(\frac{a}{b}\right)^*$ is also tr. over k_0. On writing

$$\eta = \frac{b^2c}{R^2}\left(\frac{y}{b}\right)^2 + \frac{d}{R^2},$$

using formula (1) (with $(y-a)/b)$ replaced by y/b) and arguing as above, we conclude that

$$v_0(b^2c) = v_0(R^2) = v_0(d). \qquad \text{... (5)}$$

It is clear from (5) that $cd \neq 0$ and $v_0(c), v_0(d)$ are in $2G_0$ as desired in Theorem 1.1. Observe that the residue field k of v is given by

$$k = k_0(t, u) \qquad \text{... (6)}$$

where

$$u = \left(\frac{y}{b}\right)^*, \quad t^2 = \eta^* = \left(\frac{b^2c}{R^2}\right)^* u^2 + \left(\frac{d}{R^2}\right)^* \qquad \text{... (7)}$$

As $(b^2c/R^2)^*$ and $(d/R^2)^*$ are non-zero by (5), it follows from [3, Corollary 2.9] that k_0 is algebraically closedin k (which is not a simple tr. extension of k_0) and hence condition (*iii*) of Theorem 1.1 is satisfied by virtue of [3, Theorem 1.4].

Since $(y/b)^*$ is tr. over k_0 and by (5), $v_0(b) = \frac{1}{2} v_0(d/c)$, we see that for any $f(y) = \Sigma a_i y^i$ in $K_0[y]$,

$$v(f(y)) = \min_i \{v_0(a_i) + i v_0(b)\}$$

is independent of the choice of the particular residually tr. prolongation v of v_0 whose residue field is not a simple tr. extension of a finite extension of k_0. Thus v is uniquely determined on $K_0(y)$. Moreover, the restriction v' of v to $K_0(y)$ has only one prolongation to the quadratic extension K of $K_0(y)$, for otherwise in view of the fundamental inequality [1, Chapter 6, §8.3, Theorem 1(*b*)], the residue field of v would be the same as the residue field $k_0((y/b)^*)$ of v'. The proof of Theorem 1.2 is now complete.

To prove the converse part of Theorem 1.1, assume that the conditions (i) - (iii) of Theorem 1.1 are satisfied. Fix elements c_1, d_1 of K_0 such that $v_0(c_1) = v_0(d_1) = 0$ and $c = c_1 a^2, d = d_1 b^2$ for some $a, b \in K_0^{\times}$, this is possible in view of conditions (*i*) and (*ii*). Then $K = K_0(x_1, y_1)$ where $x_1 = x/b$, $y = ya/b$ are related by $x_1^2 = c_1 y_1^2 + d_1$. We shall construct a prolongation v of v_0 to K such that the residue field k of v is $k_0(t, u)$, where $t = x_1^*$ (i.e., the v-residue of x_1) and $u = y_1^*$ satisfy $t^2 = c_1^* u^2 + d_1^*$.

Let v' denote the Gaussian valuation on $K_0(y_1)$ extending v_0 which is given by

$$v'(\Sigma a_i y_1^i) = \min_i (v_0(a_i)), a_i \in K_0.$$

Since $x_1^{*2} = c_1^* y_1^{*2} + d_1^*$ is not a square in the residue field $k_0(y_1^*)$ of v', it follows from the fundamental inequality [1, Chapter 6, §8.3, Theorem 1(*b*)] that v' has a unique prolongation to the quadratic extension $K = K_0(x_1, y_1)$ of $K_0(y_1)$. If v denotes this unique prolongation, then its residue field is clearly $k_0(x_1^*, y_1^*)$ which is not a simple tr. extension of k_0 in view of Theorem 1.4 of [3] and condition (*iii*) of Theorem 1.1.

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