Domain Wall and Periodic Solutions of Coupled Asymmetric Double Well Models

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Abstract:

Coupled asymmetric double well ($a\phi^2 - b\phi^3 + c\phi^4$) one-dimensional potentials arise in the context of first order phase transitions both in condensed matter physics and field theory. Here we provide an exhaustive set of exact periodic solutions of such a coupled asymmetric model in terms of elliptic functions (domain wall arrays) and obtain single domain wall solutions in specific limits. We also calculate the energy and interaction between solitons for various solutions. Both topological (kink-like at $T = T_c$) and nontopological (pulse-like for $T \neq T_c$) domain wall solutions are obtained. We relate some of these solutions to domain walls in hydrogen bonded materials and also in the field theory context. As a byproduct, we also obtain a new one parameter family of kink solutions of the uncoupled asymmetric double well model.

1 Introduction

First order phase transitions are typically modeled by a triple well (ϕ^6) free energy. However, if a third order term becomes symmetry allowed, one need not go to the sixth order term for the transition to be of first order. Instead an asymmetric double well (or ϕ^2-ϕ^3-ϕ^4) free energy is sufficient to drive the transition (with or without an external field [1, 2]). This situation occurs in body-centered cubic (bcc) to face-centered cubic (fcc) reconstructive structural phase transitions in crystals, the ω-phase transition in various elements and alloys [2], isotropic to nematic phase transition in liquid crystals [3] and in the hydrogen chains in hydrogen-bonded materials [4, 5, 6, 7].

Similar potentials arise in the field theory contexts as well with application to domain walls in carbon nanotubes and fullerenes [8]. For triangular or hexagonal symmetry crystals two different order parameters (e.g. strain and shuffle) can couple with each other being described by an asymmetric double well [9, 10]. We consider this situation in detail here and obtain several exact domain wall solutions of a coupled asymmetric double well model. Note that in the uncoupled limit, the well known solutions are the (i) kink solution (when the condition $b_1^2 = 4a_1c_1$ is satisfied, see below) and (ii) the pulse solutions around the false vacuum both above and below T_c. In this paper we generalize the well known kink solution of the uncoupled asymmetric double well model by obtaining a one parameter family of kink solutions of the same model. We also obtain a few other solutions in the uncoupled limit, which to the best of our knowledge are not known in the literature.

Lattice dynamical models of entropy driven first order transitions modeled by asymmetric double well potentials have been studied previously using both molecular dynamics and mean field theory [11, 12]. Exactly solvable asymmetric double well potentials are also known [13]. There are many other physical contexts in which either the quantum mechanical eigenvalue problem or the domain wall solutions of the asymmetric double well potentials have been studied [14, 15, 16, 17, 18, 19, 20]. Recently, we have obtained a large number of exact periodic domain wall solutions for the coupled ϕ^4 [21] and the coupled ϕ^6 [22] models. Here we apply the same procedure to coupled asymmetric double well models.

The paper is organized as follows. In the next section (Sec. II) we present the coupled model and obtain the relevant equations of motion. In Sec. III we obtain several solutions including the kink lattice ("bright-bright" soliton) solution at the transition temperature ($T = T_c$). Section IV deals with the pulse lattice ("dark-dark") solutions both above and below T_c. We also obtain the energy of the soliton solutions and the asymptotic interaction between the solitons. In Sec. V we consider a different variant of the model and obtain a few solutions of the corresponding coupled model. Finally, in Sec. VI we conclude with remarks on physical relevance of these solutions and their comparison with the coupled ϕ^6 and ϕ^4 models.

2 Model

We consider a potential that is asymmetric in the two fields due to a linear-quadratic coupling in addition to a biquadratic coupling. The potential we consider is given by ($c_1 > 0$, $c_2 \geq 0$)

$$V = (a_1\phi^2 - b_1\phi^3 + c_1\phi^4) + (a_2\psi^2 - b_2\psi^3 + c_2\psi^4) + d\phi\psi^2 + e\phi^2\psi^2\,, \tag{1}$$

where $a_{1,2}, b_{1,2}, c_{1,2}, d, e$ are material or system dependent parameters. The static field equations that follow from here are

$$\begin{aligned}\phi_{xx} &= 2a_1\phi - 3b_1\phi^2 + 4c_1\phi^3 + d\psi^2 + 2e\phi\psi^2,\\ \psi_{xx} &= 2a_2\psi - 3b_2\psi^2 + 4c_2\psi^3 + 2d\phi\psi + 2e\phi^2\psi\,.\end{aligned} \tag{2}$$

Observe that as long as $d \neq 0$, the two field equations are asymmetric and hence bright-dark and dark-bright solitons would be distinct.

For the standard uncoupled model ($d = e = 0$) one usually takes $a_1 > 0$ as it then corresponds to a model for first order transition. The sign of b_1 merely decides if the other minimum is to the right or left of $\phi = 0$. Hence without any loss of generality, one usually restricts to $b_1 > 0$ and we will primarily stick to this choice (i.e. $a_1, b_1, c_1 > 0$) in this paper except in Sec. V. In this case it is easy to show that while $\phi = 0$ is the only minimum in case $b_1^2 < (32/9)a_1c_1$, for $(32/9)a_1c_1 < b_1^2 < 4a_1c_1$, $\phi = 0$ is the absolute minimum while $\phi = \phi_c$ is the local minimum, whereas for $b_1^2 > 4a_1c_1$, the opposite is true. Here

$$\phi_c = \frac{[3b_1 + \sqrt{(9b_1^2 - 32a_1c_1)}]}{8c_1}\,, \tag{3}$$

while the local maximum of the potential is always at ϕ_m and is given by

$$\phi_m = \frac{[3b_1 - \sqrt{(9b_1^2 - 32a_1c_1)}]}{8c_1}\,. \tag{4}$$

At $b_1^2 = 4a_1c_1$ we have two degenerate minima at $\phi = 0$ and $\phi = \phi_c$. For the uncoupled model, it is well known that while at $T = T_c$ (i.e. $b_1^2 = 4a_1c_1$), one has a kink solution [1, 2], for $T > T_c$ as well as for $T < T_c$ one has a pulse solution around the local minimum [1, 2], in a way related to the decay of the false vacuum.

It is perhaps not well appreciated that even for $a_1 < 0$, the uncoupled model corresponds to a model for first order transition. In particular, in that case while $\phi = 0$ is always the local maximum, for $b_1 > 0$, ϕ_{c1} is the absolute minimum while ϕ_{c2} is the local minimum, while for $b_1 < 0$, ϕ_{c2} and ϕ_{c1} interchange their roles. Here $\phi_{c1,c2}$ are defined by

$$\phi_{c1} = \frac{[3b_1 + \sqrt{(9b_1^2 + 32a_1c_1)}]}{8c_1}\,, \tag{5}$$

$$\phi_{c2} = \frac{[3b_1 - \sqrt{(9b_1^2 + 32a_1c_1)}]}{8c_1}\,. \tag{6}$$

At $b_1 = 0$ they are degenerate (the usual $\phi^4 - \phi^2$ double well potential).

In this paper, we primarily focus on the case when $a_1 > 0$ and without any loss of generality take $b_1 > 0$. Needless to say that from the consideration of stability, c_1 is always taken to be positive throughout this paper. However, in Sec. V we also write down a few solutions of the coupled model in case $a_1 < 0$.

We now write down the periodic soliton solutions of the coupled continuum asymmetric ϕ^4 models. We will also consider the solutions in the (physically interesting) case of $b_2 = c_2 = 0$. We will see that, corresponding to the uncoupled limit of $b_1^2 = 4a_1c_1$ one has ten periodic soliton solutions in the coupled case which in the hyperbolic limit correspond to three bright-bright, two dark-dark and four dark-bright and bright-dark solutions, while one solution becomes a constant in this limit. In the physically interesting case of $b_2 = c_2 = 0$, only six out of these ten solutions survive. On the other hand, for $T > (<)\ T_c$, we are able to obtain only one solution of this coupled model which, depending on the value of the parameters, is related to the pulse solution around the false vacuum (local minimum), both above and below T_c. It is interesting to note that some of the solutions exist only in the coupled model but not in the uncoupled limit.

For static solutions the energy is given by

$$E = \int \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\psi}{dx}\right)^2 + V(\phi,\psi)\right] dx\,, \tag{7}$$

where the limits of integration are from $-\infty$ to ∞ in the case of hyperbolic solutions (i.e., single solitons) on the full line. On the other hand, in the case of periodic solutions (i.e. soliton lattices), the limits are from $-K(m)$ to $+K(m)$ or from $-2K(m)$ to $+2K(m)$ depending on the period of the corresponding elliptic function. Here $K(m)$ [and $E(m)$ below] denote the complete elliptic integral of the first (and second) kind [23]. Using equations of motion, one can show that for all of our solutions

$$V(\phi,\psi) = \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\psi}{dx}\right)^2\right] + C\,, \tag{8}$$

where the constant C in general varies from solution to solution. Hence the energy $\hat{E} = E - \int C\,dx$ is given by

$$\hat{E} \equiv E - \int C\,dx = \int \left[\left(\frac{d\phi}{dx}\right)^2 + \left(\frac{d\psi}{dx}\right)^2\right] dx. \tag{9}$$

Below we will give explicit expressions for energy in the case of several of the periodic solutions (and hence the corresponding hyperbolic solutions). In each case we also provide an expression for the constant C.

3 Solutions Corresponding to $b_1^2 = 4a_1c_1$ (in the Analogous Uncoupled Case)

In this and the next section we consider the model with $a_1, b_1, c_1 > 0$. We look for the most general periodic solutions in terms of the Jacobi elliptic functions $\mathrm{sn}(x,m)$, $\mathrm{cn}(x,m)$ and $\mathrm{dn}(x,m)$ [23] which in the decoupled limit correspond to $b_1^2 = 4a_1c_1$ (i.e. $T = T_c$).

Solution I

It is not difficult to show that

$$\phi = F + A\mathrm{sn}[D(x+x_0),m]\,, \quad \psi = G + B\mathrm{sn}[D(x+x_0),m]\,, \tag{10}$$

is an exact solution provided the following eight coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 + dG^2 + 2eFG^2 = 0\,, \tag{11}$$

$$2a_1A - 6b_1AF + 12c_1F^2A + 2BdG + 2eAG^2 + 4eFBG = -(1+m)AD^2\,, \tag{12}$$

$$-3b_1A^2 + 12c_1FA^2 + dB^2 + 2eFB^2 + 4eABG = 0\,, \tag{13}$$

$$2c_1A^2 + eB^2 = mD^2\,, \tag{14}$$

$$2a_2G - 3b_2G^2 + 4c_2G^3 + 2dFG + 2eGF^2 = 0\,, \tag{15}$$

$$2a_2B - 6b_2BG + 12c_2G^2B + 2dAG + 2dFB + 4eAFG + 2eBF^2 = -(1+m)BD^2\,, \tag{16}$$

$$-3b_2B^2 + 12c_2GB^2 + 2dAB + 2eGA^2 + 4eABF = 0\,, \tag{17}$$

$$2c_2B^2 + eA^2 = mD^2\,. \tag{18}$$

Here A and B denote the amplitudes of the "kink lattice", F and G are constants, D is an inverse characteristic length and x_0 is the (arbitrary) location of the kink. Five of these equations determine the five unknowns A, B, D, F, G while the other three equations, give three constraints between the eight parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e$. In particular, A and B are given by

$$A^2 = \frac{mD^2(2c_2 - e)}{(4c_1c_2 - e^2)}\,, \quad B^2 = \frac{mD^2(2c_1 - e)}{(4c_1c_2 - e^2)}\,. \tag{19}$$

It may be noted here that in case both $F, G = 0$ then no solution exists so long as $a_1 > 0$. However, a solution exists in case $G = 0, F \neq 0$ or when $F = 0, G \neq 0$, both of which we now discuss one by one.

G=0, F≠0:

In this case A, B are again given by Eq. (19) while F and D are given by

$$F \equiv \phi_m = \frac{[3b_1 - \sqrt{(9b_1^2 - 32a_1c_1)}]}{8c_1}\,,$$
$$D^2 = \frac{\sqrt{(9b_1^2 - 32a_1c_1)}\left[3b_1 - \sqrt{(9b_1^2 - 32a_1c_1)}\right]}{8(1+m)c_1}\,. \tag{20}$$

Further there are three constraints given by

$$\frac{2(d+2eF)}{3b_2} = \sqrt{\frac{(2c_1 - e)}{(2c_2 - e)}}\,, \quad (6b_1c_1 - 4c_1d - 3eb_1)F = 2(2a_2c_1 + 4a_1c_1 - ea_1)\,,$$
$$27b_2^2(b_1 - 4c_1F) = 4(d+2eF)^2\,. \tag{21}$$

It is worth noting that the solution exists only when the value of F corresponds to ϕ_m as given by Eq. (4). From one of the constraint [Eq.(21)] it follows that $b_1 > 4c_1F$, i.e. $b_1^2 > 4a_1c_1$.

Special case of $e^2 = 4c_1c_2$

One can show that the solution (10) exists even in case $e^2 = 4c_1c_2$. It turns out that such a solution exists only if

$$2c_1 = 2c_2 = e\,. \tag{22}$$

In this case, A and B are determined from the relations

$$A^2 + B^2 = \frac{mD^2}{e}\,, \quad 3b_2B = 2(d + 2eF)A\,, \tag{23}$$

while D, F are still given by Eq. (20) and the two constraints are

$$27b_2^2(b_1 - 2eF) = 4(d + 2eF)^2\,, \quad a_1 + a_2 + dF = 0\,. \tag{24}$$

Interesting case of $b_2 = c_2 = 0$

Finally, let us discuss the physically interesting case of $b_2 = c_2 = 0$. In this case, solution (10) with $G = 0, F \neq 0$ exists only if $e > 2c_1$. In particular, in this case Eq. (10) with $G = 0$ is a solution to the field Eq. (2) provided

$$A^2 = \frac{ma_1}{(1+m)e}\,, \quad B^2 = \frac{(e-2c_1)ma_1}{(1+m)e^2}\,, \quad D^2 = \frac{a_1}{(1+m)}\,, \quad F = \sqrt{\frac{a_1}{4c_1}}\,, \tag{25}$$

and further if the following three constraints are satisfied

$$b_1^2 = 4a_1c_1\,, \quad d^2 = 2e(a_1 + 2a_2)\,, \quad b_1d + 2a_1e = 0\,. \tag{26}$$

F=0, G≠0:

Let us now discuss the solution in case $F = 0$ but $G \neq 0$. In this case solution (10) exists only if $d = 0$ and $e < 0$. We therefore write $e = -|e|$ and A, B are now given by

$$A^2 = \frac{mD^2(2c_2 + |e|)}{(4c_1c_2 - e^2)}\,, \quad B^2 = \frac{mD^2(2c_1 + |e|)}{(4c_1c_2 - e^2)}\,. \tag{27}$$

while G and D are given by

$$G \equiv \psi_m = \frac{[3b_2 - \sqrt{(9b_2^2 - 32a_2c_2)}]}{8c_2}\,,$$

$$D^2 = \frac{\sqrt{(9b_2^2 - 32a_2c_2)}\left[3b_2 - \sqrt{(9b_2^2 - 32a_2c_2)}\right]}{8(1+m)c_2}\,. \tag{28}$$

Further there are three constraints given by

$$3b_1 A = -4|e|BG\,, \quad 3B^2(b_2 - 4c_2 G) = -2|e|GA^2,$$
$$3(2c_2 + |e|)b_2 G = 2(2a_1 c_2 + 4a_2 c_2 + |e|a_2)\,. \tag{29}$$

Note that one of the constraint implies that $b_2 \le 4c_2 G$ which in turn implies that $b_2^2 < 4a_2 c_2$.

It is easily checked that *no* solution of the form (10) exists when either $e^2 = 4c_1 c_2$ or if $b2 = c_2 = 0$.

m=1

In the limiting case $G = 0, F \neq 0, m = 1$, we have a bright-bright soliton solution given by

$$\phi = F + A\tanh[D(x + x_0)]\,, \quad \psi = B\tanh[D(x + x_0)]\,, \tag{30}$$

with A, B, F and D given by Eqs. (19) and (20) with $m = 1$ while the three constraints are again given by Eq. (21). A similar solution also exists in case $F = 0, G \neq 0$.

Uncoupled Case

For completeness, it may be worthwhile to write down the solution of the uncoupled field Eq. (2), i.e. when $d = e = 0$. It is easily shown that in this case the solution (10) reduces to [1, 2]

$$\phi = \sqrt{\frac{a_1}{4c_1}}\left(1 + \frac{2m}{(1+m)}\mathrm{sn}\left[\sqrt{\frac{a_1}{(1+m)}}(x + x_0), m\right]\right), \tag{31}$$

and one is at the transition temperature $T = T_c$ since the parameters a_1, b_1, c_1 satisfy $b_1^2 = 4a_1 c_1$. At $m = 1$, this reduces to the well known kink solution [1, 2]

$$\phi = \sqrt{\frac{a_1}{4c_1}}\left[1 + \tanh(\sqrt{\frac{a_1}{2}}[x + x_0])\right]. \tag{32}$$

Energy: Corresponding to the periodic solution (10) with $G = 0$ the energy is same irrespective of whether $F = 0, G \neq 0$ or $G = 0, F \neq 0$ or if both of them are zero. Only the value of C is different for the different cases. For example, for $G = 0, F \neq 0$ the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = \frac{2(A^2 + B^2)D}{3m}[(1+m)E(m) - (1-m)K(m)]\,,$$
$$C = F^2[a_1 - b_1 F + c_1 F^2] - \frac{1}{2}(A^2 + B^2)D^2\,. \tag{33}$$

On the other hand, in case $F = 0, G \neq 0$ then C is given by

$$C = G^2[a_2 - b_2 G + c_2 G^2] - \frac{1}{2}(A^2 + B^2)D^2\,. \tag{34}$$

On using the expansion formulas for $E(m)$ and $K(m)$ around $m = 1$ as given in [23]

$$K(m) = \ln\left(\frac{4}{\sqrt{1-m}}\right) + \frac{(1-m)}{4}\left[\ln\left(\frac{4}{\sqrt{1-m}}\right) - 1\right] + \dots\,, \tag{35}$$

$$E(m) = 1 + \frac{(1-m)}{2}\left[\ln\left(\frac{4}{\sqrt{1-m}}\right) - \frac{1}{2}\right] + \dots\,, \tag{36}$$

for m near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-bright) soliton solution [Eq. (30)] plus the interaction energy. We find

$$\hat{E} = E_{kink} + E_{int} = (A^2 + B^2)D\left[\frac{4}{3} + \frac{(1-m)}{3}\right]. \tag{37}$$

The interaction energy vanishes at exactly $m = 1$, as it should.

Solution II

A different type of solution (i.e. a pulse lattice) given by

$$\phi = F + A\mathrm{cn}[D(x + x_0), m]\,, \quad \psi = G + B\mathrm{cn}[D(x + x_0), m]\,, \tag{38}$$

is an exact solution provided the following eight coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 + dG^2 + 2eFG^2 = 0\,, \tag{39}$$

$$2a_1A - 6b_1AF + 12c_1F^2A + 2BdG + 2eAG^2 + 4eFBG = (2m-1)AD^2\,, \tag{40}$$

$$-3b_1A^2 + 12c_1FA^2 + dB^2 + 2eFB^2 + 4eABG = 0\,, \tag{41}$$

$$2c_1A^2 + eB^2 = -mD^2\,, \tag{42}$$

$$2a_2G - 3b_2G^2 + 4c_2G^3 + 2dFG + 2eGF^2 = 0\,, \tag{43}$$

$$2a_2B - 6b_2BG + 12c_2G^2B + 2dAG + 2dFB + 4eAFG + 2eBF^2 = (2m-1)BD^2\,, \tag{44}$$

$$-3b_2B^2 + 12c_2GB^2 + 2dAB + 2eGA^2 + 4eABF = 0\,, \tag{45}$$

$$2c_2B^2 + eA^2 = -mD^2\,. \tag{46}$$

Notice that two of these equations are meaningful only if $e < 0$ since from stability considerations, $c_1 > 0, c_2 \geq 0$. Hence, we write $e = -|e|$.

Five of these equations determine the five unknowns A, B, D, F, G while the other three equations, give three constraints between the eight parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e$. In particular, A and B are given by

$$A^2 = \frac{mD^2(2c_2 + |e|)}{(e^2 - 4c_1c_2)}\,, \quad B^2 = \frac{mD^2(2c_1 + |e|)}{(e^2 - 4c_1c_2)}\,. \tag{47}$$

Unlike the previous case, it turns out that in this case a solution exists both when $F = G = 0$, $G = 0$ or when $F \neq 0$ or if $F = 0, G \neq 0$ and we discuss all three cases one by one.

F=G=0

In this case A, B are again given by Eq. (47) while D, the characteristic inverse length, is given by

$$D^2 = \frac{2a_1}{(2m - 1)}\,, \tag{48}$$

while the three constraints are

$$a_1 = a_2\,, \quad 4d^3 = 27b_1b_2^2\,, \quad \frac{3b_1}{d} = \sqrt{\frac{(2c_1 + |e|)}{(2c_2 + |e|)}}\,. \tag{49}$$

Since $a_1 > 0$, such a solution exists only if $m > 1/2$. It is easily checked that such a solution (with $F = G = 0$) does not exist in case either $b_2 = c_2 = 0$ or if $e^2 = 4c_1c_2$.

What happens if $m = 1/2$? It is easily checked that in that case (38) with $F = G = 0$ is still a solution provided $a_1 = a_2 = 0$, D is undetermined, A, B are given by Eq. (47) and the two remaining constraints are given by Eq. (49).

G=0, F≠0:

In this case (38) is a solution with A, B again given by Eq. (47) while F and D are given by

$$F \equiv \phi_c = \frac{[3b_1 + \sqrt{(9b_1^2 - 32a_1c_1)}]}{8c_1}\,,$$
$$D^2 = \frac{\sqrt{(9b_1^2 - 32a_1c_1)}\,[3b_1 + \sqrt{(9b_1^2 - 32a_1c_1)}]}{8(2m - 1)c_1}\,. \tag{50}$$

Further there are three constraints given by

$$\frac{(2|e|F - d)}{3(4c_1F - b_1)} = \frac{(2c_1 + |e|)}{(2c_2 + |e|)}\,, \quad (6b_1c_1 - 4c_1d + 3|e|b_1)F = 2(2a_2c_1 + 4a_1c_1 + |e|a_1)\,,$$
$$27b_2^2(4c_1F - b_1) = 4(2|e|F - d)^2\,. \tag{51}$$

What happens if $m = 1/2$? It is easily checked that in that case (38) with $G = 0$ is still a solution provided A, B are given by Eq. (47), D is undetermined while the constraints are

$$3b_1 F = 4a_1\,, \quad 9b_1^2 = 32a_1c_1\,, \quad 3b_2 B = 2dA\,, \quad (2|e|F - d)B^2 = 4c_1 F A^2\,, \quad a_2 + dF = |e|F^2\,. \tag{52}$$

Interestingly enough, there is also a solution in case $m < 1/2$. In particular, it is easily checked that (38) is a solution with A, B again given by Eq. (47) while F and D are now given by

$$F \equiv \phi_m = \frac{[3b_1 - \sqrt{(9b_1^2 - 32a_1c_1)}]}{8c_1}\,,$$
$$D^2 = \frac{\sqrt{(9b_1^2 - 32a_1c_1)}\,[3b_1 - \sqrt{(9b_1^2 - 32a_1c_1)}]}{8(1-2m)c_1}\,. \tag{53}$$

Further the three constraints are again given by Eq. (51).

Interesting case of $b_2 = c_2 = 0$

Finally, let us discuss the physically interesting case of $b_2 = c_2 = 0$. It is easy to see that now a solution of the form (38) with $G = 0$ is possible only if $m < 1/2$. In particular, such a solution exists if

$$A^2 = \frac{mD^2}{|e|}\,, \quad B^2 = \frac{mD^2(2c_1 + |e|)}{e^2}\,, \quad D^2 = \frac{a_1}{(1-2m)}\,, \tag{54}$$

and if further the following constraints are satisfied

$$b_1^2 = 4a_1c_1\,, \quad db_1 = 2|e|a_1\,, \quad d^2 = -2|e|(a_1 + 2a_2)\,. \tag{55}$$

Thus with $a_1 > 0$, a solution of the form (38) with $b_2 = c_2 = 0$ can exist only if $m < 1/2$ and $a_2 < 0$.

It is easily checked that no solution is however possible in the special case of $4c_1c_2 = e^2$.

F=0, G≠0:

In this case (38) is a solution only if $d = 0$ with A, B again given by Eq. (47) while G and D are given by

$$G \equiv \psi_c = \frac{[3b_2 + \sqrt{(9b_2^2 - 32a_2c_2)}]}{8c_2}\,,$$
$$D^2 = \frac{\sqrt{(9b_2^2 - 32a_2c_2)}\,[3b_2 + \sqrt{(9b_2^2 - 32a_2c_2)}]}{8(2m-1)c_2}\,. \tag{56}$$

Further there are three constraints given by

$$3b_1 A = -4|e|BG\,, \quad (4c_2 G - b_2)B^2 = 2|e|GA^2\,, \quad 2a_1 - 2|e|G^2 = (2m-1)D^2\,. \tag{57}$$

Note that this solution exists only if $m > 1/2$.

What happens if $m = 1/2$? It is easily checked that in that case (38) with $F = 0$ is still a solution provided A, B are given by Eq. (47), D is undetermined and the constraints are

$$3b_1 A = -4|e|BG\,, \quad a_1 = |e|G^2\,, \quad b_2 B^2 = 4|e|GA^2\,, \quad 3b_2 G = 4a_2\,, \quad 9b_2^2 = 32a_2 c_2\,. \tag{58}$$

Interestingly enough, there is also a solution in case $m < 1/2$. In particular, it is easily checked that (38) is a solution with A, B again given by Eq. (47) while G and D are now given by

$$G \equiv \psi_m = \frac{[3b_2 - \sqrt{(9b_2^2 - 32a_2c_2)}]}{8c_2}\,,$$
$$D^2 = \frac{\sqrt{(9b_2^2 - 32a_2c_2)}\,[3b_2 - \sqrt{(9b_2^2 - 32a_2c_2)}]}{8(1-2m)c_2}\,. \tag{59}$$

The constraints are again given by Eq. (57). Note that one of the constraint can only be satisfied if $b_2^2 < 4a_2c_2$. Even in this case, no solution of the form (38) with $F = 0, G \neq 0$ exists either when $e^2 = 4c_1c_2$ or when $b_2 = c_2 = 0$.

m=1

In this limiting case we have a dark-dark soliton solution given by

$$\phi = F + A\text{sech}[D(x+x_0)]\,, \quad \psi = G + B\text{sech}[D(x+x_0)]\,, \tag{60}$$

where F and/or G may or may not be zero depending on the solution that we are considering.

It is worth emphasizing that while solution (38) exists in the coupled case, there is *no* analogous solution to the corresponding uncoupled model, i.e. no solution is possible in the case of the uncoupled field Eq. (2) i.e. when $d = e = 0$, so long as $c_1 > 0$.

Energy: Corresponding to the periodic solution (38) the energy is same but only the value of C changes depending on if $F = G = 0$ or $G = 0, F \neq 0$ or $F = 0, G \neq 0$. For example, if $G = 0, F \neq 0$ then the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = \frac{2(A^2+B^2)D}{3m}[(2m-1)E(m) + (1-m)K(m)]\,,$$

$$C = F^2[a_1 - b_1 F + c_1 F^2] - \frac{1}{2}(1-m)(A^2+B^2)D^2\,. \tag{61}$$

On the other hand, if $F = 0, G \neq 0$ then C is given by

$$C = G^2[a_2 - b_2 G + c_2 G^2] - \frac{1}{2}(1-m)(A^2+B^2)D^2\,, \tag{62}$$

while if $F = G = 0$ then C is simply given by

$$C = -\frac{1}{2}(1-m)(A^2+B^2)D^2\,. \tag{63}$$

On using the expansion formulas for $E(m)$ and $K(m)$ around $m = 1$ as given by Eqs. (35) and (36), for m near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. (60)] plus the interaction energy. We find

$$\hat{E} = E_{pulse} + E_{int} = (A^2+B^2)D\left[\frac{2}{3} - \frac{5(1-m)}{6} + (1-m)\ln(\frac{4}{\sqrt{1-m}})\right]\,. \tag{64}$$

Note that this solution exists only when $e < 0$ and $e^2 > 4c_1c_2$. The interaction energy vanishes at exactly $m = 1$, as it should.

Solution III

Yet another pulse lattice solution given by

$$\phi = F + A\mathrm{dn}[D(x+x_0),m]\,, \quad \psi = G + B\mathrm{dn}[D(x+x_0),m]\,, \tag{65}$$

is an exact solution provided the following eight coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 + dG^2 + 2eFG^2 = 0\,, \tag{66}$$

$$2a_1A - 6b_1AF + 12c_1F^2A + 2BdG + 2eAG^2 + 4eFBG = (2-m)AD^2\,, \tag{67}$$

$$-3b_1A^2 + 12c_1FA^2 + dB^2 + 2eFB^2 + 4eABG = 0\,, \tag{68}$$

$$2c_1A^2 + eB^2 = -D^2\,, \tag{69}$$

$$2a_2G - 3b_2G^2 + 4c_2G^3 + 2dFG + 2eGF^2 = 0\,, \tag{70}$$

$$2a_2B - 6b_2BG + 12c_2G^2B + 2dAG + 2dFB + 4eAFG + 2eBF^2 = (2-m)BD^2\,, \tag{71}$$

$$- 3b_2B^2 + 12c_2GB^2 + 2dAB + 2eGA^2 + 4eABF = 0\,, \tag{72}$$

$$2c_2B^2 + eA^2 = -D^2\,. \tag{73}$$

Notice that two of these equations are meaningful only if $e < 0$, since from stability considerations, $c_1 > 0, c_2 \geq 0$. Hence we write $e = -|e|$.

Five of these equations determine the five unknowns A, B, D, F, G while the other three equations, give three constraints between the eight parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e$. In particular, A and B are given by

$$A^2 = \frac{D^2(2c_2 + |e|)}{(e^2 - 4c_1c_2)}\,, \quad B^2 = \frac{D^2(2c_1 + |e|)}{(e^2 - 4c_1c_2)}\,. \tag{74}$$

Akin to the solution II, in this case too, a solution exists when $F = G = 0$ or $G = 0, F \neq 0$ or when $F = 0, G \neq 0$ and we discuss all the cases one by one.

F=G=0

In this case A, B are again given by Eq. (74) while D is given by

$$D^2 = \frac{2a_1}{(2-m)}\,, \tag{75}$$

and the three constraints are

$$a_1 = a_2\,, \quad 4d^3 = 27b_1b_2^2\,, \quad \frac{3b_1}{d} = \sqrt{\frac{(2c_1 + |e|)}{(2c_2 + |e|)}}\,. \tag{76}$$

It may be noted that such a solution with $F = G = 0$ will not exist in case either $b_2 = c_2 = 0$ or if $e^2 = 4c_1c_2$.

G=0, F≠0:

In this case A, B are again given by Eq. (74) while F and D are given by

$$F \equiv \phi_c = \frac{3b_1 + \sqrt{(9b_1^2 - 32a_1c_1)}}{8c_1}\,,$$
$$D^2 = \frac{\sqrt{(9b_1^2 - 32a_1c_1)}\,[3b_1 + \sqrt{(9b_1^2 - 32a_1c_1)}]}{8(2-m)c_1}\,. \tag{77}$$

Further there are three constraints given by

$$\frac{(2|e|F - d)}{3(4c_1F - b_1)} = \frac{(2c_1 + |e|)}{(2c_2 + |e|)}\,, \quad (6b_1c_1 - 4c_1d + 3|e|b_1)F = 2(2a_2c_1 + 4a_1c_1 + |e|a_1)\,,$$
$$27b_2^2(4c_1F - b_1) = 4(2|e|F - d)^2\,. \tag{78}$$

It is worth noting that since $b_1 - 4c_1F < 0$, hence $F > d/2|e|$. It is easily checked that no solution is however possible in case $b_2 = c_2 = 0$ or if $4c_1c_2 = e^2$.

m=1

In this limiting case we have the same dark-dark soliton solution as given by Eq. (60).

F=0, G≠0:

Solution (65) with $F = 0, G \neq 0$ is only possible if $d = 0$. In this case A, B are again given by Eq. (74) while G and D are given by

$$G \equiv \psi_c = \frac{3b_2 + \sqrt{(9b_2^2 - 32a_2c_2)}}{8c_2}\,,$$
$$D^2 = \frac{\sqrt{(9b_2^2 - 32a_2c_2)}\,[3b_2 + \sqrt{(9b_2^2 - 32a_2c_2)}]}{8(2-m)c_2}\,. \tag{79}$$

Further there are three constraints given by

$$3b_1A = 4|e|BG\,,\quad 2a_1 - 2|e|G^2 = (2-m)D^2\,,\quad 3(4c_2G - b_2)B^2 = 2|e|GA^2\,. \tag{80}$$

Again, it is easily checked that no solution is however possible in case $b_2 = c_2 = 0$ or if $4c_1c_2 = e^2$.

It may be noted that, as in the previous case, while there exists a solution (65) in the coupled case, no such solution exists in the corresponding uncoupled case, i.e. no solution exists in the case of the uncoupled field Eq. (2) i.e. when $d = e = 0$, so long as $c_1 > 0$.

Energy: Corresponding to the periodic solution (65), the energy is same but only C is different depending on if $F = G = 0$, or $G = 0, F \neq 0$ or $F = 0, G \neq 0$. For example, with $G = 0, F \neq 0$, the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = \frac{2(A^2+B^2)D}{3}[(2-m)E(m) - (1-m)K(m)]\,,$$
$$C = F^2[a_1 - b_1F + c_1F^2] + \frac{1}{2}(1-m)(A^2+B^2)D^2\,. \tag{81}$$

On the other hand, if $F = 0, G \neq 0$, then C is given by

$$C = G^2[a_2 - b_2G + c_2G^2] + \frac{1}{2}(1-m)(A^2+B^2)D^2\,, \tag{82}$$

while if $F = G = 0$ then C is simply given by

$$C = \frac{1}{2}(1-m)(A^2+B^2)D^2\,, \tag{83}$$

On using the expansion formulas for $E(m)$ and $K(m)$ around $m = 1$ as given by Eqs. (35) and (36), for m near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-dark) soliton solution [Eq. (60)] plus the interaction energy. We find

$$\hat{E} = E_{pulse} + E_{int} = (A^2 + B^2)D\left[\frac{2}{3} - \frac{(1-m)}{2} - (1-m)\ln\left(\frac{4}{\sqrt{1-m}}\right)\right] . \tag{84}$$

Note that this solution exists only when $e < 0, e^2 > 4c_1c_2$. The interaction energy vanishes at exactly $m = 1$, as it should. Unlike the dark-dark cn − cn and dn − dn solutions, it turns out that the (mixed) dark-dark periodic soliton solutions of the form dn − cn and cn − dn, do not exist.

Solution IV

We now discuss two bright-dark and two dark-bright periodic soliton solutions. We will see that for these four solutions G is necessarily zero while F is necessarily nonzero. Let us discuss these solutions one by one. In particular, it is easily shown that

$$\phi = F + A\mathrm{sn}[D(x + x_0), m] , \quad \psi = G + B\mathrm{cn}[D(x + x_0), m] , \tag{85}$$

is an exact solution provided

$$G = 0 , \quad b_2 = 0 , \quad d + 2eF = 0 , \tag{86}$$

and the following six coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 = 0 , \tag{87}$$

$$2a_1 - 6b_1F + 12c_1F^2 + 2eB^2 = -(1 + m)D^2 , \tag{88}$$

$$-3b_1A^2 + 12c_1FA^2 = 0 , \tag{89}$$

$$2c_1A^2 - eB^2 = mD^2 , \tag{90}$$

$$2a_2 + 2dF + 2e(A^2 + F^2) = (2m - 1)D^2 , \tag{91}$$

$$2c_2B^2 - eA^2 = -mD^2 . \tag{92}$$

Three of these equations determine the three unknowns A, B, D while the other three equations, give three constraints between the seven parameters $a_{1,2}, b_1, c_{1,2}, d, e$. In particular, A and B are given by

$$A^2 = \frac{mD^2(e - 2c_2)}{(e^2 - 4c_1c_2)} , \quad B^2 = \frac{mD^2(2c_1 - e)}{(e^2 - 4c_1c_2)} , \tag{93}$$

while the inverse characteristic length D is given by

$$D^2 = \frac{(e^2 - 4c_1c_2)a_1}{[(1+m)(e^2 - 4c_1c_2) + 2em(2c_1 - e)]} \,. \tag{94}$$

Further, the three constraints are

$$b_1^2 = 4a_1c_1 \,, \quad 2a_1e + db_1 = 0 \,, \quad (2m-1)D^2 = 2a_2 + 2eA^2 + dF \,. \tag{95}$$

m=1

In this limiting case we have a bright-dark soliton solution given by

$$\phi = F + A\tanh[D(x+x_0)] \,, \quad \psi = B\mathrm{sech}[D(x+x_0)] \,, \tag{96}$$

with A, B and D given by Eqs. (93) and (94) with $m = 1$ while the three constraints are again given by Eq. (95).

Special case of $e^2 = 4c_1c_2$

One can show that the solution (85) exists even in case $e^2 = 4c_1c_2$. It turns out such a solution exists only if Eqs. (22) and (86) are satisfied. In this case, A, B and D are given by

$$A^2 = \frac{[ma_1 - 2(1-m)a_2]}{2em} \,, \quad B^2 = \frac{[ma_1 - 2(1+m)a_2]}{2em} \,, \quad D^2 = \frac{2a_2}{m} \,, \tag{97}$$

while the constraints are

$$d = -b_1 \,, \quad b_1^2 = 4a_1c_1 \,. \tag{98}$$

Interesting case of $b_2 = c_2 = 0$

Finally, let us discuss the physically interesting case of $b_2 = c_2 = 0$. In this case Eq. (85) with $G = 0$ is a solution to the field Eqs. (2) provided

$$A^2 = \frac{m(d^2 - 4ea_2)}{2e^2} \,, \quad B^2 = \frac{(2c_1 - e)(d^2 - 4ea_2)}{2e^3} \,, \quad D^2 = \frac{(d^2 - 4ea_2)}{2e} \,, \quad F = -\frac{d}{2e} \,, \tag{99}$$

and further if the following three constraints are satisfied

$$b_1^2 = 4a_1c_1 \,, \quad (d^2 - 4ea_2)[4mc_1 + (1-m)e] = 2a_1e^2 \,, \quad b_1d + 2a_1e = 0 \,. \tag{100}$$

Energy: Corresponding to the periodic solution (85) the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = \frac{2D}{3m}\bigg([(2-m)mB^2 + (1+m)A^2]E(m) - (1-m)(A^2+2B^2)K(m)\bigg)\,,$$
$$C = F^2[a_1 - b_1F + c_1F^2] - \frac{1}{2}A^2D^2 + B^2[a_2 + dF + eF^2 + c_2B^2]\,. \tag{101}$$

On using the expansion formulas for $E(m)$ and $K(m)$ around $m = 1$ as given by Eqs. (35) and (36), for m near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-dark) soliton solution [Eq. (96)] plus the interaction energy. We find

$$\hat{E} = E_{soliton} + E_{int} = D\left[\frac{2}{3}(2A^2+B^2) + \frac{(1-m)}{6}(2A^2+3B^2) - (1-m)B^2\ln\left(\frac{4}{\sqrt{1-m}}\right)\right]\,. \tag{102}$$

Note that this solution exists when either (i) $2c_1 \ge e \ge 2c_2$ and $e^2 \ge 4c_1c_2$ or if (ii) $2c_2 \ge e \ge 2c_1$ and $4c_1c_2 \ge e^2$ or if (iii) $c_2 = 0, 2c_1 > e$. The interaction energy vanishes at exactly $m = 1$, as it should.

Solution V

It is easy to show that another such (kink- and pulse-like) solution is

$$\phi = F + A\mathrm{sn}[D(x+x_0),m]\,, \quad \psi = G + B\mathrm{dn}[D(x+x_0),m]\,. \tag{103}$$

This is an exact solution provided

$$G = 0\,, \quad b_2 = 0\,, \quad d + 2eF = 0\,, \tag{104}$$

and the following six coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 = 0\,, \tag{105}$$

$$2a_1 - 6b_1F + 12c_1F^2 + 2eB^2 = -(1+m)D^2\,, \tag{106}$$

$$-3b_1A^2 + 12c_1FA^2 = 0\,, \tag{107}$$

$$2c_1A^2 - emB^2 = mD^2\,, \tag{108}$$

$$2a_2 + 2dF + 2e(\frac{A^2}{m} + F^2) = (2-m)D^2\,, \tag{109}$$

$$2mc_2B^2 - eA^2 = -mD^2\,. \tag{110}$$

Three of these equations determine the three unknowns A, B, D while the other three equations, give three constraints between the seven parameters $a_{1,2}, b_1, c_{1,2}, d, e$. In particular, A and B are given by

$$A^2 = \frac{mD^2(e-2c_2)}{(e^2-4c_1c_2)}\,, \quad B^2 = \frac{D^2(2c_1-e)}{(e^2-4c_1c_2)}\,, \tag{111}$$

while D is given by

$$D^2 = \frac{(e^2-4c_1c_2)a_1}{[(1+m)(e^2-4c_1c_2)+2e(2c_1-e)]}\,. \tag{112}$$

Further, the three constraints are

$$b_1^2 = 4a_1c_1\,, \quad 2a_1e + db_1 = 0\,, \quad (2-m)mD^2 = 2a_2m + 2eA^2 + mdF\,. \tag{113}$$

m=1

In this limiting case we have a bright-dark soliton solution given by Eq. (96) with A, B and D given by Eqs. (93) and (94) with $m = 1$ while the three constraints are again given by Eq. (95).

Special case of $e^2 = 4c_1c_2$

One can show that the solution (103) exists even in case $e^2 = 4c_1c_2$. It turns out such a solution exists only if Eqs. (22) and (104) are satisfied. In this case, A, B and D are given by

$$A^2 = \frac{m[a_1+2(1-m)a_2]}{2e}\,, \quad B^2 = \frac{[a_1-2(1+m)a_2]}{2e}\,, \quad D^2 = 2a_2\,, \tag{114}$$

while the constraints are

$$d = -b_1\,, \quad b_1^2 = 4a_1c_1\,. \tag{115}$$

Interesting case of $b_2 = c_2 = 0$

Finally, let us discuss the physically interesting case of $b_2 = c_2 = 0$. In this case Eq. (103) with $G = 0$ is a solution to the field Eq. (2) provided

$$A^2 = \frac{(d^2-4ea_2)}{2e^2}\,, \quad B^2 = \frac{(2c_1-e)(d^2-4ea_2)}{2me^3}\,, \quad D^2 = \frac{(d^2-4ea_2)}{2me}\,, \quad F = -\frac{d}{2e}\,, \tag{116}$$

and further if the following three constraints are satisfied

$$b_1^2 = 4a_1c_1\,, \quad (d^2-4ea_2)[4c_1-(1-m)e] = 2ma_1e^2\,, \quad b_1d + 2a_1e = 0\,. \tag{117}$$

Energy: Corresponding to the periodic solution (103) the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = \frac{2D}{3m}\Big([(2m-1)B^2 + (1+m)A^2]E(m) - (1-m)(A^2 - B^2)K(m)\Big)\,,$$
$$C = F^2[a_1 - b_1 F + c_1 F^2] - \frac{1}{2}A^2 D^2 + B^2[a_2 + dF + eF^2 + c_2 B^4]\,. \quad (118)$$

On using the expansion formulas for $E(m)$ and $K(m)$ around $m = 1$ as given by Eqs. (35) and (36), for m near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (bright-dark) soliton solution [Eq. (96)] plus the interaction energy. We find

$$\hat{E} = E_{soliton} + E_{int} = D\left[\frac{2}{3}(2A^2 + B^2) + \frac{(1-m)}{6}(2A^2 - 5B^2) + (1-m)B^2 \ln\left(\frac{4}{\sqrt{1-m}}\right)\right]\,. \quad (119)$$

Note that this solution exists when either (i) $2c_1 \ge e \ge 2c_2$ and $e^2 \ge 4c_1c_2$ or if (ii) $2c_2 \ge e \ge 2c_1$ and $4c_1c_2 \ge e^2$ or if (iii) $c_2 = 0, 2c_1 > e$. The interaction energy vanishes at exactly $m = 1$, as it should.

Solution VI

We now discuss two periodic solutions which at $m = 1$ reduce to dark-bright soliton solutions. In particular, it is easily shown that

$$\phi = F + A\,\mathrm{cn}[D(x + x_0), m]\,, \quad \psi = G + \mathrm{sn}[D(x + x_0), m]\,, \quad (120)$$

is an exact solution provided

$$G = 0\,, \quad b_2 = 0\,, \quad d + 2eF = 0\,, \quad (121)$$

and the following six coupled equations are satisfied

$$2a_1 F - 3b_1 F^2 + 4c_1 F^3 = 0\,, \quad (122)$$

$$2a_1 - 6b_1 F + 12c_1 F^2 + 2eB^2 = (2m-1)D^2\,, \quad (123)$$

$$-3b_1 A^2 + 12c_1 F A^2 = 0\,, \quad (124)$$

$$2c_1 A^2 - eB^2 = -mD^2\,, \quad (125)$$

$$2a_2 + 2dF + 2e(A^2 + F^2) = -(1+m)D^2\,, \quad (126)$$

$$2c_2 B^2 - eA^2 = mD^2\,. \quad (127)$$

Three of these equations determine the three unknowns A, B, D while the other three equations, give three constraints between the seven parameters $a_{1,2}, b_1, c_{1,2}, d, e$. In particular, A and B are given by

$$A^2 = \frac{mD^2(e-2c_2)}{(4c_1c_2-e^2)}\,, \quad B^2 = \frac{mD^2(2c_1-e)}{(4c_1c_2-e^2)}\,, \tag{128}$$

while D is given by

$$D^2 = \frac{(4c_1c_2-e^2)a_1}{[(4c_1c_2-e^2)+4mc_1(e-2c_2)]}\,. \tag{129}$$

Further, the three constraints are

$$b_1^2 = 4a_1c_1\,, \quad 2a_1e + db_1 = 0\,, \quad -(1+m)D^2 = 2a_2 + 2eA^2 + dF\,. \tag{130}$$

m=1

In this limiting case we have a dark-bright soliton solution given by

$$\phi = F + A\text{sech}[D(x+x_0)]\,, \quad \psi = B\tanh[D(x+x_0)]\,, \tag{131}$$

with A, B and D given by Eqs. (128) and (129) with $m = 1$ while the three constraints are given by Eq. (130).

Special case of $e^2 = 4c_1c_2$

One can show that the solution (120) exists even in case $e^2 = 4c_1c_2$. It turns out such a solution exists only if Eqs. (22) and (121) are satisfied. In this case, A, B and D are given by

$$A^2 = \frac{(ma_1+2a_2)}{2em}\,, \quad B^2 = \frac{[ma_1-2(2m-1)a_2]}{2em}\,, \quad D^2 = -\frac{2a_2}{m}\,, \tag{132}$$

while the constraints are

$$d = -b_1\,, \quad b_1^2 = 4a_1c_1\,. \tag{133}$$

Finally, let us discuss the physically interesting case of $b_2 = c_2 = 0$. In this case, a solution does not exist since whereas $A^2 > 0$ requires that $e < 0$ but that in turn makes $B^2 < 0$, since from stability considerations, $c_1 > 0$.

Energy: Corresponding to the periodic solution (120) the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = \frac{2D}{3m}\Big([(2m-1)A^2 + (1+m)B^2]E(m) - (1-m)(B^2-A^2)K(m)\Big)\,,$$
$$C = F^2[a_1 - b_1F + c_1F^2] + \frac{1}{2}(1-m)A^2D^2 + B^2[a_2 + dF + eF^2 + c_2B^2]\,. \tag{134}$$

On using the expansion formulas for $E(m)$ and $K(m)$ around $m = 1$ as given by Eqs. (35) and (36), for m near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-bright) soliton solution [Eq. (131)] plus the interaction energy. We find

$$\hat{E} = E_{soliton} + E_{int} = D\left[\frac{2}{3}(2B^2 + A^2) + \frac{(1-m)}{6}(2B^2 - 5A^2) + (1-m)A^2 \ln\left(\frac{4}{\sqrt{1-m}}\right)\right] . \qquad (135)$$

Note that this solution exists when either (i) $2c_1 \geq e \geq 2c_2$ and $4c_1c_2 \geq e^2$ or if (ii) $2c_2 \geq e \geq 2c_1$ and $e^2 \geq 4c_1c_2$. The interaction energy vanishes at exactly $m = 1$, as it should.

Solution VII

Another periodic solution which at $m = 1$ reduces to the dark-bright soliton solution is given by

$$\phi = F + A\text{dn}[D(x + x_0), m] , \quad \psi = G + B\text{sn}[D(x + x_0), m] , \qquad (136)$$

provided

$$G = 0 , \quad b_2 = 0 , \quad d + 2eF = 0 , \qquad (137)$$

and the following six coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 = 0 , \qquad (138)$$

$$2a_1 - 6b_1F + 12c_1F^2 + \frac{2eB^2}{m} = (2-m)D^2 , \qquad (139)$$

$$-3b_1A^2 + 12c_1FA^2 = 0 , \qquad (140)$$

$$2mc_1A^2 - eB^2 = -mD^2 , \qquad (141)$$

$$2a_2 + 2dF + 2e(A^2 + F^2) = -(1+m)D^2 , \qquad (142)$$

$$2c_2B^2 - meA^2 = mD^2 . \qquad (143)$$

Three of these equations determine the three unknowns A, B, D while the other three equations, give three constraints between the seven parameters $a_{1,2}, b_1, c_{1,2}, d, e$. In particular, A and B are given by

$$A^2 = \frac{D^2(e - 2c_2)}{(4c_1c_2 - e^2)} , \quad B^2 = \frac{mD^2(2c_1 - e)}{(4c_1c_2 - e^2)} , \qquad (144)$$

while D is given by

$$D^2 = \frac{(4c_1c_2 - e^2)a_1}{[m(4c_1c_2 - e^2) + 4c_1(e - 2c_2)]} . \qquad (145)$$

Further, the three constraints are

$$b_1^2 = 4a_1c_1\,, \quad 2a_1e + db_1 = 0\,, \quad -(1+m)D^2 = 2a_2 + 2eA^2 + dF\,. \tag{146}$$

m=1

In this limiting case we have a dark-bright soliton solution given by Eq. (131).

Special case of $e^2 = 4c_1c_2$

One can show that the solution (136) exists even in case $e^2 = 4c_1c_2$. It turns out such a solution exists only if Eqs. (22) and (137) are satisfied. In this case, A, B and D are given by

$$A^2 = \frac{(a_1 + 2ma_2)}{2e}\,, \quad B^2 = \frac{[a_1 - 2(2-m)a_2]}{2e}\,, \quad D^2 = -2a_2\,, \tag{147}$$

while the constraints are

$$d = -b_1\,, \quad b_1^2 = 4a_1c_1\,. \tag{148}$$

Finally, let us discuss the physically interesting case of $b_2 = c_2 = 0$. As in the previous case, a solution does not exist since whereas $A^2 > 0$ requires that $e < 0$ but that in turn makes $B^2 < 0$, since from stability considerations, $c_1 > 0$.

Energy: Corresponding to the periodic solution (136) the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = \frac{2D}{3m}\Big([(2-m)mA^2 + (1+m)B^2]E(m) - (1-m)(B^2+2A^2)K(m)\Big)\,,$$
$$C = F^2[a_1 - b_1F + c_1F^2] + \frac{1}{2}(1-m)A^2D^2 + \frac{B^2}{m^2}[ma_2 + mdF + meF^2 + c_2B^4]\,. \tag{149}$$

On using the expansion formulas for $E(m)$ and $K(m)$ around $m = 1$ as given by Eqs. (35) and (36), for m near one, the energy of the periodic solution can be rewritten as the energy of the corresponding hyperbolic (dark-bright) soliton solution [Eq. (131)] plus the interaction energy. We find

$$\hat{E} = E_{soliton} + E_{int} = D\left[\frac{2}{3}(2B^2 + A^2) + \frac{(1-m)}{6}(2B^2 + 3A^2) - (1-m)A^2\ln\left(\frac{4}{\sqrt{1-m}}\right)\right]\,. \tag{150}$$

Note that this solution exists when either (i) $2c_1 \geq e \geq 2c_2$ and $4c_1c_2 \geq e^2$ or if (ii) $2c_2 \geq e \geq 2c_1$ and $e^2 \geq 4c_1c_2$. The interaction energy vanishes at exactly $m = 1$, as it should.

Solution VIII

We now present three solutions, which to the best of our knowledge were not known before even in the uncoupled limit. Two of these solutions, in the uncoupled limit are kink-type while one is a pulse type solution. Let us discuss these solutions one by one.

One of these periodic solutions is given by

$$\phi = F + \frac{A\mathrm{sn}[D(x+x_0),m]}{1+B\mathrm{dn}[D(x+x_0),m]}\,, \quad \psi = G + \frac{H\mathrm{sn}[D(x+x_0),m]}{1+B\mathrm{dn}[D(x+x_0),m]}\,, \tag{151}$$

provided

$$B = 1\,, \tag{152}$$

and the following eight coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 + dG^2 + 2eFG^2 = 0\,, \tag{153}$$

$$2a_1A - 6b_1FA + 12c_1F^2A + 2dGH + 4eFGH + 2eAG^2 = -\frac{A(2-m)D^2}{2}\,, \tag{154}$$

$$-3b_1A^2 + 12c_1FA^2 + dH^2 + 2eFH^2 + 4eAGH = 0\,, \tag{155}$$

$$8c_1A^2 + 4eH^2 = m^2D^2\,, \tag{156}$$

$$2a_2G - 3b_2G^2 + 4c_2G^3 + 2dFG + 2eGF^2 = 0\,, \tag{157}$$

$$2a_2H - 6b_2GH + 12c_2G^2H + 2dFH + 2dAG + 4eFGA + 2eHF^2 = -\frac{H(2-m)D^2}{2}\,, \tag{158}$$

$$-3b_2H^2 + 12c_2GH^2 + 2dAH + 2eGA^2 + 4eAFH = 0\,, \tag{159}$$

$$8c_2H^2 + 4eA^2 = m^2D^2\,. \tag{160}$$

Five of these equations determine the five unknowns A, H, D, F, G while the other three equations, give three constraints between the eight parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e$. In particular, A and H are given by

$$A^2 = \frac{m^2D^2(2c_2-e)}{4(4c_1c_2-e^2)}\,, \quad H^2 = \frac{m^2D^2(2c_1-e)}{4(4c_1c_2-e^2)}\,. \tag{161}$$

It turns out that there is no solution to these field equations with $F = G = 0$ so long as $a_1 > 0$. However, there are solutions when either $G = 0, F \neq 0$ or $F = 0, G \neq 0$ which we discuss one by one.

G=0, F $\neq$ 0

In this case A, H are again given by Eq. (161), $B = 1$ while F and D are given by

$$F \equiv \phi_m = \frac{[3b_1 - \sqrt{(9b_1^2 - 32a_1c_1)}]}{8c_1} \,,$$
$$D^2 = \frac{\sqrt{(9b_1^2 - 32a_1c_1)}[3b_1 - \sqrt{(9b_1^2 - 32a_1c_1)}]}{4(2-m)c_1} \,. \tag{162}$$

Further there are three constraints given by Eq. (21).

Special case of $e^2 = 4c_1c_2$

One can show that the solution (151) with $G = 0$, $a_1 = b_1F$ exists even in case $e^2 = 4c_1c_2$ provided if Eq. (22) is satisfied. In this case, A and H can be determined from the relations

$$A^2 + H^2 = \frac{m^2D^2}{4e} \,, \quad 3b_2H = 2(d + 2eF)A \,, \tag{163}$$

F, D are still given by (162), while the two constraints are given by Eq. (24).

Interesting case of $b_2 = c_2 = 0$

Finally, let us discuss the physically interesting case of $b_2 = c_2 = 0$. In this case Eq. (151) with $G = 0$ is a solution to the field Eq. (2) provided

$$A^2 = \frac{m^2a_1}{2e(2-m)} \,, \quad H^2 = \frac{(e - 2c_1)m^2a_1}{2e^2(2-m)} \,, \quad D^2 = \frac{2a_1}{(2-m)} \,, \quad F = \sqrt{\frac{a_1}{4c_1}} \,, \tag{164}$$

while the three constraints are given by Eq. (26).

Thus the solution with $G = 0, F \neq 0$ exists when $2c_1 \geq e, 2c_2 \geq e$ and $4c_1c_2 \geq e^2$.

F=0, G $\neq$ 0

Solution (151) is a solution with $F = 0, G \neq 0$ provided $d = 0$, $B = 1$ and $e < 0$, we therefore write $e = -|e|$. In this case A, H are given by Eq. (161) while G and D are given by

$$G \equiv \psi_m = \frac{[3b_2 - \sqrt{(9b_2^2 - 32a_2c_2)}]}{8c_2} \,,$$
$$D^2 = \frac{\sqrt{(9b_2^2 - 32a_2c_2)}[3b_2 - \sqrt{(9b_2^2 - 32a_2c_2)}]}{4(2-m)c_2} \,. \tag{165}$$

Further there are three constraints given by

$$3b_1A = -4|e|GH \,, \quad 3H^2(b_2 - 4c_2G) = -2|e|GA^2 \,, \quad 3(2c_2 + |e|)b_2G = 2(2a_1c_2 + 4a_2c_2 + |e|a_2) \,. \tag{166}$$

For $F=0, G\neq 0$, no solution of the form (151) exists in case either $e^2=4c_1c_2$ or if $b_2=c_2=0$.

m=1

In this limiting case we have a bright-bright soliton solution given by

$$\phi = F + A\tanh\left[\sqrt{\frac{a_1}{2}}(x+x_0)\right]\,, \quad \psi = H\tanh\left[\sqrt{\frac{a_1}{2}}(x+x_0)\right]\,, \tag{167}$$

with A and H given by Eq. (161), $D^2=2a_1, B=1$ while the three constraints are again given by Eq. (21). **Uncoupled Case**

For completeness, it may be worthwhile to write down the solution of the uncoupled field Eq. (2), i.e. when $d=e=0$. It is easily shown that in this case the solution (151) reduces to

$$\phi = \sqrt{\frac{a_1}{4c_1}}\left[1+\left(\frac{m}{2-m}\right)\frac{\mathrm{sn}\left(\sqrt{\frac{2a_1}{2-m}}(x+x_0),m\right)}{\left[1+\mathrm{dn}\left(\sqrt{\frac{2a_1}{2-m}}(x+x_0),m\right)\right]}\right]\,, \tag{168}$$

and one is at the transition temperature $T=T_c$ since the parameters a_1, b_1, c_1 satisfy $b_1^2=4a_1c_1$. At $m=1$, this reduces to the well known kink solution (32).

Energy: Corresponding to the periodic solution (151), while the energy is same but the value of C is different for solutions with either $G=0, F\neq 0$ or $F=0, G\neq 0$. For example, in case $G=0, F\neq 0$ the energy $\hat{E}$ and the constant C are given by

$$\begin{aligned}\hat{E} &= (A^2+H^2)D\int_{-2K}^{2K}\frac{\mathrm{cn}^2(y,m)}{[1+\mathrm{dn}(y,m)]^2}dy\,,\\ C &= [a_1-b_1F+c_1F^2]F^2-\frac{1}{8}(A^2+H^2)D^2\,,\end{aligned} \tag{169}$$

where $y=D(x+x_0)$. On the other hand, when $F=0, G\neq 0$, the constant C is given by

$$C = [a_2-b_2G+c_2G^2]G^2-\frac{1}{8}(A^2+H^2)D^2\,. \tag{170}$$

Unfortunately, we are unable to solve the integral (169) analytically and hence are unable to calculate the corresponding soliton interaction energy. However, the energy of the corresponding hyperbolic (bright-bright) soliton solution [Eq. (167)] is easily calculated. We find

$$E_{soliton} = \sqrt{\frac{16a_1}{18}}(A^2+H^2)D\,. \tag{171}$$

Solution IX

We now present a periodic solution, which at $m = 1$ reduces to a constant but is nontrivial for arbitrary m. This periodic solution is given by

$$\phi = F + \frac{A\text{cn}[D(x+x_0),m]}{\sqrt{1-m} + B\text{dn}[D(x+x_0),m]}\,, \quad \psi = G + \frac{H\text{cn}[D(x+x_0),m]}{\sqrt{1-m} + B\text{dn}[D(x+x_0),m]}\,, \tag{172}$$

provided Eq. (152) holds (i.e. $B = 1$) and the following eight coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 + dG^2 + 2eFG^2 = 0\,, \tag{173}$$

$$2a_1A - 6b_1FA + 12c_1F^2A + 2dGH + 4eFGH + 2eAG^2 = -\frac{A(2-m)D^2}{2}\,, \tag{174}$$

$$-3b_1A^2 + 12c_1FA^2 + dH^2 + 2eFH^2 + 4eAGH = 0\,, \tag{175}$$

$$8c_1A^2 + 4eH^2 = m^2D^2\,, \tag{176}$$

$$2a_2G - 3b_2G^2 + 4c_2G^3 + 2dFG + 2eGF^2 = 0\,, \tag{177}$$

$$2a_2H - 6b_2GH + 12c_2G^2H + 2dFH + 2dAG + 4eFGA + 2eHF^2 = -\frac{H(2-m)D^2}{2}\,, \tag{178}$$

$$-3b_2H^2 + 12c_2GH^2 + 2dAH + 2eGA^2 + 4eAFH = 0\,, \tag{179}$$

$$8c_2H^2 + 4eA^2 = m^2D^2\,. \tag{180}$$

It is rather remarkable that the eight field equations that we obtain here are identical to those obtained for the solution VIII. As a result, all the solutions obtained in that case continue to be true even in this case. In particular, as in the previous case, there does not exist a solution to Eq. (172) in case $F = G = 0$. On the other hand, when $G = 0, F \neq 0$ or when $F = 0, G \neq 0$, the solutions exist with the parameters as given for solution VIII.

m=1

In this limit, (unlike the solution VIII) the solution (172) with $G = 0, F \neq 0$ goes over to a constant solution, given by

$$\phi = F + A\,, \quad \psi = H\,, \tag{181}$$

where A, H are given by Eq. (161) while $F = \phi_m$ with ϕ_m given by Eq. (4). Similar conclusions are also obviously valid for the solution with $F = 0, G \neq 0$.

Uncoupled Case

For completeness, it may be worthwhile to write down the solution of the uncoupled field Eq. (2), i.e. when $d = e = 0$. It is easily shown that in this case the solution (172) reduces to

$$\phi = \sqrt{\frac{a_1}{4c_1}} \left(1 \pm \frac{m}{\sqrt{(2-m)}} \frac{\mathrm{cn}\left[\sqrt{\frac{2a_1}{2-m}}(x+x_0), m\right]}{\left(\sqrt{1-m} + \mathrm{dn}\left[\sqrt{\frac{2a_1}{2-m}}(x+x_0), m\right]\right)} \right), \tag{182}$$

and one is at the transition temperature $T = T_c$ since the parameters a_1, b_1, c_1 satisfy $b_1^2 = 4a_1c_1$. At $m = 1$, ϕ is a constant, which is either 0 or $\phi_c = \sqrt{\frac{a_1}{c_1}}$, i.e. at $m = 1$, the field ϕ stays at one of the degenerate minima of the potential.

Energy: Corresponding to the periodic solution (172), while the energy is same, the value of C differs depending on if $G = 0, F \neq 0$ or if $F = 0, G \neq 0$. For example, if $G = 0, F \neq 0$, the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = (1-m)(A^2+H^2)D \int_{-2K}^{2K} \frac{\mathrm{sn}^2(y,m)}{[\sqrt{1-m} + \mathrm{dn}(y,m)]^2} dy\,,$$
$$C = [a_1 - b_1F + c_1F^2]F^2 - \frac{1}{8}(A^2+H^2)D^2\,, \tag{183}$$

where $y = D(x+x_0)$. ON the other hand, if $F = 0, G \neq 0$, then C is given by

$$C = [a_2 - b_2G + c_2G^2]G^2 - \frac{1}{8}(A^2+H^2)D^2\,, \tag{184}$$

Unfortunately, we are unable to solve the integral (183) analytically and hence are unable to calculate the corresponding soliton interaction energy. However, the energy of the corresponding solution [Eq. (181)] is of course zero, being a constant solution.

Solution X

We now discuss a periodic solution which in the uncoupled limit and even at $m = 1$ is distinct from the well known kink solution (32). This periodic solution is given by

$$\phi = F + \frac{A\mathrm{sn}[D(x+x_0), m]}{1 + B\mathrm{sn}[D(x+x_0), m]}\,, \quad \psi = G + \frac{H\mathrm{sn}[D(x+x_0), m]}{1 + B\mathrm{sn}[D(x+x_0), m]}\,, \tag{185}$$

provided the following eight coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 + dG^2 + 2eFG^2 = -2BAD^2\,, \tag{186}$$

$$2a_1A - 6b_1FA + 12c_1F^2A + 2dGH + 4eFGH + 2eAG^2 = A[6B^2 - (1+m)]D^2\,, \tag{187}$$

$$-3b_1A^2 + 12c_1FA^2 + dH^2 + 2eFH^2 + 4eAGH = 3AB[(1+m) - 2B^2]D^2\,, \tag{188}$$

$$2c_1A^2 + eH^2 = (1-B^2)(m-B^2)D^2\,, \tag{189}$$

$$2a_2G - 3b_2G^2 + 4c_2G^3 + 2dFG + 2eGF^2 = -2BHD^2\,, \tag{190}$$

$$2a_2H - 6b_2GH + 12c_2G^2H + 2dFH + 2dAG + 4eFGA + 2eHF^2 = H[6B^2 - (1+m)]D^2\,, \tag{191}$$

$$-3b_2H^2 + 12c_2GH^2 + 2dAH + 2eGA^2 + 4eAFH = 3HB[(1+m) - 2B^2]D^2\,, \tag{192}$$

$$2c_2H^2 + eA^2 = (1-B^2)(m-B^2)D^2\,. \tag{193}$$

Six of these equations determine the six unknowns A, B, H, D, F, G while the other two equations, give two constraints between the eight parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e$. In particular, A and H are given by

$$A^2 = \frac{(1-B^2)(m-B^2)D^2(2c_2 - e)}{(4c_1c_2 - e^2)}\,, \quad H^2 = \frac{(1-B^2)(m-B^2)D^2(2c_1 - e)}{(4c_1c_2 - e^2)}\,. \tag{194}$$

Unfortunately, it is not very easy to identify a case when the analysis is somewhat simpler. One possible case is when $F = 0$. In that case, it is easily seen that five of the above equations determine the five unknowns A, B, G, H, D while the remaining three equations give three constraints. Further, solutions also exist when $2c_1 = 2c_2 = e$ as well as when $b_2 = c_2 = 0$. However, because of lack of simplicity of these solutions, we do not write them here explicitly.

Uncoupled case at $m = 1$

There is one case when the analysis is simple and an explicit solution can be written down. In particular, in the uncoupled case, at $m = 1$ it is easily shown that

$$\phi = \sqrt{\frac{a_1}{4c_1}}(1+B)\left(\frac{1 + \tanh\left[\sqrt{\frac{a_1}{2}}(x+x_0)\right]}{1 + B\tanh\left[\sqrt{\frac{a_1}{2}}(x+x_0)\right]}\right)\,, \tag{195}$$

is an exact solution to the above equations. Note that in this case

$$b_1^2 = 4a_1c_1\,, \quad A = F(1-B)\,, \quad Fb_1 = a_1(1+B)\,, \quad 2D^2 = a_1\,, \tag{196}$$

so that one is again at $T = T_c$. Thus one has generalized the well known kink solution at $T = T_c$ by obtaining a one parameter family of kink solutions, characterized by the parameter B. In the limit $B = 0$

this solution goes over to the well known kink solution but is different otherwise. In particular, while as $x \to \pm\infty$, both solutions (32) and (195) go to $\phi_c \equiv \sqrt{\frac{a_1}{c_1}}$ and 0 respectively, the difference is in how they behave at $x = 0$. In particular, while the kink solution (32) goes to $\phi_{max} \equiv \sqrt{\frac{a_1}{2c_1}}$, the new kink solution (196) goes to $\phi = (1+B)\sqrt{\frac{a_1}{2c_1}}$. In other words, with the new solution, the center of the kink can be anywhere between 0 and ϕ_c, while for the conventional kink solution (32), the center of the kink is always at ϕ_{max}. However, the energy of all the kink solutions is identical, i.e. $E_{kink} = \frac{a_1^{3/2}}{3\sqrt{2}c_1}$ and is independent of B.

Energy: Corresponding to the periodic solution (185) the energy $\hat{E}$ and the constant C are given by

$$\hat{E} = (A^2+H^2)D\int_{-2K}^{2K} \frac{\text{cn}^2(y,m)\text{dn}^2(y,m)}{[1+B\text{sn}(y,m)]^4}dy\,,$$
$$C = \left(F+\frac{A}{B}\right)^2\left[a_1 - b_1\left(F+\frac{A}{B}\right) + c_1\left(F+\frac{A}{B}\right)^2 + e\left(G+\frac{H}{B}\right)^2\right]$$
$$+\left(G+\frac{H}{B}\right)^2\left[a_2 - b_2\left(G+\frac{H}{B}\right) + c_2\left(G+\frac{H}{B}\right)^2 + d\left(F+\frac{A}{B}\right)\right]\,, \tag{197}$$

where $y = D(x+x_0)$. Unfortunately, we are unable to solve this integral analytically and hence are unable to calculate the corresponding soliton interaction energy. However, the energy of the corresponding hyperbolic (bright-bright) soliton solution [Eq. (195)]

$$\phi = F + \frac{A\tanh[D(x+x_0)]}{1+B\tanh[D(x+x_0)]}\,,\quad \psi = G + \frac{H\tanh[D(x+x_0)]}{1+B\tanh[D(x+x_0)]}\,, \tag{198}$$

is easily computed. We find that

$$E_{soliton} = \frac{4(A^2+H^2)D}{3(1-B^2)^2}\,. \tag{199}$$

Note that this solution exists when $2c_1 \geq e, 2c_2 \geq e$ and $4c_1c_2 \geq e^2$.

4 Solutions when $b_1^2 \neq 4a_1c_1$ (in the Analogous Uncoupled Case): Solution XI

So far, we have considered ten solutions, all of which, in the uncoupled limit correspond to $b_1^2 = 4a_1c_1$, i.e. $T = T_c$. Now we will discuss a case when the uncoupled limit at $m = 1$ corresponds to $T \neq T_c$, i.e. one is either above or below the transition temperature T_c and as a consequence one has a pulse lattice solution. In particular, it is easily shown that

$$\phi = F + \frac{A\mathrm{dn}[D(x+x_0),m]}{1+B\mathrm{dn}[D(x+x_0),m]}\,, \quad \psi = G + \frac{H\mathrm{dn}[D(x+x_0),m]}{1+B\mathrm{dn}[D(x+x_0),m]}\,, \tag{200}$$

is an exact solution to the field Eq. (2) provided the following eight coupled equations are satisfied

$$2a_1F - 3b_1F^2 + 4c_1F^3 + dG^2 + 2eFG^2 = 2(1-m)BAD^2\,, \tag{201}$$

$$2a_1A - 6b_1FA + 12c_1F^2A + 2dGH + 4eFGH + 2eAG^2 = A[2-m-6(1-m)B^2]D^2\,, \tag{202}$$

$$-3b_1A^2 + 12c_1FA^2 + dH^2 + 2eFH^2 + 4eAGH = -3AB[(2-m) - 2(1-m)B^2]D^2\,, \tag{203}$$

$$2c_1A^2 + eH^2 = (B^2-1)[1-(1-m)B^2]D^2\,, \tag{204}$$

$$2a_2G - 3b_2G^2 + 4c_2G^3 + 2dFG + 2eGF^2 = 2(1-m)BHD^2\,, \tag{205}$$

$$2a_2H - 6b_2GH + 12c_2G^2H + 2dFH + 2dAG + 4eFGA + 2eHF^2 = H[(2-m) - 6(1-m)B^2]D^2\,, \tag{206}$$

$$-3b_2H^2 + 12c_2GH^2 + 2dAH + 2eGA^2 + 4eAFH = -3HB[(2-m) - 2(1-m)B^2]D^2\,, \tag{207}$$

$$2c_2H^2 + eA^2 = (B^2-1)[1-(1-m)B^2]D^2\,. \tag{208}$$

Six of these equations determine the six unknowns A, B, H, D, F, G while the other two equations, give two constraints between the eight parameters $a_{1,2}, b_{1,2}, c_{1,2}, d, e$. In particular, A and H are given by

$$A^2 = \frac{(B^2-1)[1-(1-m)B^2]D^2(2c_2-e)}{(4c_1c_2-e^2)}\,, \quad H^2 = \frac{(B^2-1)[1-(1-m)B^2]D^2(2c_1-e)}{(4c_1c_2-e^2)}\,, \tag{209}$$

from where it follows that B^2 must satisfy the bound

$$1 < B^2 < \frac{1}{1-m}\,. \tag{210}$$

So far, we have not been able to obtain explicit solutions to these equations in an elegant form except at $m = 1$ which we now present.

m=1

In case $m = 1$, the solution (200) goes over to the pulse solution

$$\phi = F + \frac{A\mathrm{sech}[D(x+x_0),m]}{1+B\mathrm{sech}[D(x+x_0),m]}\,, \quad \psi = G + \frac{H\mathrm{sech}[D(x+x_0),m]}{1+B\mathrm{sech}[D(x+x_0),m]}\,, \tag{211}$$

where A, H are now given by

$$A^2 = \frac{(B^2-1)D^2(2c_2-e)}{(4c_1c_2-e^2)}\,, \quad H^2 = \frac{(B^2-1)D^2(2c_1-e)}{(4c_1c_2-e^2)}\,. \tag{212}$$

Several solutions are possible in this case, which we discuss one by one.

(i) $F = G = 0, m = 1$

In this case one finds that while A, H are given by Eq. (212), the inverse characteristic length D is given by

$$D^2 = 2a_1\,, \quad a_1 = a_2\,, \tag{213}$$

and B is determined from the relation

$$dH^2 + 6ABa_1 = 3b_1A^2\,. \tag{214}$$

Further, the parameters satisfy the constraint

$$(2d+3b_1)(2c_2-e) - d(2c_1-e) = 3b_2\sqrt{(2c_1-e)(2c_2-e)}\,. \tag{215}$$

Interesting case of $b_2 = c_2 = 0$

In this physically interesting case, one finds that

$$A^2 = \frac{(B^2-1)D^2}{e}\,, \quad H^2 = \frac{(B^2-1)D^2(2c_1-e)}{e^2}\,, \tag{216}$$

while Eq. (213) is still valid. However, B is now given by a simpler expression

$$B^2 = \frac{2d^2}{(2d^2-9ea_1)}\,, \tag{217}$$

and the constraint has a simpler form

$$d(e+2c_1) + 3b_1e = 0\,. \tag{218}$$

Special case of $4c_1c_2 = e^2$

The above solution continues to exist even in the case $4c_1c_2 = e^2$ provided $2c_1 = 2c_2 = e$. Further, while Eq. (213) is still valid, A, H are no more given by Eq. (212), but they satisfy the constraint relations

$$e(A^2+H^2) = (B^2-1)D^2\,, \quad (2d+3b_1)A^2 = dH^2 + 3b_2HA\,, \quad 2(dA+3Ba_1) = 3b_2H\,, \tag{219}$$

from where one can determine A, H and B.

(ii) $G = 0, F \neq 0, m = 1$

In this case one finds that while A, H are given by Eq. (212), D and F are now given by

$$D^2 = 3b_1 F - 4a_1\,, \quad F \equiv \phi_c = \frac{3b_1 + \sqrt{(9b_1^2 - 32a_1c_1)}}{8c_1}\,. \tag{220}$$

From the remaining three equations, one can determine B and further one has two constraints between the various parameters. This solution is also valid in case $b_2, c_2 = 0$ or when $2c_1 = 2c_2 = e$ with appropriate constraints which can be easily worked out.

(iii) $F = 0, G \neq 0, m = 1$

In this case one finds that the solution exists only if $d = 0$. Further, A, H are still given by Eq. (212), while D and G are now given by

$$D^2 = 3b_2 G - 4a_2\,, \quad G \equiv \psi_c = \frac{3b_2 + \sqrt{(9b_2^2 - 32a_2c_2)}}{8c_2}\,. \tag{221}$$

From the remaining three equations, one can determine B and further one has two constraints between the various parameters. This solution is however not valid either in case $b_2, c_2 = 0$ or when $2c_1 = 2c_2 = e$.

Uncoupled case with $m = 1$

Finally, for completeness, we write down the solution in the uncoupled case when $m = 1$. In this case, it is not difficult to show that one of the solution is given by

$$\phi = \frac{A\mathrm{sech}[\sqrt{2a_1}(x + x_0)]}{1 + B\mathrm{sech}[\sqrt{2a_1}(x + x_0)]}\,, \tag{222}$$

where

$$B = \frac{b_1}{\sqrt{b_1^2 - 4a_1c_1}}\,, \quad A = \frac{2a_1}{\sqrt{b_1^2 - 4a_1c_1}}\,. \tag{223}$$

The solution (222) can also be written in the form [24]

$$\phi = \frac{2a_1}{b_1 + \sqrt{(b_1^2 - 4a_1c_1)}\, cosh[\sqrt{2a_1}(x + x_o)]}\,. \tag{224}$$

Note that this solution exists provided $b_1^2 > 4a_1c_1$, i.e. below the transition temperature $T < T_c$. As pointed out at several places in the literature, this solution is a pulse solution around the local minimum $\phi = 0$.

The uncoupled equations also have another solution given by

$$\phi = F + \frac{A\text{sech}[D(x+x_0), m]}{1 + B\text{sech}[D(x+x_0), m]}\,, \tag{225}$$

where

$$F \equiv \phi_c = \frac{3b_1 + \sqrt{(9b_1^2 - 32a_1c_1)}}{8c_1}\,, \quad D^2 = 3b_1F - 4a_1\,,$$
$$B = \frac{4c_1F - b_1}{\sqrt{(b_1^2 - 2b_1c_1F)}}\,, \quad A = -\frac{3b_1F - 4a_1}{\sqrt{(b_1^2 - 2b_1c_1F)}}\,. \tag{226}$$

Note that this solution is valid only if $b_1^2 < 4a_1c_1$, i.e. above the transition temperature $T > T_c$. This follows from the fact that $b_1^2 > 2b_1c_1F$ so that A, B are real. As pointed out at several places in the literature, this solution is a pulse solution around the local minimum $\phi = \phi_c \equiv F$ where ϕ_c is as given by Eq. (3).

Summarizing, we have obtained eleven solutions to the coupled Eq. (2) in case $a_1 > 0$ out of which ten solutions, for the corresponding uncoupled limit, are at $T = T_c$ (i.e. $b_1^2 = 4a_1c_1$) while one solution is for $T > T_C$ or $T < T_c$ depending on the value of the parameters.

5 Solutions when $a_1 < 0$

Before concluding this paper, we consider a few solutions of the coupled Eq. (2) in case $a_1 < 0$. As pointed out in Sec. II, the uncoupled model with $a_1 < 0$ is also a model for first order transition with the temperature now being controlled by the parameter b_1. We will see that most of the solutions which we have obtained in the case $a_1 > 0$, are also valid when $a_1 < 0$ but with suitable modifications of parameters.

Solution I with $F = G = 0$

In Sec. III we saw that with $a_1 > 0$, solution I as given by Eq. (10) is not possible in case $F = G = 0$. However, it turns out that if $a_1 < 0$ then such a solution is indeed possible. A and B are still given by Eq. (19) while D is now given by

$$D^2 = \frac{2|a_1|}{(1+m)}\,, \tag{227}$$

while the three constraints are

$$a_1 = a_2\,, \quad 27b_2^2b_1 = 4d^3\,, \quad \frac{3b_1}{d} = \frac{(2c_1 - e)}{(2c_2 - e)}\,. \tag{228}$$

Solution I with $G=0, F \neq 0$

Similarly, by looking at Eqs. (20) and (21) it is easy to convince oneself that solution (10) with $G=0$ and $F=\phi_{c1}$ or $F=\phi_{c2}$ exists in case $a_1<0$ and the parameters satisfy essentially similar relations to Eqs. (19) to (21) with appropriate modifications. Further, such solutions continue to exist in the special case of $e^2=4c_1c_2$. However, such solutions do not exist in the physically interesting limit of $b_2=c_2=0$. Note that ϕ_{c1}, ϕ_{c2} are as defined by Eqs. (5) and (6).

Solution I with $F=0, G \neq 0$

Similarly, by looking at Eqs. (28) and (29) it is easy to convince oneself that solution (10) with $F=0$ and $G=\psi_m$ exists in case $a_1<0$ and the parameters satisfy essentially similar relations to Eqs. (27) to (29) with appropriate modifications. But, such a solution does not exist either when $e^2=4c_1c_2$ or when $b_2=c_2=0$. Note that ψ_m, ψ_c are essentially same as ϕ_m, ϕ_c but with the replacement of a_1, b_1, c_1 by a_2, b_2, c_2.

Solution II with $F=G=0$

In Sec. III we had seen that solution (38) with $F=G=0$ and $m<1/2$ is not possible in case $a_1>0$. It is however easy to see that if $a_1<0$, then such a solution is clearly possible (with $m<1/2$) provided relations (47) to (49) are satisfied. However, like other solutions of type II (and also III) with $a_1>0$, no solutions of type II with $a_1<0$ are possible in case either $e^2=4c_1c_2$ or if $b_2=c_2=0$.

Solution II with $G=0, F \neq 0$

Similarly, by looking at Eqs. (50) and (51) it is easy to convince oneself that solution (38) with $m>1/2$, $G=0$ and $F=\phi_{c1}$ or $F=\phi_{c2}$ exists in case $a_1<0$ and the parameters satisfy essentially similar relations to Eqs. (47), (50) and (51) with appropriate modifications.

Solution II with $F=0, G \neq 0$

Similarly, by looking at Eqs. (56) and (57) it is easy to convince oneself that solution (38) with $m<1/2$, $F=d=0$ and $G=\psi_m$ exists in case $a_1<0$ and the parameters satisfy essentially similar relations to Eqs. (56), (57) and (59) with appropriate modifications.

Solution III with $G=0, F \neq 0$

Looking at Eqs. (77) and (78) it is easy to convince oneself that solution (65) with $G=0$ and $F=\phi_{c1}$

or $F = \phi_{c2}$ exists for any m $(0 \le m \le 1)$, in case $a_1 < 0$ and the parameters satisfy essentially similar relations to Eqs. (74), (77) and (78) with appropriate modifications.

Solution VIII with $F = G = 0$

In Sec. III we observed that with $a_1 > 0$, solution VIII as given by Eq. (151) is not possible in case $F = G = 0$. However, it turns out that if $a_1 < 0$ then such a solution is indeed possible. A and H are still given by Eq. (161) while the inverse characteristic length D is now given by

$$D^2 = \frac{4|a_1|}{(2-m)}\,, \tag{229}$$

while the three constraints are

$$a_1 = a_2\,, \quad 27b_2^2 b_1 = 4d^3\,, \quad \frac{3b_1}{d} = \frac{(2c_1 - e)}{(2c_2 - e)}\,. \tag{230}$$

Solution VIII with $F = 0, G \neq 0$

While no solutions are possible for $G = 0, F \neq 0$ in case $a_1 < 0$, it turns out that solutions are indeed possible in case $F = 0, G \neq 0, a_1 < 0$ provided $d = 0$ and $B = 1$. In this case A, H are given by Eq. (161) while G and D are given by Eq. (165). Further there are three constraints given by

$$3b_1 A = 4eGH\,, \quad 3H^2(b_2 - 4c_2 G) = 2eGA^2\,, \quad 4(|a_1 - eG^2) = (2-m)D^2\,. \tag{231}$$

Solution IX with $F = G = 0$

In Sec. III we also observed that with $a_1 > 0$, solution IX as given by Eq. (172) is not possible in case $F = G = 0$. However, it turns out that if $a_1 < 0$ then such a solution is indeed possible. A and H are still given by Eq. (161) while D and the three constraints are again given by Eqs. (229) and (230), respectively.

Solution IX with $F = 0, G \neq 0$

Since solutions VIII and IX satisfy same field equations, it is then clear that a solution with $F = 0, G \neq 0, a_1 < 0$ will also exist in this case.

Solution XI

In the last section we obtained a pulse-like periodic solution in case $a_1 > 0$. It is amusing to note that the same solution continues to be valid even when $a_1 < 0$. In particular, note that in the uncoupled limit and at $m = 1$ the solutions for $a_1 < 0$ are again pulse solutions around the local minimum. Specifically, if

$b_1 > (<)\ 0$, then the pulse solution is around ϕ_{c2} (ϕ_{c1}). In both cases, the solution is given by Eq. (225) but with F now being either ϕ_{c1} or ϕ_{c2} where ϕ_{c1}, ϕ_{c2} are as given by Eqs. (5) and (6), respectively, and the values of other parameters like D, B, A, H are again given by Eq. (226) but with appropriate modification.

6 Conclusion

We have systematically provided an exhaustive set of exact periodic domain wall solutions for a coupled asymmetric double well model (with and) without an external field. We found ten solutions at the transition temperature ($T = T_c$) and one solution each for $T > T_c$ and $T < T_c$. For the (physically interesting) special case when $b_2 = c_2 = 0$ there is no nonlinearity in the ψ field. However, due to the coupling with the ϕ field which has explicit nonlinearity, an effective nonlinearity is induced in the ψ field thus enabling soliton-like solutions. When we set the coupling parameters $d = e = 0$ the soliton solutions cease to exist in the ψ field, as expected (for $b_2 = c_2 = 0$). We emphasize that the solutions found here are quite different from the ones we found for the first order transition in the coupled ϕ^6 model [22] or the second order transition in coupled symmetric double well (ϕ^4) model [21].

It would be instructive to explore whether the different solutions reported here in Sec. III are completely disjoint or if there are any possible bifurcations linking them via, for instance, analytical continuation. We have not tried to carry out an explicit stability analysis of various periodic solutions. However, the energy calculations and interaction energy between solitons (for $m \sim 1$) could provide useful insight in this direction. Unlike the coupled ϕ^4 and ϕ^6 cases [21, 22], we have not yet succeeded in finding a solvable discrete analog of an asymmetric double well model.

Our results are relevant for a wide range of physical situations including structural transformations (with coupling of strain and shuffle modes) [9, 10], liquid crystals [3], hydrogen bonded chains [4, 5, 6, 7] and field theoretic contexts [8]. For structural phase transformations the solutions provide novel domain wall arrays, e.g. periodic antiphase boundaries and twin boundaries. Similarly, for hydrogen bonded chains these solutions represent new periodic nonlinear excitations.

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