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PT-Invariant Periodic Potentials with a Finite Number of Band Gaps

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Abstract: We obtain the band edge eigenstates and the mid-band states for the complex, PT-invariant generalized associated Lamé potentials $V^{PT}(x) = -a(a+1)m\,\mathrm{sn}^2(y,m) - b(b+1)m\,\mathrm{sn}^2(y+K(m),m) - f(f+1)m\,\mathrm{sn}^2(y+K(m)+iK'(m),m) - g(g+1)m\,\mathrm{sn}^2(y+iK'(m),m)$, where $y \equiv ix+\beta$, and there are four parameters a,b,f,g. This work is a substantial generalization of previous work with the associated Lamé potentials $V(x) = a(a+1)m\mathrm{sn}^2(x,m) + b(b+1)m\mathrm{sn}^2(x+K(m),m)$ and their corresponding PT-invariant counterparts $V^{PT}(x) = -V(ix+\beta)$, both of which involving just two parameters a,b. We show that for many integer values of a,b,f,g, the PT-invariant potentials $V^{PT}(x)$ are periodic problems with a finite number of band gaps. Further, using supersymmetry, we construct several additional, new, complex, PT-invariant, periodic potentials with a finite number of band gaps. We also point out the intimate connection between the above generalized associated Lamé potential problem and Heun's differential equation.

1 Introduction

In the past few years, Bender and others [1, 2] have looked at several complex potentials with PT-symmetry and have shown that the energy eigenvalues are real when PT-symmetry is unbroken, whereas they occur in complex conjugate pairs when PT-symmetry is spontaneously broken. However, there have been relatively few papers discussing periodic potentials with PT-symmetry [3, 4]. Recently, we [4] have constructed several new classes of analytically solvable, complex, PT-invariant, periodic potentials with the special feature that they possess just a finite number of band gaps. The purpose of this paper is to substantially increase this list of solvable periodic potentials.

A few years ago, we obtained the band edges of the associated Lamé (AL) potentials [5]

$$\begin{aligned} V(x) &= a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\,\mathrm{sn}^2(x+K(m),m) \\ &= a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2(x,m)}\,. \end{aligned} \tag{1}$$

Here, $\mathrm{sn}\,(x,m)$, $\mathrm{cn}\,(x,m)$, $\mathrm{dn}\,(x,m)$ are Jacobi elliptic functions with elliptic modulus parameter m ($0 \le m \le 1$). They are doubly periodic functions with periods $[4K(m), i2K'(m)]$, $[4K(m), 2K(m) + i2K'(m)]$, $[2K(m), i4K'(m)]$ respectively [13], where $K(m) \equiv \int_0^{\pi/2} d\theta[1 - m\sin^2\theta]^{-1/2}$ denotes the complete elliptic integral of the first kind, and $K'(m) \equiv K(1-m)$. For simplicity, from now on, we will not explicitly display the modulus parameter m as an argument of Jacobi elliptic functions. It was shown that the AL potentials with integral values of a, b are periodic potentials with a finite number of band gaps [6]. We also constructed and studied the PT-invariant potentials $V^{PT}(x) \equiv -V(ix+\beta)$ obtained from the AL potentials via the anti-isospectral transformation of variables $x \to ix + \beta$ [4].

In this paper, we make a substantial generalization of our previous work. We consider the four parameter family of generalized associated Lamé (GAL) potentials

$$\begin{aligned} V(x) &= a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\,\mathrm{sn}^2(x+K(m),m) \\ &\quad + f(f+1)m\,\mathrm{sn}^2(x+K(m)+iK'(m),m) + g(g+1)m\,\mathrm{sn}^2(x+iK'(m),m) \\ &= a(a+1)m\,\mathrm{sn}^2(x,m) + b(b+1)m\frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2(x,m)} + f(f+1)\frac{\mathrm{dn}^2(x,m)}{\mathrm{cn}^2(x,m)} + g(g+1)\frac{1}{\mathrm{sn}^2(x,m)}\,. \end{aligned} \tag{2}$$

In contrast to the AL potentials of eq. (1) where there are two parameters a, b and the two terms correspond to real translations of the independent variable x by 0 and $K(m)$, the GAL potentials of eq. (2) have four parameters a, b, f, g and the four terms correspond to complex translations of the independent variable x by $0, K(m), K(m) + iK'(m), iK'(m)$. Although the GAL potentials are real, they do have singularities on the real axis coming from the zeros of the Jacobi elliptic functions $\mathrm{sn}\,(x)$ and $\mathrm{cn}\,(x)$ in the last two terms. Consequently, we will focus our attention on the PT-invariant versions of the GAL potentials, which are given by

$$\begin{aligned} V^{PT}(x) &= -a(a+1)m\mathrm{sn}^2(y,m) - b(b+1)m\mathrm{sn}^2(y+K(m),m) \\ &\qquad - f(f+1)m\mathrm{sn}^2(y+K(m)+iK'(m),m) - g(g+1)m\mathrm{sn}^2(y+iK'(m),m) \\ &= -a(a+1)m\mathrm{sn}^2(y,m) - b(b+1)m\frac{\mathrm{cn}^2(y,m)}{\mathrm{dn}^2(y,m)} - f(f+1)\frac{\mathrm{dn}^2(y,m)}{\mathrm{cn}^2(y,m)} - g(g+1)\frac{1}{\mathrm{sn}^2(y,m)} \\ &\equiv [a(a+1), b(b+1), f(f+1), g(g+1)]\,, \end{aligned} \tag{3}$$

where

$$y = ix + \beta\,, \tag{4}$$

with β being an arbitrary constant. We shall frequently use the notation $[a(a+1), b(b+1), f(f+1), g(g+1)]$ to denote $V^{PT}(x)$. In this notation, PT-invariant ordinary Lamé potentials are denoted by $[a(a+1), 0, 0, 0]$, and PT-invariant AL potentials are denoted by $[a(a+1), b(b+1), 0, 0]$. Here, the arbitrary constant β is chosen so as to avoid the singularities of the Jacobi elliptic functions on the real axis. We show that several of these periodic potentials for specific integer values of a, b, f, g have a finite number of band gaps. Looking at the symmetry of these potentials, we are in fact tempted to conjecture that many (and perhaps all) of these potentials with integral values of the parameters a, b, f, g also have a finite number of band gaps. It would be nice if this conjecture could be proved.

In addition, we also discover a huge class of mid-band states when at least one of the parameters a, b, f, g is half integral. As a special case, we find some new mid-band eigenstates of the associated Lamé potentials. Further, we show that the Schrödinger equation for the generalized AL potential is intimately connected with the celebrated Heun's differential equation [7]. In fact, using the exact solutions obtained in this paper, one can immediately obtain the corresponding solutions of Heun's equation. In another related paper [8], we use this connection and discover a wide class of new quasi-periodic solutions of Heun's equation.

Finally, using the exact eigenstates of the GAL potentials (3) and the machinery of supersymmetric quantum mechanics [9], we construct several more potentials with finite band-gaps. There is one important point involved here using which we are in fact able to construct many more supersymmetric partner potentials corresponding to a given potential. The key point to note is that normally, in supersymmetric quantum mechanics [9], given a potential $V_-(x)$, the ground state wave function $\psi_0(x)$ is used to construct the superpotential $W(x) = -\psi_0'(x)/\psi_0(x)$, which then yields the supersymmetric (SUSY) partner potential $V_+(x) = W^2 + W'$. If one uses any excited state wave function $\psi(x)$ of $V_-(x)$ to construct a superpotential $W(x)$, then the original potential $V_-(x)$ is recovered correctly (by construction), but the corresponding partner potential $V_+(x)$ turns out to be singular on the real x-axis due to the zeros of the excited state wave function $\psi(x)$. However, as has been noticed recently [10], if we consider PT-symmetric complex potentials, then the singularity is not on the real axis. Besides, as we have stressed previously [4, 11], in the case of doubly periodic potentials composed of Jacobi elliptic functions, both $V(x)$ and $V^{PT}(x)$ can be simultaneously periodic even though their periods are different. In this way, by starting from the analytically solvable Lamé and associated Lamé potentials and using the excited state band edges of the corresponding PT-symmetric potentials, we discover a wide range of new, analytically solvable, complex PT-invariant periodic potentials with a finite number of band gaps. As an illustration, we discuss a few of these potentials in detail.

The plan of the paper is the following. In Sec. 2 we discuss the PT-invariant GAL potentials (3) in some detail and obtain band edges as well as mid-band states of several of these potentials. As a byproduct we also obtain some new solutions of the AL potentials (which we had missed in earlier work [5, 6]). Further, we show that the class of potentials $[a(a+1), 0, 0, g(g+1)]$ have finite number of band-gaps in case a, g are integers. In Sec. 3 we start from the energy eigenstates obtained in Sec. 2 and using both the ground state as well as excited state wave functions, obtain new periodic PT-invariant potentials with a finite number of band gaps. In Sec. 4 we briefly discuss the connection between the solutions of the potentials (3) and Heun's differential equation.

2 Solutions for the Generalized Associated Lamé (GAL) Potentials

A few years ago, we obtained analytic solutions of the associated Lamé potentials (1) [5, 6] and showed that when a, b are integers, then the resulting potentials all had a finite number of band gaps. The purpose of this section is to show that the complex PT-invariant GAL potentials as given by eq. (3) are also quasi-exactly solvable. In particular, we show that the band edges or mid-band states of these problems can be obtained depending on whether $a+b+f+g$ (or $a-b-f-g$) is an integer or an arbitrary non-integer number. It should be noted that we are considering PT-invariant potentials (3), since the corresponding real potentials (2) are singular on the real axis.

It may be worthwhile explaining the underlying basic idea here, even though it has been well established by us before [4]. Note that if $\psi(x)$ is a solution of the Schrödinger equation for the real potential $V(x)$ with energy E, then $\psi(ix+\beta)$ is a solution of the Schrödinger equation for the complex potential $-V(ix+\beta)$ with energy $-E$, where β is an arbitrary nonzero constant. The new potential $-V(ix+\beta)$, generated by the anti-isospectral transformation $x \to ix+\beta$ [12], is clearly PT-symmetric and will be denoted by $V^{PT}(x)$. Further, if $\psi(x)$ and $\psi(ix+\beta)$ satisfy appropriate boundary conditions, they are eigenfunctions of $V(x)$ and $V^{PT}(x)$ respectively. The ordering of energy levels for $V^{PT}(x)$ is the opposite of the ordering of energy levels for $V(x)$.

In this paper, our main focus is on the Schrödinger equation ($\hbar = 2m = 1$)

$$-\frac{d^2}{dx^2}\psi(x) + V^{PT}(x)\psi(x) = E\psi(x)\,, \tag{5}$$

where $V^{PT}(x)$ is the potential given by eq. (3). Eq. (5) is called the generalized associated Lamé equation, and we are seeking its eigenstates and mid-band states.

2.1 Symmetries

At this stage, it is worth pointing out the symmetries of the PT-invariant GAL potential (3) and hence the corresponding Schrödinger equation (5).

1. The potential (3) and hence the Schrödinger eq. (5) remains unchanged when any one (or more) of the four parameters a, b, f, g change to $-a-1, -b-1, -f-1, -g-1$ respectively.

2. Under the translation $y \to y + K(m)$, the GAL potential $[a(a+1), b(b+1), f(f+1), g(g+1)]$ goes to the potential $[b(b+1), a(a+1), g(g+1), f(f+1)]$. Hence, both GAL potentials must have the same energy eigenvalues and the corresponding energy eigenfunctions are simply related: $y \to y + K(m)$, i.e.

$$E^{PT}(b, a, g, f; m) = E^{PT}(a, b, f, g; m)\,, \quad \psi(y, b, a, g, f; m) \propto \psi(y + K(m), a, b, f, g; m)\,. \tag{6}$$

3. Similarly, by considering the translations $y \to y + K(m) + iK'(m)$, and $y \to y + iK'(m)$, it is easy to show that

$$E^{PT}(f, g, a, b; m) = E^{PT}(a, b, f, g; m)\,, \quad \psi(y, f, g, a, b; m) \propto \psi(y + K(m) + iK'(m), a, b, f, g; m)\,. \tag{7}$$

$$E^{PT}(g, f, b, a; m) = E^{PT}(a, b, f, g; m)\,, \quad \psi(y, g, f, b, a; m) \propto \psi(y + iK'(m), a, b, f, g; m)\,. \tag{8}$$

Thus, once we obtain the eigenvalues and eigenfunctions of a given GAL potential $[a(a+1), b(b+1), f(f+1), g(g+1)]$, then we immediately know the eigenvalues and eigenfunctions of three other potentials: $[b(b+1), a(a+1), g(g+1), f(f+1)]$, $[f(f+1), g(g+1), a(a+1), b(b+1)]$ and $[g(g+1), f(f+1), b(b+1), a(a+1)]$. Therefore, it suffices to present results for only one of the four potentials.

2.2 Duality Relations

We shall now derive some remarkable relations relating the quasi-exactly solvable eigenvalues and eigenfunctions (corresponding either to the band edges or mid-band states) of two GAL potentials at two different values m and $1-m$ of the modulus parameter.

To that purpose we start from the Schrödinger eq. (5) for the PT-invariant GAL potential (3). On using the relations [13]

$$\begin{aligned}
&\sqrt{m}\,\mathrm{sn}(y,m) = -\mathrm{dn}\,[iy + K'(m) + iK(m), 1-m]\,, \\
&\mathrm{dn}(y,m) = \sqrt{1-m}\,\mathrm{sn}\,[iy + K'(m) + iK(m), 1-m]\,, \\
&\sqrt{m}\,\mathrm{cn}(y,m) = i\sqrt{1-m}\,\mathrm{cn}\,[iy + K'(m) + iK(m), 1-m]\,,
\end{aligned} \tag{9}$$

and defining a new variable $w = iy + K'(m) + iK(m)$, the Schrödinger eq. (5) takes the form

$$\psi''(w) - [a(a+1)(1-m)\mathrm{sn}^2(w,1-m) + g(g+1)(1-m)\frac{\mathrm{cn}^2(w,1-m)}{\mathrm{dn}^2(w,1-m)} f(f+1)\frac{\mathrm{dn}^2(w,1-m)}{\mathrm{cn}^2(w,1-m)}$$
$$+b(b+1)\frac{!}{\mathrm{sn}^2(w,1-m)}]\psi(w) = -[a(a+1)+b(b+1)+f(f+1)+g(g+1)+E]\psi(w)\,. \qquad (10)$$

On comparing eqs. (5) and (10) we then have the remarkable relations

$$E^{PT}(a,b,f,g,m) = -[a(a+1)+b(b+1)+f(f+1)+g(g+1)] - E^{PT}(a,g,f,b,1-m)\,,$$
$$\psi(y,m) \propto \psi(iy + K'(m) + iK(m), 1-m)\,, \qquad (11)$$

which is valid for the QES states corresponding to either the band edges or mid-band states. Note that here, a,b,f,g can be arbitrary (real) numbers and are not restricted to integer values. This is a very powerful relation which has several interesting consequences. One immediate important consequence of eq. (11) is that for arbitrary integer values of a,g, the potential $[a(a+1),0,0,g(g+1)]$ has only a finite number of band-gaps. This happens because, for $f=g=0$, one has

$$E^{PT}(a,b,0,0,m) = -[a(a+1)+b(b+1)] - E^{PT}(a,0,0,g=b,1-m)\,, \qquad (12)$$

so that both the potentials must have the same number of band-edges and band-gaps and we have already proved [6] that the AL potentials have finite number of band gaps in case a,b are integers.

2.3 QES Solutions

Let us now seek solutions of the Schrödinger eq. (5) for the PT-invariant GAL potential (3). On making the ansatz

$$\psi(x) = \mathrm{dn}^{-b}(y)\mathrm{sn}^{-g}(y)\mathrm{cn}^{-f}(y)\phi(y)\,, \quad y = ix + \beta\,, \qquad (13)$$

it is easily shown that ϕ satisfies the equation

$$\phi''(y) + 2[mb\frac{\mathrm{sn}(y)\mathrm{cn}(y)}{\mathrm{dn}(y)} - g\frac{\mathrm{cn}(y)\mathrm{dn}(y)}{\mathrm{sn}(y)} + f\frac{\mathrm{dn}(y)\mathrm{sn}(y)}{\mathrm{cn}(y)}]\phi'(y) + [Qm\mathrm{sn}^2(y) - R]\phi(y) = 0\,, \qquad (14)$$

where

$$Q = (b+g+f)(b+g+f-1) - a(a+1)\,, \quad R = E + (f+g)^2 + m(g+b)^2\,. \qquad (15)$$

It is well known [14] that this is a quasi-exactly solvable (QES) problem. We shall now systematically consider solutions of eq. (14) for several special cases and then finally consider the most general case.

2.4 $b = f = g = 0$

The simplest possibility is when three out of the four parameters a, b, f, g are zero. For example, when $b = f = g = 0$, then the problem reduces to the PT-invariant version of the well studied Lamé potential problem. We might add here that, instead of a, if any one of the other parameters b, f, g is nonzero, one still has a potential which is strictly isospectral to the PT-invariant Lamé potential. It may be noted that while the Lamé potential is a periodic potential with (real) period $2K(m)$, the PT-invariant Lamé potential has real period $2K'(m)$. Further, the band edge eigenvalues, eigenfunctions and the discriminant Δ of $V^{PT}(x)$ are related to those of Lamé potential by [4]

$$E_j^{PT}(m) = -E_{2a-j}(m)\,, \quad \psi_j^{PT}(x,m) \;\propto\; \psi_{2a-j}(ix+\beta,m)\;, \quad j = 0,1,2,\ldots,2a$$
$$\Delta^{PT}(E,m) = \Delta[E + a(a+1), 1-m]\,. \tag{16}$$

From eq. (11), it follows that the PT-invariant Lamé band-edge eigenvalues and eigenfunctions, for integral a satisfy the remarkable relations ($j = 0, 1, 2, ...2a$)

$$E_j^{PT}(m) = -a(a+1) - E_{2a-j}^{PT}(m)\,, \quad \psi_j(y,m) \propto \psi_{2a-j}(iy + K'(m) + iK(m), 1-m)\,. \tag{17}$$

We would like to add here that even the mid-band states satisfy (for half-integral a) relations analogous to (17):

$$E_j(m) = a(a+1) - E_{a-1/2-j}(m)\,, \quad \psi_j(y,m) \propto \psi_{a-1/2-j}(iy + K'(m) + iK(m), 1-m)\,, \tag{18}$$

where $j = 0, 1, 2, ..., a - 1/2$. Note the remarkable fact that for any integer a, all bands and band gaps exchange their role as one goes from the Lamé potential to its PT-invariant version $V^{PT}(x)$ [4].

The next simple possibility is when two of the four parameters a, b, f, g are zero. Here there are three distinct possibilities which we discuss one by one.

2.5 $f = g = 0$

In this case the problem reduces to the PT-invariant AL potential which we have already discussed at great length [5, 6]. Note that if either a or b is zero (or -1), then this potential reduces to the PT-invariant of the Lamé potential. As previously shown by us [6], for arbitrary integral values of a and b, AL potentials are exactly solvable problems with finite number of band-gaps for which one can write down the form of all the band edge eigenfunctions, as we do below. We note here that when $a > b$ are both integers, then there are precisely a bound bands (some of which are unusual in that both the band edges are of the same period), same (a) number of band gaps and all the $2a+1$ band edges are analytically known beyond which there is a continuum band extending to $E = \infty$. Note that if $b > a$, then also there are b bound bands and b band gaps and the corresponding eigenfunctions are simply obtained from the $a > b$ case by the transformation $x \to x + K(m)$ while the $a = b$ case essentially corresponds to the Lamé potential $[a(a+1), 0, 0, 0]$. Without any loss of generality, we shall only consider AL potentials with $a > b$.

The form of the $2a+1$ band edge eigenfunctions of the AL potential depends on whether $a - b$ is an odd or an even integer. For example, when $b = a - 2p - 1\,(p = 0, 1, 2, ...)$, then there are:

p eigenstates of the form $\text{sn}(x)\text{cn}(x)\text{dn}(x)F_{p-1}[\text{sn}^2(x)]$,

$p+1$ eigenstates of the form $\text{dn}^{a-2p}(x)F_p[\text{sn}^2(x)]$,

$a-p$ eigenstates of the form $\text{cn}(x)\text{dn}^{2p+1-a}(x)F_{a-p-1}[\text{sn}^2(x)]$,

$a-p$ eigenstates of the form $\text{sn}(x)(\text{dn}^{2p+1-a}(x)F_{a-p-1}[\text{sn}^2(x)]$.

On the other hand, when $b = a - 2p$, there are:

p eigenstates of the form $\text{cn}(x)\text{dn}^{a-2p+1}(x)F_{p-1}[\text{sn}^2(x)]$,

p eigenstates of the form $\text{sn}(x)\text{dn}^{a-2p+1}(x)F_{p-1}[\text{sn}^2(x)]$,

$a-p$ eigenstates of the form $\text{sn}(x)\text{cn}(x)\text{dn}^{2p-a}(x)F_{a-p-1}[\text{sn}^2(x)]$,

$a-p+1$ eigenstates of the form $\text{dn}^{2p-a}(x)F_{a-p}[\text{sn}^2(x)]$.

Here $F_n[\text{sn}^2(x)]$ denotes a polynomial in $\text{sn}^2(x)$ of order n.

We would like to re-state here that all the eigenstates of the PT-invariant version of the AL potentials are immediately obtained from the known eigenfunctions of the associated Lamé problem and the ordering of energy levels of these is the opposite of the corresponding AL problem. Hence, this is also an exactly solvable problem with a finite number (a) of band gaps and $2a+1$ known band edges when both a, b are integers.

2.6 $b = f = 0$

Following our discussion for the AL case, without any loss of generality we assume here that $a > g$. In this case, one obtains $n+1$ QES solutions when $a+g = n$ (or $g - a = n+1$) with $n = 0, 1, 2, ...$ The QES solutions for $n = 0, 1, 2, 3, 4$ are given in Table 1. In particular, for any choice of $a(a+1)$, Table 1 lists the eigenstates for various values of $g(g+1)$. The general form of these eigenfunctions is obtained from the corresponding AL eigenfunctions as given in Table 3 of [5] by simply interchanging $\mathrm{dn}(y)$ and $\mathrm{sn}(y)$.

A few remarks are in order.

1. Since we are considering the case ($b = f = 0$), the duality relation (11) takes the form

$$E^{PT}(a,0,0,g;m) = -[a(a+1) + g(g+1)] - E^{PT}(a, b = g, 0, 0; 1-m)\,,$$
$$\psi(y,a,0,0,g;m) \propto \psi(iy + K'(m) + iK(m), a, b = g, 0, 0; 1-m)\,, \tag{19}$$

   Using Table 3 of ref. [5] and this duality relation, it is straightforward to obtain all the QES eigenstates, thereby providing an independent check on the results given in Table 1. Further, it follows that for arbitrary integer values of a and g, $[a(a+1), 0, 0, g(g+1)]$ is an exactly solvable potential problem with a finite number (a) of band-gaps. From the duality relation (19), it follows that for integer values of a, g

$$E_j^{PT}(a,0,0,g;m) = -[a(a+1) + g(g+1)] + E_j(a, b = g, 0, 0; 1-m)\,, \tag{20}$$

   and hence the corresponding discriminants Δ are related by

$$\Delta^{PT}(E, m; a, 0, 0, g) = \Delta[E + a(a+1) + g(g+1), 1-m; a, b = g, 0, 0]\,. \tag{21}$$

2. Following the structure of the eigenfunctions of the AL potentials as given above, it is now straightforward to write down the general form of the eigenfunctions for arbitrary value of n. However, to obtain the corresponding eigenvalues, one needs to solve cubic and higher order equations.

3. Under the transformation $y \to y + iK'(m)$ followed by the interchange of a and g (note $b = f = 0$), the Schrödinger eq. (5) for the GAL potential (3) remains unchanged. Thus it follows that under

the interchange of a with g, the eigenvalue spectrum must remain unaltered. Clearly, this is only possible if either the energy eigenvalues remain unchanged under this transformation, or if two of the eigenvalues go into each other. It is easy to verify from Table 1 that the eigenvalues corresponding to the eigenfunctions of period $2iK'(m)$ remain unaltered under $a \to g$ while the other eigenvalues go into each other under this transformation.

4. Similarly, From Table 3 of [5], it is easy to check that for the AL potentials (1), the eigenvalues corresponding to the eigenfunctions of period $2K(m)$ remain unaltered under $a \to b$ while the other eigenvalues go into each other under this transformation. This happens because the AL potentials remain unaltered under the transformation $y \to y + K(m)$ followed by the interchange of a with b.

Summarizing, we have discovered new exactly solvable potential problems with a finite number of band gaps when a, g are arbitrary integers. In fact everything about these potentials can be derived from previously known results for AL potentials.

2.7 $b = g = 0$

In this case, one obtains $n + 1$ QES solutions when $a + f = n$ with $n = 0, 1, 2, ...$ The solutions for $n = 0, 1, 2, 3, 4$ are given in Table 2. In particular, for any choice of $a(a + 1)$, Table 2 lists the eigenstates for various values of $f(f + 1)$. The general form of these eigenfunctions is simply obtained from the corresponding AL eigenfunctions as given in Table 2 of [5] by interchanging $\mathrm{dn}(y)$ and $\mathrm{cn}(y)$.

Some comments are in order at this stage.

1. The form of eigenfunctions for arbitrary value of n is easily written down following the structure of the AL eigenfunctions given in the last section.

2. From eq. (11) it follows that the potential (3) with $b = g = 0$ is a self-dual potential, satisfying

$$E^{PT}_{j_1}(a, f, m) = -[a(a + 1) + f(f + 1)] - E^{PT}_{j_2}(a, f, 1 - m) \,. \tag{22}$$

Using Table 2, it is easily checked that indeed this is true, for any values of a, f. In particular, whereas δ_5, δ_8 are invariant under $m \to 1 - m$, $\delta_6 \leftrightarrow \delta_7$ under the same transformation.

3. Under the transformation $y \to y + K(m) + iK'(m)$ followed by the interchange of a and f (note $b = g = 0$), the Schrödinger eq. (5) for the GAL potential (3) remains unchanged. Thus it follows that under the interchange of a with f, the eigenvalue spectrum must remain unaltered. Clearly, this is only possible if either the energy eigenvalues remain unchanged under this transformation, or if two of the eigenvalues go into each other. It is easy to verify from Table 2 that the eigenvalues corresponding to the eigenfunctions of period $2K(m) + 2iK'(m)$ remain unaltered under $a \to f$ while the other eigenvalues go into each other under this transformation. In particular, while δ_5, δ_8 are invariant under $a \leftrightarrow f$, $\delta_6 \leftrightarrow \delta_7$ under the same transformation.

2.8 $f = 0$

Let us consider the case when only one out of the four parameters a, b, f, g is zero. As an illustration, we discuss the case $f = 0$. In fact, as described below, once we know the eigenstates of this problem, the eigenstates of the other three problems corresponding to either b or g or a equal to zero are immediately obtainable, since the four potentials are related by translations of the independent variable.

For the case $f = 0$, one obtains $\frac{n+2}{2}(\frac{n+1}{2})$ QES solutions when n is even (odd). Here $a + b + g = n$ with $n = 0, 1, 2, ...$ The QES solutions for $n = 0, 1, 2, 3$ are given in Table 3. In particular, for any choice of $a(a+1)$, Table 3 lists the eigenstates for various values of $(b+g)(b+g+1)$.

Some remarks are appropriate.

1. By looking at the structure of the QES eigenfunctions in Table 3, it is easy to write down the nature of eigenfunctions for the general case.

2. From Table 3, it is easily checked that the duality relation

$$E^{PT}(a, b, g, m) = -[a(a+1) + b(b+1) + g(g+1)] - E^{PT}(a, g, b, 1-m) \,. \tag{23}$$

is indeed satisfied. In particular, both δ_9, δ_{10} are invariant under $b \leftrightarrow g$ followed by $m \to 1 - m$.

3. Under the transformation $y \to y + K(m)$ followed by the interchange of a and b, and replacing g by f, the Schrödinger eq. (5) for the GAL potential (3) with $f = 0$ goes over to the Schrödinger equation

for the GAL potential (3) with $g = 0$. Hence, under the interchange of a and b and replacing g by f, all the energy eigenvalues of the potential (3) with $f = 0$ must go over into those of (3) with $g = 0$, while the corresponding eigenfunctions are simply obtained from Table 3 by replacing y by $y+K(m)$.

4. Using similar reasoning it also follows that under the interchange of a with g and replacing b by f, all the energy eigenvalues of the GAL potential (3) with $f = 0$ go over to those of potential (3) with $b = 0$ while the corresponding eigenfunctions are obtained from Table 3 by replacing y by $y+iK'(m)$. And finally, under the interchange of b with g and replacing a by f, all the energy eigenvalues of the GAL potential (3) with $f = 0$ go over to those of potential (3) with $a = 0$, while the corresponding eigenfunctions are easily obtained from Table 3 by replacing y by $y + K(m) + iK'(m)$.

2.9 The General Case: $a,\ b,\ f,\ g$ All Nonzero

Finally, let us discuss the most general case when all the four parameters are nonzero. In this case one obtains $n + 1$ solutions when $a + b + f + g = 2n$ with $n = 0, 1, 2, ...$ The QES solutions for $n = 0, 1$ are given in Table 4.

1. It is easy to see that in the general case, the eigenfunction is of the form

$$\psi = \mathrm{sn}^{-g}(y)\mathrm{cn}^{-f}(y)\mathrm{dn}^{-b}(y)\sum_{k=0}^{n} A_k \mathrm{sn}^{2k}(y)\,, \tag{24}$$

while the corresponding eigenvalues are solutions of a $n + 1$'th order equation.

2. It can be checked from Table 4 that δ_{11} is invariant under $b \leftrightarrow g$ followed by $m \to 1 - m$.

3. The GAL potential (3) and hence the corresponding Schrödinger eq. (5) is invariant under the transformation $y \to y + K(m)$ followed by the interchange of a with b and f with g. Hence, under the interchange of a with b and f with g, all the eigenvalues of the GAL system must either remain invariant or go into each other. In fact it is easily checked from Table 4 that all the eigenvalues are invariant under the interchange of a with b and f with g. Extending this argument, in fact one finds that all the eigenvalues are also invariant under $a \leftrightarrow f, b \leftrightarrow g$ as well as under $a \leftrightarrow g, b \leftrightarrow f$.

2.10 Mid-Band States

So far we have discussed the results for the PT-invariant GAL potentials, which give eigenvalues and eigenfunctions corresponding to the band edges. It may be noted that in all these cases, while a, b, f, g need not be integers, either $a+b+f+g$ or $a-b-f-g$ is always integral. We now show that when at least one of a, b, f, g is half-integral and either $a+b+f+g$ and/or $a-b-f-g$ is an arbitrary number (being an integer is of course a very special case here), then one can obtain doubly degenerate eigenstates which correspond to mid-band states. In fact depending on whether we want b or f or g to be half-integral (with the other two parameters being integral), we need to use different trial solutions. Therefore, we shall consider all three cases one by one.

Case 1: b half-integral

We start from eq. (14) and further substitute the ansatz

$$\phi(y) = [\text{cn}(y) + i\text{sn}(y)]^t Z(y)\,, \tag{25}$$

where t is any real number. After lengthy but straightforward algebra, one can show that $Z(y)$ satisfies the equation

$$\begin{aligned}
&Z''(y) + [2it\text{dn}(y) + 2mb\frac{\text{sn}(y)\text{cn}(y)}{\text{dn}(y)} - 2g\frac{\text{cn}(y)\text{dn}(y)}{\text{sn}(y)} + 2f\frac{\text{dn}(y)\text{sn}(y)}{\text{cn}(y)}]Z'(y) \\
&+[-(R+t^2) + (Q+t^2)m\text{sn}^2(y) - 2itg\frac{\text{cn}(y)}{\text{sn}(y)} + 2itf(1-m)\frac{\text{sn}(y)}{\text{cn}(y)} \\
&+imt(2b+2f+2g-1)\text{sn}(y)\text{cn}(y)]Z(y) = 0\,,
\end{aligned} \tag{26}$$

where R and Q are as given by eq. (15). Not surprisingly, $Z(y)$ =constant is a solution with energy $E = -(4t^2+m)/4$ provided $f = g = 0, b = 1/2, a = t - 1/2$ (i.e. $b+f+g = 1/2$).

One can build solutions for higher values of $b+f+g$ from here. In particular, for $b+f+g = 2M+1/2$, we consider the ansatz ($M = 0, 1, 2, ...$)

$$Z(y) = \sum_{k=0}^{M} A_k \text{sn}^{2k}(y) + \text{cn}(y)\text{sn}(y)\sum_{k=0}^{M-1} B_k \text{sn}^{2k}(y)\,, \tag{27}$$

while if $b+f+g = 2M+3/2$ then we consider the ansatz ($M = 0, 1, 2, ...$)

$$Z(y) = \text{cn}(y)\sum_{k=0}^{M} A_k \text{sn}^{2k}(y) + \text{sn}(y)\sum_{k=0}^{M} B_k \text{sn}^{2k}(y)\,, \tag{28}$$

Substitution into eq. (26) and simplification yields analytic expressions for the energy eigenvalues and eigenfunctions for arbitrary M for $b = 1/2$ and $b = 3/2$. In particular, for $b = 1/2$, we find that

$$b = 1/2, f = p, f + g = N, a = t - 1/2\,, \quad E = -[t^2 + m(g + b)^2]\,, \tag{29}$$

where both f, g are nonnegative integers satisfying $f + g = N$ with $N = 0, 1, 2, ...$

Similarly, when $b = 3/2, a = t - 1/2, f = p, f + g = N$ we find that

$$E = m(2g + 1) - [1 + t^2 + m(g + b)^2] \pm \sqrt{(2g + 1)^2 m^2 + 4m(N + 1)(f - g) + 4(1 - m)t^2} \tag{30}$$

where, f and g are again nonnegative integers. In all these cases, the corresponding eigenfunctions have the form as given above in eqs. (27) and (28). For small values of N, the explicit coefficients A_k, B_k appearing in the eigenfunction expressions can be easily written down. For example, for $b = 1/2$ and $N = 1$, the eigenfunction is $Z(y) = A\mathrm{cn}(y) + B\mathrm{sn}(y)$ with $\frac{B}{A} = it$ in case $f = 1, g = 0$ while $\frac{B}{A} = i$ in case $g = 1, f = 0$.

For the special case of $f = g = 0$ and $t \neq 1/2$, these results represent the generalization of results obtained by us previously [6] in the case of AL potential. Further, for $f = g = 0, t = 1/2$, the results obtained above match with the energy eigenvalue expressions obtained in ref. [6] (as they should!).

Several comments can be readily made.

1. Since, in the variable y, the GAL potential (3) has period $2K(m)$ as well as $2iK'(m)$, hence $\psi(y)$ and $\psi(y + 2K(m))$ as well as $\psi(y + 2iK'(m))$ are all eigenfunctions of GAL equation with the same eigenvalue. As a consequence, $\phi(y) = [\mathrm{cn}(y) - i\mathrm{sn}(y)]^t Z(y)$ is also the eigenfunction with the *same* eigenvalue. Thus for any nonintegral t, each level is doubly-degenerate. The same remark also applies to the other two solutions (when f or g is half integral) discussed below.

2. There is one remarkable symmetry associated with eq. (26). In particular, notice that this equation is invariant under $t \to -t$ followed by $i \to -i$ (where $i = \sqrt{-1}$). But under this transformation, the ansatz (25) becomes

$$\phi(y) = [\mathrm{cn}(y, m) - i\mathrm{sn}(y, m)]^{-t}\,, \tag{31}$$

Hence it follows that the energy eigenvalues must be independent of sign of t, i.e. they must be a function of t^2. Similar remarks also apply in the other two cases discussed below (i.e. when f, g are half-integral).

3. For integral t, both a, b are half integral and these solutions reduce to those discussed in the last section and in that case they correspond to QES band edge eigenstates.

4. Here we have obtained solutions $\psi(y)$ in which $a = t - 1/2, f = p, g = N - p$ and $b = 1/2$ or $3/2$. In view of the symmetries of the GAL potentials, we then also have solutions $\psi(y + K(m))$ with the same energy in case $b = t - 1/2, g = p, f = N - p$ and a is either $1/2$ or $3/2$. Similarly we have solutions $\psi(y + K(m) + iK'(m))$ with the same energy in case $f = t - 1/2, a = p, b = N - p$ and $g = 1/2$ or $3/2$. Further, we also have solutions $\psi(y + iK'(m))$ with the same energy in case $g = t - 1/2, a = N - p, b = p$ and $f = 1/2$ or $3/2$.

Case 2: f half-integral

We start from eq. (14) and further substitute the ansatz

$$\phi(y) = [\mathrm{dn}(y) + ik\mathrm{sn}(y)]^t Z(y)\,, \tag{32}$$

where t is any real number and $k = \sqrt{m}$. After some lengthy but straightforward algebra, one finds that $Z(y)$ satisfies the equation

$$\begin{aligned}
&Z''(y) + [2ikt\mathrm{cn}(y) + 2mb\frac{\mathrm{sn}(y)\mathrm{cn}(y)}{\mathrm{dn}(y)} - 2g\frac{\mathrm{cn}(y)\mathrm{dn}(y)}{\mathrm{sn}(y)} + 2f\frac{\mathrm{dn}(y)\mathrm{sn}(y)}{\mathrm{cn}(y)}]Z'(y) \\
&+[-(R + mt^2) + (Q + t^2)m\mathrm{sn}^2(y) - 2itkg\frac{\mathrm{dn}(y)}{\mathrm{sn}(y)} - 2iktb(1-m)\frac{\mathrm{sn}(y)}{\mathrm{dn}(y)} \\
&+ikt(2b + 2f + 2g - 1)\mathrm{sn}(y)\mathrm{dn}(y)]Z(y) = 0\,,
\end{aligned} \tag{33}$$

where R and Q are as given by eq. (15). Not surprisingly, $Z(y)$ =constant is a solution with energy $E = -(4mt^2 + 1)/4$ provided $b = g = 0, f = 1/2, a = t - 1/2$ (i.e. $b + f + g = 1/2$).

One can build solutions for higher values of $b+f+g$ from here. In particular, in case $b+f+g = 2M+1/2$, we consider the ansatz ($M = 0, 1, 2, ...$)

$$Z(y) = \sum_{k=0}^{M} A_k \mathrm{sn}^{2k}(y) + \mathrm{sn}(y)\mathrm{dn}(y) \sum_{k=0}^{M-1} B_k \mathrm{sn}^{2k}(y)\,, \tag{34}$$

while if $b + f + g = 2M + 3/2$ then we consider the ansatz ($M = 0, 1, 2, ...$)

$$Z(y) = \mathrm{dn}(y) \sum_{k=0}^{M} A_k \mathrm{sn}^{2k}(y) + \mathrm{sn}(y) \sum_{k=0}^{M} B_k \mathrm{sn}^{2k}(y)\,, \tag{35}$$

On substituting this ansatz in eq. (33) and making algebraic simplifications, we obtain analytic expressions for the energy eigenvalues and eigenfunctions for arbitrary M for $f=1/2$ and $f=3/2$. In particular, for $f=1/2$, we find that

$$f=1/2\,,\ b+g=N\,,\ a=t-1/2\,,\quad E=-[mt^2+(g+f)^2]\,, \tag{36}$$

where both b, g are nonnegative integers satisfying $b+g=N$ with $N=0,1,2,....$

Similarly, when $f=3/2, a=t-1/2, b+g=N$ we find that

$$E=(2g+1)-[(1+t^2)m+(g+f)^2]\pm\sqrt{(2g+1)^2+4m(N+1)(f-g)-4m(1-m)t^2} \tag{37}$$

where, b and g are again nonnegative integers. In all these cases, the corresponding eigenfunctions have the form as given above in eqs. (34) and (35). For small values of N, the explicit coefficients A_k, B_k in the eigenfunction expressions can be easily written down. Further, as in the half-integral b case, one can write down three more solutions with the same energy.

Case 3: g half-integral

We start from eq. (14) and further substitute the ansatz

$$\phi(y)=[\mathrm{dn}(y)+k\mathrm{cn}(y)]^t Z(y)\,, \tag{38}$$

where t is any real number. After algebraic simplification, it is easy to show that $Z(y)$ satisfies the equation

$$\begin{aligned}&Z''(y)+[-2kt\mathrm{sn}(y)+2mb\frac{\mathrm{sn}(y)\mathrm{cn}(y)}{\mathrm{dn}(y)}-2g\frac{\mathrm{cn}(y)\mathrm{dn}(y)}{\mathrm{sn}(y)}+2f\frac{\mathrm{dn}(y)\mathrm{sn}(y)}{\mathrm{cn}(y)}]Z'(y)\\&+[-R+(Q+t^2)m\mathrm{sn}^2(y)-2ktb\frac{cn(y)}{\mathrm{dn}(y)}-2ktf\frac{\mathrm{dn}(y)}{\mathrm{cn}(y)}\\&+kt(2b+2f+2g-1)\mathrm{cn}(y)\mathrm{dn}(y)]Z(y)=0\,,\end{aligned} \tag{39}$$

where R and Q are as given by eq. (15). Not surprisingly, $Z(y)=$ constant is a solution with energy $E=-(1+m)/4$ provided $b=f=0, g=1/2, a=t-1/2$ (i.e. $b+f+g=1/2$).

One can build solutions for higher values of $b+f+g$ from here. In particular, in case $b+f+g=2M+1/2$, we consider the ansatz $(M=0,1,2,...)$

$$Z(y)=\sum_{k=0}^{M}A_k\mathrm{sn}^{2k}(y)+\mathrm{cn}(y)\mathrm{dn}(y)\sum_{k=0}^{M-1}B_k\mathrm{sn}^{2k}(y)\,, \tag{40}$$

while if $b+f+g = 2M+3/2$ then we consider the ansatz ($M = 0, 1, 2, ...$)

$$Z(y) = \mathrm{cn}(y,m)\sum_{k=0}^{M} A_k \mathrm{sn}^{2k}(y) + \mathrm{dn}(y)\sum_{k=0}^{M} B_k \mathrm{sn}^{2k}(y)\,, \tag{41}$$

Substituting this ansatz in eq. (39) and simplifying, one gets analytic expressions for the energy eigenvalues and eigenfunctions for arbitrary M for $b = 1/2$ and $b = 3/2$. In particular, for $b = 1/2$, we find that

$$g = 1/2\,,\ \ b+f = N\,,\ \ a = t-1/2\,,\quad E = -[(f+g)^2 + m(g+b)^2]\,, \tag{42}$$

where both b, f are nonnegative integers satisfying $b+f = N$ with $N = 0, 1, 2, ...$

Similarly, when $g = 3/2, a = t-1/2, b+f = N$ we find that

$$E = 1+2f+(2b+1)m - [(f+g)^2 + m(g+b)^2] \pm \sqrt{(1-m)[(2f+1)^2-(2b+1)^2m]+4mt^2} \tag{43}$$

where, b and f are again nonnegative integers. In all these cases, the corresponding eigenfunctions have the form as given above in eqs. (40) and (41). For small values of N, the coefficients A_k, B_k appearing in the eigenfunctions can be easily written down. Further, as in the half-integral b case, one can write down three more solutions with the same energy.

3 Supersymmetry and Potentials with a Finite Number of Band Gaps

We shall now start with the ground state as well as the excited state eigenfunctions of various PT-invariant GAL potentials discussed in the last section and using supersymmetry obtain the corresponding SUSY partner potentials. In this manner, we obtain many new periodic potentials $V_+(x)$ with a finite number of band gaps. As emphasized in the introduction, unlike real potentials, if we take a complex PT-invariant potential, then even if we start with an excited state wave function and calculate the corresponding superpotential W, the singularities in W and hence $V_+(x)$ are not on the real axis, and do not cause problems.

3.1 Supersymmetry Partners of PT-Invariant Lamé Potentials

The simplest case is when only one parameter, (say a) is nonzero. This gives the PT-invariant Lamé potential

$$V(x) = -a(a+1)m\text{sn}^2(y)\,. \tag{44}$$

For concreteness, take $a = 1$, which yields $V(x) = -2m\text{sn}^2(y)$. Here, the three band edge eigenfunctions (in order of increasing energy eigenvalues) are $\text{sn}(y), \text{cn}(y), \text{dn}(y)$. It is easily computed that corresponding to these three eigenstates, the corresponding partner potentials (up to a constant) are $V_+(x) = -2m\text{sn}^2(y + K(m)), -2m\text{sn}^2(y + iK'(m)), -2m\text{sn}^2(y + K(m) + iK'(m))$ which are all strictly isospectral potentials to the original Lamé potential. Thus, in this case we do not obtain any new solvable potentials by using supersymmetry.

Now consider the case $a = 2$. All the five band edge eigenvalues and eigenfunctions of the PT-invariant Lamé potential $V(x) = -6m\text{sn}^2(y)$ have already been given by us in Table 4 of ref. [11]. Starting from any of the five band edge eigenfunctions and calculating the corresponding superpotentials, we obtain five different supersymmetric partner potentials all of which have the same band edge energy eigenvalues as given in Table 4 of ref. [11]. In Table 5 we have given the expressions for these five different strictly isospectral potentials. It is worth noting that out of these five potentials, three are self-isospectral - they are the PT-invariant GAL potentials $[2, 2, 2, 0]$. Hence, truly speaking, we only have three genuinely different potentials, all having the same band edge energies. For each of these cases, using the formalism of supersymmetric quantum mechanics [9], we can easily obtain expressions for the corresponding five eigenstates. Now, again by starting from these eigenfunctions, we can construct still different partner potentials but with identical band edges. In this way, one could construct a large number of periodic potentials with five band edges and two band gaps, all strictly isospectral to the PT-invariant Lamé potential (44) with $a = 2$.

Similarly, if we consider the PT invariant Lamé potential (44) with $a = 3$, then we have 7 band edge eigenfunctions and eigenvalues all of which are analytically known and are given in Table 1 of ref. [4]. Again, using supersymmetry, we can obtain seven different partner potentials V_+ all with the same band edge eigenvalues. By starting from any one of them and using other eigenfunctions recursively, we can in principle construct a huge class of new isospectral potentials. Particular mention may be made of the case when we start from the eigenfunction $\mathrm{sn}(y)\mathrm{cn}(y)\mathrm{dn}(y)$ of the potential $V(x) = -12m\,\mathrm{sn}^2(y)$. It is easily shown that the corresponding partner potential V_+ (up to a constant) is given by

$$V(x) = -m[6\mathrm{sn}^2(y) + 2\mathrm{sn}^2(y + K(m)) + 2\mathrm{sn}^2(y + iK'(m)) + 2\mathrm{sn}^2(y + K(m) + iK'(m))]\,. \qquad (45)$$

Thus, we see that the PT-invariant GAL potential $[6, 2, 2, 2]$ has precisely three bands, three band gaps and seven band edges, since it is the supersymmetric partner of the PT-invariant Lamé potential (44) with $a = 3$. The process described above is readily extended to any Lamé potential with integer a. We can start from any of the $2a + 1$ band edges and obtain the corresponding supersymmetric partner potentials all having the same band edges.

We have shown that the SUSY partners of the PT-invariant Lamé potentials $[6, 0, 0, 0]$ and $[12, 0, 0, 0]$ are the potentials $[2, 2, 2, 0]$ and $[6, 2, 2, 2]$ respectively. What about the higher Lamé potentials? In this connection, it is amusing to notice that the band edges of the PT-invariant Lamé potential $[20, 0, 0, 0]$ and the potential $[6, 6, 6, 2]$ (which follow from Table 4) are identical. For example, out of 9 band edges, the 6 band edge energy eigenvalues of $[20, 0, 0, 0]$ are given by

$$E = -5(m+2) \pm \sqrt{4m^2 - 9m + 9}\,, \quad E = -5(1+m) \pm 2\sqrt{4m^2 + m + 4}\,, \quad E = 5(1+2m) \pm 2\sqrt{9m^2 - 9m + 4}\,. \qquad (46)$$

It is easily seen from Table 4 that exactly the same eigenvalues are obtained when a, b, f, g take the values $(2, 2, -3, 1), (2, -3, 2, 1), (-3, 2, 2, 1)$. Similarly, one can show that the three remaining eigenvalues of $[20, 0, 0, 0]$ satisfy the same cubic equation as $[6, 6, 6, 2]$ when a, b, f, g take the values $(2, 2, 2, -2)$.

In fact, one can show that the number (and structure) of band edges of the PT-invariant Lamé potential $[2a(2a+1), 0, 0, 0]$ is same as the QES states of the potential $[a(a+1), a(a+1), a(a+1), (a-1)a]$. For example, for this PT-invariant Lamé potential it is well known that out of the $4a+1$ band edges of the Lamé potential, a states each are of the form $\mathrm{cn}(y)\mathrm{sn}(y)F_{a-1}(\mathrm{sn}^2(y))$, $\mathrm{cn}(y)\mathrm{dn}(y)F_{a-1}(\mathrm{sn}^2(y))$, $\mathrm{dn}(y)\mathrm{sn}(y)F_{a-1}(\mathrm{sn}^2(y))$, while the remaining $a+1$ states are of the form $F_a(\mathrm{sn}^2(y))$. Using Table 4, it is easily shown that there are again $4a + 1$ QES states of the potential $[a(a + 1), a(a + 1), a(a + 1), (a - 1)a]$, out of which a QES states each are obtained when a, b, f, g are of the form $a, a, a - 1, a - 1$, or $a, a - 1, a, a - 1$, or $a - 1, a, a, a - 1$, while $a+1$ QES states are obtained when a, b, f, g are of the form $a, a, a, -a$. In fact we believe that all the band edge eigenvalues of the potentials $[2a(2a+1), 0, 0, 0]$ and $[a(a+1), a(a+1), a(a+1), (a-1)a]$ are identical. While this is easily shown for low values of a, at the moment, a general proof is still lacking.

Similarly, one can show that the number (as well as the structure) of band edges of the PT-invariant Lamé potential $[(2a-1)2a, 0, 0, 0]$ is the same as the QES states of the potential $[a(a+1), (a-1)a, (a-1)a, (a-1)a]$. For example, it is well known that out of the $4a-1$ band edges of the PT-invariant Lamé potential, a states each are of the form $\text{cn}(y)F_{a-1}(\text{sn}^2(y))$, $\text{dn}(y)F_{a-1}(\text{sn}^2(y))$, $\text{sn}(y)F_{a-1}(\text{sn}^2(y))$, while the remaining $a-1$ states are of the form $\text{sn}(y)\text{cn}(y)\text{dn}(y)F_{a-2}(\text{sn}^2(y)$. Using Table 4, it is easily shown that there are $4a-1$ QES states of the potential $[a(a+1), (a-1)a, (a-1)a, (a-1)a]$, out of which a QES states each are obtained when a, b, f, g are of the form $a, -a, a-1, a-1$, or $a, a-1, -a, a-1$, or $a, a-1, a-1, -a$, while $a-1$ QES states are obtained when a, b, f, g are of the form $-a-1, a-1, a-1, a-1$. In fact we believe that all the band edge eigenvalues of the potentials $[(2a-1)2a, 0, 0, 0]$ and $[a(a+1), (a-1)a, (a-1)a, (a-1)a]$ are identical. While this is easily shown for low values of a, a general proof is not available.

On the basis of these results, we then conjecture that the potentials $[a(a+1), a(a+1), a(a+1), (a-1)a]$, for integer a, have the same band edges as the Lamé potential $[2a(2a+1), 0, 0, 0]$ and hence these potentials also have precisely $2a$ band gaps and $(4a+1)$ band edges, all of which are known in principle. Further, the potentials $[a(a+1), (a-1)a, (a-1)a, (a-1)a]$ have the same band edges as the Lamé potential $[(2a-1)2a, 0, 0, 0]$ and hence are also potentials with a finite number $(2a-1)$ of band gaps. It would be nice to have a general proof.

3.2 Supersymmetry Partners of PT-Invariant Associated Lamé Potentials

We start our discussion with the $a=2$, $b=1$ associated Lamé potential and its corresponding PT-invariant potential $V^{PT}(x) = -6m\text{sn}^2(y) - 2m\text{cn}^2(y)/\text{dn}^2(y)$. All five band-edge eigenvalues and eigenfunctions for this potential have been given by us in Table 3 of ref. [11]. As established previously [5, 11], this is a self-isospectral potential and hence using the band edge eigenfunction $\text{dn}^2(y)$ does not give any new partner potential. However, if instead we use the remaining four band edge eigenfunctions, then one gets four new SUSY partner potentials which are strictly isospectral to the PT-invariant [6,2,0,0] potential.

Let us now consider the PT-invariant AL potential $[a(a+1),(a-2)(a-1),0,0]$, i.e. the potential (3) with $b=a-2, f=g=0$. As shown by us [5], one of its exact band edge eigenfunction is $\psi(x)=\text{cn}(y)\text{dn}^{a-1}(y)$. It is easy to see that the corresponding partner potential V_+ (up to a constant) is the potential $[(a-1)a,(a-1)a,2,0]$. Thus we immediately conclude that the PT-invariant potential $[(a-1)a,(a-1)a,2,0]$ is strictly isospectral to the PT-invariant AL potential $[a(a+1),(a-2)(a-1),0,0]$. In the special case when both a,b are integers, in view of our results on AL potentials [6], it then follows that the GAL potential $[(a-1)a,(a-1)a,2,0]$ has a band gaps and a bands, out of which $b=a-2$ bands are rather unusual.

Note that if instead we use $\psi(x)=\text{sn}(y)\text{dn}^{a-1}(y,m)$, which is also one of the exact eigenfunctions of the above AL potential, then nothing new is obtained. In particular, the corresponding partner potential is $[(a-1)a,(a-1)a,0,2]$ which is strictly isospectral to the potential $[(a-1)a,(a-1)a,2,0]$.

Let us now consider the PT-invariant AL potential $[a(a+1),(a-3)(a-2),0,0]$, i.e. the potential (3) with $b=a-3, f=g=0$. As shown by us [5], one of its exact band edge eigenfunction is $\psi(x)=\text{sn}(y)\text{cn}(y)\text{dn}^{a-2}(y)$. It is easy to see that the corresponding partner potential V_+ (up to a constant) is the potential $[(a-1)a,(a-2)(a-1),2,2]$ in the notation of (3). Thus we immediately conclude that when a,b are integers, then this PT-invariant potential is strictly isospectral to the AL potential $[a(a+1),(a-3)(a-2),0,0]$, has a band gaps and a bands, out of which $b=a-3$ bands are rather unusual.

We can generalize the above arguments. In particular, we find that the number (and even structure) of the potentials $[(a-p)(a-p+1),(a-p-1)(a-p),p(p+1),p(p+1)]$ is the same as the AL potentials $[a(a+1),(a-2p-1)(a-2p),0,0]$. For example, as remarked in the previous section, if $b=a-2p-1 (p=0,1,2,...)$, then there are p eigenstates of the form $\text{sn}(y)\text{cn}(y)\text{dn}(y)F_{p-1}(\text{sn}^2(y))$, $p+1$ eigenstates of the form $\text{dn}^{a-2p}(y)F_p(\text{sn}^2(y))$, $a-p$ eigenstates of the form $\text{cn}(y)\text{dn}^{2p+1-a}(y)F_{a-p-1}(\text{sn}^2(y))$ and also $a-p$ eigenstates of the form $\text{sn}(y)\text{dn}^{2p+1-a}(y)F_{a-p-1}(\text{sn}^2(y))$. Using Table 4 it is easy to show that for the GAL potential, $[(a-p)(a-p+1),(a-p-1)(a-p),p(p+1),p(p+1)]$, there are p eigenstates of the form $\text{sn}^{-p}(y)\text{cn}^{-p}(y)\ \text{dn}^{1+p-a}(y)F_{p-1}(\text{sn}^2(y))$, $p+1$ eigenstates of the form $\text{dn}^{a-p}(y)\text{cn}^{-p}(y)\ \text{sn}^{-p}(y)F_p(\text{sn}^2(y))$, $a-p$ eigenstates of the form $\text{cn}^{p+1}(y)\text{sn}^{-p}(y)\ \text{dn}^{p+1-a}(y)\ F_{a-p-1}(\text{sn}^2(y))$ and also $a-p$ eigenstates of the form $\text{sn}^{p+1}(y)\text{cn}^{-p}(y)\ \text{dn}^{p+1-a}(y)F_{a-p-1}(\text{sn}^2(y))$. In fact we believe that all the band edge eigenvalues of the potentials $[a(a+1),(a-2p)(a-2p+1),0,0]$ and $[(a-p)(a-p+1),(a-p-1)(a-p),p(p+1),p(p+1)]$

are identical. While this is easily shown for low values of a and p, a general proof is still lacking.

Similarly, we can show that the number (and even structure) of the potentials $[(a-p)(a-p+1),(a-p)(a-p+1),p(p+1),(p-1)p]$ is the same as the AL potentials $[a(a+1),(a-2p)(a-2p+1),0,0]$. In particular, for the AL potential, as shown in Sec. 2, when $b=a-2p$, then there are p eigenstates of the form $\mathrm{cn}(x)\mathrm{dn}^{a-2p+1}(x)F_{p-1}(\mathrm{sn}^2(x))$, p eigenstates of the form $\mathrm{sn}(x)\mathrm{dn}^{a-2p+1}(x)F_{p-1}(\mathrm{sn}^2(x))$, $a-p$ eigenstates of the form $\mathrm{sn}(x)\mathrm{cn}(x)\mathrm{dn}^{2p-a}(x)F_{a-p-1}(\mathrm{sn}^2(x))$, and $a-p+1$ eigenstates of the form $\mathrm{dn}^{2p-a}(x)F_{a-p}(\mathrm{sn}^2(x))$. It is easily shown that for the potential $[(a-p)(a-p+1),(a-p)(a-p+1),p(p+1),(p-1)p]$, there are $4a-1$ QES states of similar form. In particular, there are p eigenstates of the form $\mathrm{sn}^{-p}(y)\mathrm{cn}^{1-p}(y)\ \mathrm{dn}^{p-a}(y)F_{p-1}(\mathrm{sn}^2(y))$, p eigenstates of the form $\mathrm{dn}^{a+1-p}(y)\mathrm{cn}^{1-p}(y)\ \mathrm{sn}^{-p}(y)F_{p-1}(\mathrm{sn}^2(y))$, $a-p+1$ eigenstates of the form $\mathrm{cn}^{p}(y)\mathrm{sn}^{-p}(y)\ \mathrm{dn}^{p-a}(y)F_{a-p}(\mathrm{sn}^2(y))$ and also $a-p$ eigenstates of the form $\mathrm{sn}^{p+1}(y)\mathrm{cn}^{1-p}(y)\ \mathrm{dn}^{p-a}(y)\ F_{a-p-1}(\mathrm{sn}^2(y))$. In fact we believe that all the band edge eigenvalues of the potentials $[a(a+1),(a-2p)(a-2p+1),0,0]$ and $[(a-p)(a-p+1),(a-p)(a-p+1),p(p+1),(p-1)p]$ are identical. While this is easily shown for low values of a and p, we don't yet have a general proof.

3.3 SUSY Partners of Potentials with $b=f=0$

Let us now consider the SUSY partners of the potential $[a(a+1),0,0,g(g+1)]$ which for integral values of a,g, is a problem with a finite number of band gaps. By exactly following the above discussion about the PT-invariant AL potential, we can construct a host of new potentials with a finite number of band gaps. For example, by starting from the potential $[6,0,0,2]$ and following the procedure as in the AL case, we can easily obtain four new SUSY partner potentials, all with two band gaps.

From Table 1 we observe that for integral a, two of the exact eigenfunctions of the potential $[a(a+1),0,0,(a-2)(a-1)]$ with a band gaps are $\mathrm{cn}(y)\mathrm{sn}^{a-2}(y)$ and $\mathrm{dn}(y)\mathrm{sn}^{a-2}(y)$. It is easily seen that if we start with either of these eigenfunctions, then the corresponding SUSY partner potential with the same finite (a) number of band gaps is the potential $[(a-1)a,2,,0,(a-1)a]$ (or its isospectral partner $[(a-1)a,0,2,(a-1)a]$).

From Table 1 we also observe that one of the exact eigenfunction of the potential $[a(a+1),0,0,(a-3)(a-2)]$ is $\mathrm{cn}(y,m)\mathrm{dn}(y,m)\mathrm{sn}^{a-3}(y,m)$. On starting with this eigenfunction, it is easily shown that the corresponding SUSY partner potential is $[(a-1)a, 2, 2, (a-2)(a-1)]$ which therefore must also be a potential with finite (a) number of band-gaps in case a is an integer.

Similarly, by starting from the finite band-gap potentials $[a(a+1), 0, 0, (a-2p-1)(a-2p)]$ as well as $[a(a+1), 0, 0, (a-2p)(a-2p+1)]$, and following the discussion in the case of PT-invariant AL potential, it is easily shown that the corresponding SUSY partners with the same (finite) number of band gaps are the potentials $[(a-p)(a-p+1), p(p+1), p(p+1), (a-p-1)(a-p)]$ and $[(a-p)(a-p+1), p(p+1), (p-1)p, (a-p)(a-p+1)]$ and respectively, where a and p are positive integers.

3.4 SUSY Partners of Potentials with $b = g = 0$

Let us now consider the SUSY partners of the PT-invariant potential $[a(a+1), 0, f(f+1), 0]$. From Table 2 we observe that two of the exact eigenfunctions of the potential $[a(a+1), 0, (a-2)(a-1), 0]$ are $\mathrm{sn}(y)\mathrm{cn}^{a-2}(y)$ and $\mathrm{dn}(y)\mathrm{cn}^{a-2}(y)$. It is easily seen that if we start with either of these eigenfunctions, then the corresponding SUSY partner potentials turn out to be $[(a-1)a, 2, (a-1)a, 0]$ or $[(a-1)a, 0, (a-1)a, 2]$. Since we know that the potentials $[a(a+1), a(a-1), 2, 0]$ as well as $[a(a+1), 2, 0, (a-1)a]$ have a finite number of band gaps, we conjecture that maybe the potential $[a(a+1), 2, (a-1)a, 0]$ also has only a finite number (a) of band gaps when a is an integer.

From Table 2 we also observe that one of the exact eigenfunctions of the potential $[a(a+1), 0, (a-3)(a-2), 0]$ is $\mathrm{sn}(y)\mathrm{dn}(y)\mathrm{cn}^{a-3}(y)$. Starting with this eigenfunction, it is easily shown that the corresponding SUSY partner potential is $[(a-1)a, 2, (a-2)(a-1), 2]$. Again, since for integer a, the potential $[(a-1)a, (a-2)(a-1), 2, 2]$ has only a finite number of band gaps, it is tempting to conjecture that the same may also be true for the potential $[(a-1)a, 2, (a-2)(a-1), 2]$.

Similarly, by starting from the finite band-gap potentials $[a(a+1), 0, (a-2p-1)(a-2p), 0]$ as well as $[a(a+1), 0, (a-2p)(a-2p+1), 0]$, and following the discussion in the case of PT-invariant AL potentials, it is easily shown that the corresponding SUSY partners with the same number of band gaps are the GAL potentials $[(a-p)(a-p+1), p(p+1), p(p+1), (a-p-1)(a-p)]$ and $[(a-p)(a-p+1), p(p+1), (p-1)p, (a-p)(a-p+1)]$ respectively when a and p are integers.

3.5 SUSY Partners of Potentials with $f = 0$

Let us now consider the SUSY partners of the potential $[a(a+1), b(b+1), 0, g(g+1)]$. From Table 3 we observe that one of the exact eigenfunctions is $\mathrm{dn}^{-b}(y)\mathrm{sn}^{-g}(y)$ when $a + b + g = 0$. If we start with this eigenfunction, then the corresponding SUSY partner potential turns out to be $[(a-1)a, (b-1)b, 0, (g-1)g]$.

From Table 3 we also observe that an exact eigenfunction of the potential $[a(a+1), b(b+1), 0, g(g+1)]$ is $\mathrm{cn}(y)\mathrm{dn}^{-b}(y)\mathrm{sn}^{-g}(y)$ when $a + b + g = 1$. Starting with this eigenfunction, it is easily shown that the corresponding SUSY partner potential is $[(a-1)a, (b-1)b, 2, (g-1)g]$.

In summary, we have discovered a large number of complex PT-invariant periodic potentials with a finite number of band gaps, many occurring when the parameters a, b, c, d have specific integer values. This leads us to make the plausible conjecture that all GAL potentials (3) for integer values of a, b, f, g have a finite number of band-gaps, but there is as yet no formal proof.

4 Heun's Equation and the Generalized Associated Lamé Equation

In this section, we point out an interesting connection between Heun's differential equation [7] and the generalized associated Lamé equation (5). This connection enables us to use the various solutions of eq. (5) obtained in this paper to write down several solutions of Heun's equation which have apparently not been studied in the mathematics literature.

The canonical form of Heun's equation is given by [7]

$$\left[\frac{d^2}{dx^2} + (\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-c})\frac{d}{dx} + \frac{\alpha\beta x - q}{x(x-1)(x-c)}\right]G(x) = 0\,, \tag{47}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, q, c$ are real parameters, except that $c \neq 0, 1$ and the first five parameters are constrained by the relation

$$\gamma + \delta + \epsilon = \alpha + \beta + 1\,. \tag{48}$$

If we make the transformation $x = \mathrm{sn}^2(y, m)$, then Heun's equation takes the form [7]

$$\begin{aligned} &F''(y) + [(1-2\epsilon)m\frac{\mathrm{sn}(y)\mathrm{cn}(y)}{\mathrm{dn}(y)} + (1-2\delta)\frac{\mathrm{sn}(y)\mathrm{dn}(y)}{\mathrm{cn}(y)} + (2\gamma-1)\frac{\mathrm{cn}(y)\mathrm{dn}(y)}{\mathrm{sn}(y)}]F'(y) \\ &-[4mq - 4\alpha\beta m\mathrm{sn}^2(y)]F(y) = 0\,, \end{aligned} \tag{49}$$

where $[G(x) \equiv F(y)]$ and $m = 1/c$. It is interesting to note that eq. (49) is very similar to the ϕ equation (14) which we have analyzed in great detail. In particular, with the identification

$$b = \frac{1}{2} - \epsilon\,,\ f = \frac{1}{2} - \delta\,,\ g = \frac{1}{2} - \gamma\,,\ b + f + g = \frac{1}{2} - \alpha - \beta, 4\alpha\beta = Q\,, 4mq = R\,, \tag{50}$$

all the results discussed above can be immediately used to obtain different solutions of Heun's equation. It turns out that using the mid-band states obtained in Sec. 2, one generates new quasi-periodic solutions of Heun's eq. (49), which we discuss in a separate publication [8].

References

[1] C.M. Bender and S. Boettcher, Phys. Rev. Lett. **80** (1998) 5243. For a recent review of this field see, C.M. Bender, D.C. Brody and H.F. Jones, Amer. Jour. Phys. **71** (2003) 1095 and references therein.

[2] C.M. Bender and S. Boettcher, Jour. Phys. **A31** (1998) L273; C.M. Bender, S. Boettcher and P.N. Meisinger, Jour. Math. Phys. **40** (1999) 2210; F.M. Fernandez, R. Guardiola, J. Ros and M. Znojil, Jour. Phys. **A32** (1999) 3105 ; M. Znojil, Jour. Phys. **A32** (1999) 4563 and Phys. Lett. **A264** (1999) 108 ; F. Cannata, G. Junker and J. Trost, Phys. Lett. **A246** (1998) 219 ; B. Bagchi and R. Roychoudhury, Jour. Phys. **A33** (2000) L1 ; P. Dorey, C. Dunning and R. Tateo, Jour. Phys. **A34** (2001) L391 and **A34** (2001) 5679; A. Khare and B.P. Mandal, Phys. Lett. **A272** (2000) 53; G. Levai and M. Znojil, Jour. Phys. **A33** (2000) 7165; B. Bagchi and C. Quesne, Phys. Lett. **A300** (2002) 18; A. Mostafazadeh, Jour. Math. Phys. **43** (2003) 205, 2814, 3944; Z. Ahmad, Phys. Lett. **A290** (2001) 19; R.N. Deb, A. Khare and B. Dutta Roy, Phys. Lett. **A307** (2003) 215.

[3] C.M. Bender, G.V. Dunne and P.N. Meisinger, Phys. Lett. **A252** (1999) 272; Z. Ahmad, Phys. Lett. **A286** (2001) 231; J.M. Cerveró, Phys. Lett. **A317** (2003) 26; J.M. Cerveró and A. Rodriguez, Jour. Phys. **A37** (2004) 10167.

[4] A. Khare and U. Sukhatme, Phys. Lett. **A324** (2004) 406.

[5] A. Khare and U. Sukhatme, Jour. Math. Phys. **40** (1999) 5473.

[6] A. Khare and U. Sukhatme, Jour. Math. Phys. **42** (2001) 5652.

[7] For an excellent up to date mathematical summary of Heun's equation, see for example, A. Ronveaux (ed.), *Heun's Differential Equation* (Oxford Univ. Press, 1995); R.S. Maier, math.CA/0408317; A. Erdélyi et al. (ed.), *Higher Transcendental Functions* (Bateman Manuscript Project) Vol. III (McGraw-Hill, 1955).

[8] A. Khare and U.P. Sukhatme, *Quasi-Periodic Solutions of Heun's Equation*, in preparation, May 2005.

[9] See, for example, F. Cooper, A. Khare and U.P. Sukhatme, *Supersymmetry in Quantum Mechanics* (World Scientific, Singapore, 2001) and detailed references contained therein.

[10] A. Sinha and P. Roy, quant-ph/0312085.

[11] A. Khare and U. Sukhatme, Jour. Phys. **A37** (2004) 10037.

[12] A. Krajewska, A. Ushveridze and Z. Walczak, Mod. Phys. Lett. **A12** (1997) 1225.

[13] For the properties of Jacobi elliptic functions, see for example, M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, 1964); I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products* (Academic Press, 2000).

[14] Y. Brihaye, S. Giller and P. Kosinski, Jour. Phys. **A28** (1995) 421.

Table 1: Energy eigenstates of PT-invariant GAL potentials with $b = f = 0$, $g = n - a$, $n = 0, 1, 2, \ldots$; $\delta_1 \equiv \sqrt{(1+m)^2(a-1)^2 - (2a-1)(2a-3)m}$, $\delta_2 \equiv \sqrt{[a-1+m(a-2)]^2 - (2a-1)(2a-5)m}$, $\delta_3 \equiv \sqrt{[a-2+m(a-1)]^2 - (2a-1)(2a-5)m}$, $\delta_4 \equiv \sqrt{(1+m)^2(a-2)^2 - (2a-1)(2a-7)m}$.

n	$g(g+1)$	E	$\mathrm{sn}^{-a}(y)\psi$
0	$(a-1)a$	$-(1+m)a^2$	1
1	$(a-2)(a-1)$	$-a^2 - m(a-1)^2$	$\frac{\mathrm{cn}(y)}{\mathrm{sn}(y)}$
1	$(a-2)(a-1)$	$-(a-1)^2 - ma^2$	$\frac{\mathrm{dn}(y)}{\mathrm{sn}(y)}$
2	$(a-3)(a-2)$	$-(1+m)(a-1)^2$	$\frac{\mathrm{cn}(y)\mathrm{dn}(y)}{\mathrm{sn}^2(y)}$
2	$(a-3)(a-2)$	$-(1+m)(a^2-2a+2) \pm 2\delta_1$	$\frac{[E+(1+m)(a-2)^2]\mathrm{sn}^2(y)+2(2a-3)}{\mathrm{sn}^2(y)}$
3	$(a-4)(a-3)$	$-(a^2-2a+2) - (a^2-4a+5)m \pm 2\delta_2$	$\frac{[[E+(a-2)^2+m(a-3)^2]\mathrm{sn}^2(y)+2(2a-5)]\mathrm{cn}(y)}{\mathrm{sn}^3(y)}$
3	$(a-4)(a-3)$	$-(a^2-4a+5) - (a^2-2a+2)m \pm 2\delta_3$	$\frac{[[E+(a-3)^2+m(a-2)^2]\mathrm{sn}^2(y)+2(2a-5)]\mathrm{dn}(y)}{\mathrm{sn}^3(y)}$
4	$(a-5)(a-4)$	$-(1+m)(a^2-4a+5) \pm 2\delta_4$	$\frac{[[E+(1+m)(a-3)^2]\mathrm{sn}^2(y)+2(2a-7)]\mathrm{cn}(y)\mathrm{dn}(y)}{\mathrm{sn}^4(y)}$

Table 2: Energy eigenstates of PT-invariant GAL potentials with parameters $b = g = 0$, $f = n - a$, $n = 0, 1, 2, \ldots$; $\delta_5 \equiv \sqrt{(a-1+m)^2 - (2a-1)m}$, $\delta_6 \equiv \sqrt{(a-1+2m)^2 - 3(2a-1)m}$, $\delta_7 \equiv \sqrt{(a-2+2m)^2 - (2a-1)m}$, $\delta_8 \equiv \sqrt{(a-2+3m)^2 - 3(2a-1)m}$.

n	$f(f+1)$	E	$\mathrm{cn}^{-a}(y)\psi$
0	$(a-1)a$	$-a^2$	1
1	$(a-2)(a-1)$	$-a^2 - m$	$\frac{\mathrm{sn}(y)}{\mathrm{cn}(y)}$
1	$(a-2)(a-1)$	$-(a-1)^2 - m$	$\frac{\mathrm{dn}(y)}{\mathrm{cn}(y)}$
2	$(a-3)(a-2)$	$-(a-1)^2 - 4m$	$\frac{\mathrm{dn}(y)\mathrm{sn}(y)}{\mathrm{cn}^2(y)}$
2	$(a-3)(a-2)$	$-(a^2+2-2a+2m) \pm 2\delta_5$	$\frac{(E+(a-2)^2)\mathrm{sn}^2(y)+2}{\mathrm{cn}^2(y)}$
3	$(a-4)(a-3)$	$-(a^2+2-2a+5m) \pm 2\delta_6$	$\frac{[(E+(a-2)^2+m)\mathrm{sn}^2(y)+6]\mathrm{sn}(y)}{\mathrm{cn}^3(y)}$
3	$(a-4)(a-3)$	$-(a^2+5-4a+5m) \pm 2\delta_7$	$\frac{[(E+(a-3)^2+m)\mathrm{sn}^2(y)+2]\mathrm{dn}(y)}{\mathrm{cn}^3(y)}$
4	$(a-5)(a-4)$	$-(a^2+5-4a+10m) \pm 2\delta_8$	$\frac{[(E+(a-3)^2+4m)\mathrm{sn}^2(y)+6]\mathrm{dn}(y)\mathrm{sn}(y)}{\mathrm{cn}^4(y)}$

Table 3: Energy eigenstates of PT-invariant GAL potentials with $f = 0$, $g = n - a - b$, $n = 0, 1, 2, \ldots$; $\delta_9 \equiv \sqrt{[(1+m)(a-1)+b]^2 - (2a-1)(2a+2b-3)m}$, $\delta_{10} \equiv \sqrt{[a+b-1+m(a-2)]^2 - (2a-1)(2a+2b-5)m}$.

n	E	$\mathrm{dn}^b(y)\mathrm{sn}^{-(a+b)}(y)\psi$
0	$-(a+b)^2 - ma^2$	1
1	$-(a+b)^2 - m(a-1)^2$	$\frac{\mathrm{cn}(y)}{\mathrm{sn}(y)}$
2	$-(1+m) - (a+b-1)^2 - m(a-1)^2 \pm 2\delta_9$	$\frac{[E+(a+b-2)^2+m(a-2)^2]\mathrm{sn}^2(y)+2(2a+2b-3)}{\mathrm{sn}^2(y)}$
3	$-(1+m) - (a+b-1)^2 - m(a-2)^2 \pm 2\delta_{10}$	$\frac{[[E+(a+b-2)^2+m(a-3)^2]\mathrm{sn}^2(y)+2(2a+2b-5)]\mathrm{cn}(y)}{\mathrm{sn}^3(y)}$

Table 4: Energy eigenstates of PT-invariant GAL potentials with $f = 2n - a - b - g$, $n = 0, 1, 2, \ldots$; $\delta_{11} \equiv \sqrt{[(a+b-1)+m(1-b-g)]^2 - (2a-1)(1-2g)m}$.

n	E	$\mathrm{sn}^g(y)\mathrm{dn}^b(y)\mathrm{cn}^{-(a+b+g)}(y)\psi$
0	$-(a+b)^2 - m(g+b)^2$	1
1	$-(a+b-1)^2 - m(b+g-1)^2 - (1+m) \pm 2\delta_{11}$	$\frac{[E+(a+b-2)^2+m(b+g)^2]\mathrm{sn}^2(y)-2(2g-1)}{\mathrm{cn}^2(y)}$

Table 5: The five supersymmetric partner potentials of the PT-invariant Lamé potential $V_-^{PT}(x) = -6m\mathrm{sn}^2(y)$. Here $y = ix + \beta$ and $\delta \equiv \sqrt{1 - m + m^2}$. All partner potentials have a period $2K'(m)$.

E	$\psi^{(-)}$	$V_+(x)$
$-2(1+m) - 2\delta$	$1 + \frac{E}{2}\mathrm{sn}^2(y)$	$6m\mathrm{sn}^2(y) + E - \frac{2E^2\mathrm{sn}^2(y)\mathrm{cn}^2(y)\mathrm{dn}^2(y)}{(1+\frac{E}{2}\mathrm{sn}^2(y))}$
$-4 - m$	$\mathrm{sn}(y)\mathrm{cn}(y)$	$-2m[\mathrm{sn}^2(y) + \mathrm{sn}^2(y + K(m) + iK'(m)) + \mathrm{sn}^2(y + iK'(m)] - E$
$-1 - 4m$	$\mathrm{sn}(y)\mathrm{dn}(y)$	$-2m[\mathrm{sn}^2(y) + \mathrm{sn}^2(y + K(m)) + \mathrm{sn}^2(y + iK'(m)] - E$
$-1 - m$	$\mathrm{cn}(y)\mathrm{dn}(y)$	$-2m[\mathrm{sn}^2(y) + \mathrm{sn}^2(y + K(m)) + \mathrm{sn}^2(y + K(m) + iK'(m)] - E$
$-2(1+m) + 2\delta$	$1 + \frac{E}{2}\mathrm{sn}^2(y)$	$6m\mathrm{sn}^2(y) + E - \frac{2E^2\mathrm{sn}^2(y)\mathrm{cn}^2(y)\mathrm{dn}^2(y)}{(1+\frac{E}{2}\mathrm{sn}^2(y))}$