

Stability of Compacton Solutions of Fifth-Order Nonlinear Dispersive Equations

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Abstract

We consider fifth-order nonlinear dispersive $K(m, n, p)$ type equations to study the effect of nonlinear dispersion. Using simple scaling arguments we show, how, instead of the conventional solitary waves like solitons, the interaction of the nonlinear dispersion with nonlinear convection generates compactons - the compact solitary waves free of exponential tails. This interaction also generates many other solitary wave structures like cuspons, peakons, tipons etc. which are otherwise unattainable with linear dispersion. Various self similar solutions of these higher order nonlinear dispersive equations are also obtained using similarity transformations. Further, it is shown that, like the third-order nonlinear $K(m, n)$ equations, the fifth-order nonlinear dispersive equations also have the same four conserved quantities and further even any arbitrary odd order nonlinear dispersive $K(m, n, p\dots)$ type equations also have the same three (and most likely the four) conserved quantities. Finally, the stability of the compacton solutions for

the fifth-order nonlinear dispersive equations are studied using linear stability analysis. From the results of the linear stability analysis it follows that, unlike solitons, all the allowed compacton solutions are stable, since the stability conditions are satisfied for arbitrary values of the nonlinear parameters.

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1 Introduction

The recent discovery [1] that solitary wave solutions supported by nonlinear wave equations may compactify under nonlinear dispersion, has shown that nonlinear dispersion can cause qualitative changes in the nature of genuinely nonlinear phenomena. Such nonlinearly dispersive partial differential equations which support compacton solutions are represented by the $K(m, n)$ equations of the form

$$u_t + a(u^m)_x + (u^n)_{3x} = 0; \quad m, n > 1 \quad (1)$$

Most of the weakly nonlinear and linear dispersion equations studied so far admit solitary waves, called solitons, that are infinite in extent. On the other hand, it has been shown that the interaction of nonlinear dispersion with nonlinear convection generates exactly compact structures, called compactons, free of exponential tails. The compacton solutions so generated have immediate applications in the study of pattern formations, as the observed stationary and dynamical patterns in nature are usually finite in extent. The interaction also generates many other nonlinear solitary wave structures like cuspons, peakons, tipons etc. [2] which are otherwise not possible in the weakly nonlinear models with linear dispersion. The compacton speed depends on its height, but unlike solitons, its width is independent of its speed. Beside the compact structure and the unusual speed-width relation, the compactons have the remarkable soliton like property that they collide elastically. However, unlike soliton collisions in an integrable system, the point at which the compactons collide is marked by the creation of low amplitude compacton-anticompacton pairs. More recently [3,4] the study of the third-order $K(m, n)$ equations was generalised by including into the equation higher order nonlinear dispersive terms, like for example, the fifth-order $K(m, n, p)$ equations of the form [3,4]

$$u_t + \beta_1(u^m)_x + \beta_2(u^n)_{3x} + \beta_3(u^p)_{5x} = 0, \quad m, n, p > 1 \quad (2)$$

This type of higher order dispersive equations are useful for describing the dynamics of various physical systems. As has been explained in [1], the compacton supporting nonlinear dispersive equations arises if one controls the effects of nonlinearity and dispersion by two independent parameters while modelling physical phenomena. In such cases, it is required to retain

the quadratic and higher order effects in dispersion. For example, one can show that the equations governing the motion of mass points in a dense chain (which leads to vibrational excitations in the chain) are a prototype of the Eq.(2) above, if one consider the effect of the higher order dispersive interactions. Similarly, for plasma ion acoustic waves, if the ion-electron charge separation is treated separately from the ions inertia, then again, considerations of the effects of the higher order dispersive interactions leads to the equations of the form as in Eq.(2) above. Beside the applications of such higher order equations in physical systems (see [3,4] for details), the studies of these generalised equations such as the fifth-order nonlinear dispersive equations as above (Eq.(2)) are motivated by the need to understand how far the concept of compact structures can be extended and how generic are the properties of the compacton solutions. A variety of explicit compact solitary wave structures of these fifth- order nonlinear dispersive equations are constructed [3,4] and numerical simulations of these equations have also revealed the existence of compact travelling breathers [4].

In the present paper we report on the study of the stability property of the compacton solutions [3,4] of these fifth-order nonlinear dispersive equation. Beside predicting about the asymptotic nature of the compacton solutions, the stability analysis is also important for the following reasons. Since the nonlinear dispersive equations represented by $K(m, n)$ equations (Eq.(1)) do not appear to be integrable [1], this suggests that the observed almost elastic collisions of the compactons are probably not due to the integrability property and thus the mechanism responsible for the compact structure, coherence and robustness of the compactons calls for a more systematic study of the nonlinear dispersive systems. Stability analysis of the compacton solutions may provide some clues regarding the almost elastic nature of the compacton collisions. Beside, the stability problem of the nonlinear dispersive equation is interesting because for such equations with higher power of the nonlinearity and nonlinear dispersion, the phenomena of collapse is possible. Further motivation for studying the stability properties of the fifth-order nonlinear dispersive equations comes from the result of the recent study on the role of the fifth-order dispersion term on the soliton stability of the usual KdV type linear dispersion equations. For example, it has been shown that [5] the solitary wave solutions of the fifth-order equation of the type

$$u_t + u^p u_x + \alpha u_{3x} + \beta u_{5x} = 0 \quad (3)$$

are unstable with respect to the collapse type instabilities, if $p \geq 4$ for $\beta = 0$ while for $\beta \neq 0$, i.e. the addition of the fifth-order term stabilizes the soliton for $p > 4$ [5]. The exact upper limit of the nonlinearity parameter p in this case is still an open question [6]. It would therefore be appropriate to examine whether the addition of the higher order nonlinear dispersion term puts any additional constraint on the conditions for the stability of the corresponding compacton solutions. Recently some attempts have been made to numerically study the stability of compacton solutions of the fifth-order nonlinear dispersive equations [7].

The rest of the paper is organised as follows: in Sec. 2 we present some general properties like various self similar solutions, conservation laws, various solitary wave structures etc. of the fifth-order nonlinear dispersive equations. In Sec. 3 we discuss the stability of the compacton solutions of these equations using linear stability analysis. A short report on this method appeared recently in an rapid communication article [8]. Finally we conclude in Sec. 4.

2 Some General Properties

The fifth-order nonlinear dispersive $K(m, n, p)$ equation of the form as in Eq.(2) is not derivable from a Lagrangian and hence does not possess the usual conservation laws of mass, energy etc. that are associated with the KdV type of equations ($m = n = p = 1$ case). However, this equation has exact compacton solutions for the nonlinearity parameters within the range $2 \leq k \leq 5$, provided $k = m = n = p$ [3,4]. Since this equation does not have a Lagrangian and hence a conserved Hamiltonian, we cannot do a linear stability analysis for the compacton solutions of this equation, as the linear stability requires a Hamiltonian, as shown below. Hence, we consider a slightly different fifth-order nonlinear dispersive equation

$$\begin{aligned}
u_t - a\delta u^{a-1}u_x + \alpha b(b-1)u^{b-2}(u_x)^3 + 4\alpha b u^{b-1}u_x u_{2x} + 2\alpha u^b u_{3x} \\
+ 3\beta c(c-1)u^{c-2}(u_x)^5 + 24\beta c u^{c-1}(u_x)^3 u_{2x} + 24\beta u^c u_x (u_{2x})^2 \\
+ 12\beta u^c (u_x)^2 u_{3x} - 2\gamma d(d-1)(d-2)u^{d-3}(u_x)^3 u_{2x} \\
- 7\gamma d(d-1)u^{d-2}u_x (u_{2x})^2 - 6\gamma d(d-1)u^{d-2}(u_x)^3 u_{3x} - 10\gamma d u^{d-1}u_{2x} u_{3x} \\
- 6\gamma d u^{d-1}u_x u_{4x} - 2\gamma u^d u_{5x} = 0
\end{aligned} \tag{4}$$

where $u(x) = \partial_x \phi(x)$. At a first glance it may appear that this equation has been created artificially. However, a closer look will immediately reveal that this equation is very similar to Eq.(2) above. It contains exactly the same terms as in Eq.(2), but only the weightage of the terms are different. Comparing term by term we see that the set of parameters m, n, p in Eq. (2) corresponds to $a = m, b + 1 = n$ and $c + 3 = d + 1 = p$ in Eq. (4). Both Eqs.(2) and (4) have compacton solutions. But the advantage of Eq.(4) is that it is possible now to write a Lagrangian for this equation leading to a Hamiltonian and thus one can do the stability analysis of the compacton solutions of Eq.(4). Thus we will present here the stability analysis of the fifth-order nonlinear dispersive equation given by Eq.(4), instead of Eq.(2). The Lagrangian corresponding to Eq.(4) is given by

$$L = \int \mathcal{L} dx$$

$$= \int dx \left[\frac{1}{2} \phi_x \phi_t - \delta \frac{(\phi_x)^{a+1}}{a+1} - \alpha (\phi_x)^b (\phi_{2x})^2 - \beta (\phi_x)^c (\phi_{2x})^4 - \gamma (\phi_x)^d (\phi_{3x})^2 \right] \quad (5)$$

It is worth remarking here that recently Cooper et al [7] have obtained compacton solutions for a slightly different fifth-order nonlinear equation which is again obtainable from a Lagrangian.

The conserved Hamiltonian H which is obtained from the Lagrangian [Eq. (5)] is

$$H = \int_{-\infty}^{\infty} [\pi \dot{\phi} - \mathcal{L}] dx, \quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2} \phi_x$$

$$= \int_{-\infty}^{\infty} \left[\delta \frac{u^{a+1}}{a+1} + \alpha u^b (u_x)^2 + \beta u^c (u_x)^4 + \gamma u^d (u_{2x})^2 \right] dx \quad (6)$$

This Hamiltonian also follows from the fact that Eq. (4) can be written in the canonical form

$$u_t = \partial_x \frac{\delta H}{\delta u} = \{u, H\} \quad (7)$$

where the Poisson bracket structure is given by

$$\{u(x), u(y)\} = \partial_x \delta(x - y) \quad (8)$$

We first consider the scaling relations between speed, width and amplitude of the travelling wave ($\xi = x + Dt$) structure of Eq. (4). Under the

scaling transformation $x \rightarrow \mu x, t \rightarrow \nu t$ and $u \rightarrow \eta u$, it can be easily shown that

$$u(x, t) \rightarrow D^{\frac{1}{a-1}} u(D^{\frac{(b+1-a)}{2(a-1)}} \xi) = D^{\frac{1}{a-1}} u(D^{\frac{(c+3-b-1)}{2(a-1)}} \xi) = D^{\frac{1}{a-1}} u(D^{\frac{(d+1-b-1)}{2(a-1)}} \xi) \quad (9)$$

Similarly Eq. (2) under scaling transformation also admits solutions of the form

$$u(x, t) = D^{\frac{1}{(m-1)}} u(D^{\frac{(p-n)}{2(m-1)}} \xi) \quad (10)$$

From Eq. (9) we see that for the case when $a = b + 1 = c + 3 = d + 1$ (which corresponds to the case of $m = n = p$ in Eq. (10)), there is a detailed balance between the convection and dispersion and as a result of which the width of the solitary wave solutions (compactons) become independent of the amplitude (or speed D).

It can also be shown that the fifth-order equations [Eq. (4)] are invariant under the stretching group of transformations [2] which supports self-similar solutions (similarity structures) of the form

$$u(x, t) = t^{\frac{2}{\Delta}} F(\zeta) \quad (11)$$

where $\zeta = xt^{-\mu}$, $\mu = \frac{b+1-a}{\Delta}$ and $\Delta = b + 3(1 - a)$ along with $c + 3 = d + 1$. Now, when $\mu = 0$ or $\Delta = 0$, the self similar solutions of the form in Eq. (11) is no longer valid. However, in that case, it can be shown that the equations [Eqs. (2) and (4)] have additional invariance under shifts in time or space (spiral symmetry) giving rise to the self similar solutions of the form

$$\begin{aligned} u(x, t) &= t^{\frac{1}{1-a}} F(x + D \log t) & \text{for } a = b + 1 = c + 3 = d + 1 \\ &= e^{-Dt} F(x e^{D(a-1)t}) & \text{for } 3a = b + 3 \end{aligned} \quad (12)$$

Similarly, it can be shown that the fifth-order nonlinear dispersive $K(m, n, p)$ equations of the form of Eq. (2) are also invariant under stretching group of transformations which admits similarity solutions of the form as in Eq. (11) for $\mu = (p - n)/\Delta$ where $\Delta = 3p - 5n + 2$ and $m = 2n - p$. Again, when $n = m$ or when $n = 3m - 2$ it can be shown that the $K(m, n, p)$ equations are invariant under the spiral group of transformations leading to self similar solutions of the form

$$\begin{aligned} u(x, t) &= t^{\frac{1}{1-m}} F(x + D \log t) & \text{for } m = n = p \\ &= e^{-Dt} F(x e^{D(m-1)t}) & \text{for } 5n = 3p + 2 \text{ and } m = 2n - p \end{aligned} \quad (13)$$

For the case when $m = n + 2$ and $p = n - 2$, substituting Eq. (11) in Eq. (2) we get the equation for $F(\xi)$ as

$$-\alpha\xi F + F^{n+2} + (F^n)_{2\xi} + (F^{n-2})_{4\xi} = \beta \quad (14)$$

where $\xi = xt^{-\alpha}$, $\alpha = \frac{1}{n+2}$ and $\beta = \text{constant}$. For the particular case of $n=2$ (in which case Eq. (2) reduces to the form of the third- order nonlinear dispersion $K(m, n)$ equation [Eq. (1)]) Eq. (14) reduces to the generalised second-order Painleve equation [2].

We now make some comment about the conserved quantities associated with these fifth-order nonlinear dispersive equations. A conservation law associated with equations of the form Eq.(2) and Eq.(4) can be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial X}{\partial x} = 0 \quad (15)$$

where Q is the density of the conserved quantity $\int_{-\infty}^{\infty} Q dx$ and X is the corresponding flux density. In an earlier paper [3] we had reported that Eq. (2) has only one conservation law. We now find that, like the $K(m, n)$ equations (Eq. (1)), the fifth-order $K(m, n, p)$ equations (Eq. (2)) also have four conservation laws for the case $m = n = p$, with four densities (Q's) same as that of Eq.(1) [1]. In fact, we are now able to show that the same three (out of four) densities (Q's) of Eq.(1) also exist for arbitrary odd $(2n + 1)$ th-order nonlinear dispersion equations (Eq. (36) in [3]). It can be checked that the Q and X values for the arbitrary odd $(2n + 1)$ th- order $K(m, m, m, \dots)$ equations are given by

$$\begin{aligned} Q_1 = u , \quad X_1 &= a_1(u^m) + (a_1 + a_3)(u^m)_{2x} + \dots \\ &+ (a_{2n-3} + a_{2n-1})(u^m)_{(2n-2)x} + a_{(2n-1)}(u^m)_{(2n)x} \\ Q_2 = u \cos x , \quad X_2 &= a_1[\sin x(u^m)_x + \cos x(u^m)_{2x}] + \dots \\ &+ a_{2n-1}[\sin x(u^m)_{(2n-1)x} + \cos x(u^m)_{(2n)x}] \\ Q_3 = u \sin x , \quad X_3 &= a_1[-\cos x(u^m)_x + \sin x(u^m)_{2x}] + \dots \\ &+ a_{(2n-1)}[-\cos x(u^m)_{(2n-1)x} + \sin x(u^m)_{(2n)x}] \end{aligned} \quad (16)$$

where a_1, a_3 etc are coefficients of various dispersive terms. We have not been able to prove in general the existence of the fourth density $Q_4 = u^{m+1}$ [1] for the arbitrary odd order nonlinear equation . However, we can show that for the third and fifth-order nonlinear dispersion equation $Q_4 = u^{m+1}$ is

the fourth density for arbitrary values of the nonlinearity parameter m . We have also checked that $Q_4 = u^{m+1}$ is also the fourth density for the ninth and eleventh-order equation for the particular value of the nonlinearity parameter $m = 2$. From the above calculations we conjecture that $Q = u^{m+1}$ is the fourth density for the arbitrary odd order nonlinear dispersion equation for arbitrary values of the nonlinearity parameter m . The origin or the symmetry associated with these unusual conservation laws of these nonlinear dispersive equations are not known at present. However, it should be noted that for the fifth-order nonlinear dispersive equation [Eq. (4)] which is derivable from a Lagrangian, there are only three conserved quantities.

It can be easily shown [3,4] that both the equations Eq.(2) and Eq.(4) support a class of one parameter family of compacton solutions of the form

$$u(\xi) = A \cos^\nu(B\xi) \quad (17)$$

for $|B\xi| \leq \pi/2$, $u(\xi) = 0$ otherwise and where $\xi = x - Dt$, $\nu = \frac{4}{(k-1)}$. The width B of the compacton solutions is independent of speed D . The compacton solutions exist for the continuous values of the nonlinearity parameter $k = a = b + 1 = c + 3 = d + 1$ in the range $2 \leq k \leq 5$. Similarly, it can be shown that the fifth-order nonlinear dispersive equations of the form of $K(m, n, p)$ equations (Eq. (2)) also have the compacton solutions of the form of Eq. (17) within the same range of the nonlinearity parameter $2 \leq k = m = n = p \leq 5$. For the third-order nonlinear dispersive equations (Eq.(1)) it has been shown that, beside compacton solutions, the interaction of the nonlinear dispersion term with the convective term gives rise to various other kinds of nonlinear localised structures such as cuspons, peakons, spikons, tipons etc. [2]. In the same way, we have been able to obtain the peakon solution of the form

$$u(x, t) = u_0[e^{-\beta|\xi|} - 1] \quad (18)$$

for the fifth-order nonlinear dispersive equation (Eq.(2)), but, only for the special case of the nonlinear parameters $m = n = p = 2$. So far we have not been able to obtain the peakon solutions (if any) for the same equations for other values of the nonlinear parameters m, n, p . Similarly, we have not yet been able to obtain other solitary wave structures (if any) such as cuspons and tipons for the fifth-order nonlinear dispersive equations (Eqs. (2),(4)).

3 Compacton Stability

In this section we discuss in detail the stability analysis of the compacton solutions of the fifth-order nonlinear dispersive equations which have been studied recently [3,4]. We use here the method of linear stability analysis to analyze the problem. As has been mentioned above, we cannot do the linear stability analysis for the compacton solutions of Eq.(2) as these equations do not have a Lagrangian. Accordingly we will study the stability of the compacton solutions for a slightly different equation (Eq.(4)) which can be derived from a Lagrangian density (Eq. (5)) and has a conserved Hamiltonian [Eq. (6)] and momentum

$$P(u) = \frac{1}{2} \int_{-\infty}^{+\infty} u^2 dx \quad (19)$$

It can be easily checked that Eq. (4) can also be derived from the variational principle $\delta(H + Dp) = 0$, where P and D denote the compacton momentum and velocity respectively. Introducing the notations

$$\begin{aligned} I_n &= \int_{-\infty}^{+\infty} u^n(x) dx, \quad J_2 = \int_{-\infty}^{+\infty} u^b(u_x)^2 dx, \quad J_3 = \int_{-\infty}^{+\infty} u^c(u_x)^4 dx \\ J_4 &= \int_{-\infty}^{+\infty} u^d(u_{2x})^2 dx \end{aligned} \quad (20)$$

we can write the Hamiltonian [Eq. (6)] and the momentum for the compactons as

$$H_c = \left[\frac{\delta}{(a+1)} I_{a+1} + \alpha J_2 + \beta J_3 + \gamma J_4 \right], \quad P_c = \frac{1}{2} I_2 \quad (21)$$

Now we consider the scaling transformation $x \rightarrow \nu x$. Under this transformation the integrals in Eq. (20) are transformed as

$$I_n(\nu) = \frac{I_n}{\nu}, \quad J_2(\nu) = \nu J_2, \quad J_3(\nu) = \nu^3 J_3 \quad \text{and} \quad J_4(\nu) = \nu^3 J_4 \quad (22)$$

such that

$$H_c(\nu) = \frac{\delta}{\nu(a+1)} I_{a+1} + \alpha \nu J_2 + \beta \nu^3 J_3 + \gamma \nu^3 J_4, \quad \text{and} \quad P_c(\nu) = \frac{P_c}{\nu} \quad (23)$$

Integrating Eq. (4) twice we get

$$2DP_c + \delta I_{a+1} + \alpha(b+2)J_2 + \beta(c+4)J_3 + \gamma(d+2)J_4 = 0 \quad (24)$$

Similarly from

$$\frac{d}{d\nu}[H(\nu) + DP_c(\nu)]_{\nu=1} = 0$$

we get,

$$-\frac{\delta}{(a+1)}I_{a+1} + \alpha J_2 + 3\beta J_3 + 3\gamma J_4 - DP_c = 0 \quad (25)$$

We eliminate J_4 and I_{a+1} from Eqs. (24) and (25) to write the Hamiltonian (Eq. (21)) as

$$\begin{aligned} H_c &= \frac{\alpha J_2}{(3a+d+5)}(2a-4b+2d-2) + \frac{J_3}{(3a+d+5)}(-8\beta+4\beta d-4c\beta) \\ &= \frac{DP_c}{(3a+d+5)}(a-d-9) \end{aligned}$$

However, as mentioned above, the compacton solutions (Eq. (17)) exist for nonlinear parameters $a = b+1 = c+3 = d+1 = k$ within the range $2 \leq k \leq 5$ for which the Hamiltonian can be written as

$$H_c = -\frac{2DP_c}{(a+1)} \quad (26)$$

Similarly Eq. (23) can be written as

$$H_c(\nu) = \frac{\alpha J_2}{2}(2\nu - 1/\nu - \nu^3) + \frac{DP_c}{4(a+1)}[\nu^3(a-1) - \frac{(a+7)}{\nu}] \quad (27)$$

Thus $H_c(1) = H_c$. Now, we consider the more general scaling transformation $u \rightarrow \mu^{\frac{1}{2}}u(\lambda x)$. Under this transformation H_c and P_c are transformed as $H(\lambda, \mu)$ and $P(\lambda, \mu)$ and

$$\begin{aligned} \Phi(\lambda, \mu) &= \frac{\delta}{\lambda(a+1)}\mu^{\frac{a+1}{2}}I_{a+1} + \alpha\lambda\mu^{\frac{b+2}{2}}J_2 + \beta\lambda^3\mu^{\frac{c+4}{2}}J_3 \\ &\quad + \gamma\lambda^3\mu^{\frac{d+2}{2}}J_4 + D\frac{\mu}{\lambda}P_c \end{aligned} \quad (28)$$

where $\Phi(\lambda, \mu) = H_c(\lambda, \mu) + DP_c(\lambda, \mu)$. The expressions $\frac{\partial \Phi}{\partial \lambda} = \frac{\partial \phi}{\partial \mu} = 0$ give the stationary points at $\mu = \lambda = 1$ (the compacton equation) and near this point, using the Taylor's series for $\mu = \lambda$ we get (the transformation in this case does not change the momentum P)

$$\begin{aligned} \delta^{(2)}\Phi(\lambda) &= \delta^{(2)}H(\lambda) = \frac{(\lambda - 1)^2}{8} \left[\frac{\delta I_{a+1}}{a+1}(a-1)(a-3) \right. \\ &\quad + \alpha J_2(b+2)(b+4) + \beta J_3(c+10)(c+8) \\ &\quad \left. + \gamma J_4(d+6)(d+8) \right] \end{aligned} \quad (29)$$

which has a definite sign. If it is positive (negative), the expression

$$H_c(\lambda) = H_c(\mu, \lambda)|_{\mu=\lambda} = \frac{\delta}{a+1} \lambda^{\frac{(a-1)}{2}} I_{a+1} + \alpha \lambda^{\frac{(b+4)}{2}} J_2 + \beta \lambda^{\frac{(c+10)}{2}} J_3 + \gamma \lambda^{\frac{(d+8)}{2}} J_4 \quad (30)$$

has a minimum (maximum) at $\lambda = 1$.

Now, let us assume that $u = u_c + v$, where $|v| \ll 1$ and the scalar product $\langle u_c, v \rangle = 0$. Substituting this in Eq. (4) and after linearization we get

$$\partial_T v = \partial_\xi \hat{L} v \quad (31)$$

where $\xi = x - Dt$, $T = t$ and the operator \hat{L} is given by

$$\begin{aligned} \hat{L} &= D + a\delta u^{a-1} - 2\alpha b u^{b-1} u_x \partial_x - \alpha b(b-1) u^{b-2} u_x^2 - 2\alpha b u^{b-1} u_{2x} - 2\alpha u^b \partial_x^2 \\ &\quad - 3\beta c(c-1) u^{c-2} u_x^4 - 12\beta c u^{c-1} u_x^3 \partial_x - 12\beta c u^{c-1} u_x^2 u_{2x} - 24\beta u^c u_x u_{2x} \partial_x \\ &\quad - 12\beta u^c u_x^2 \partial_x^2 + 3\gamma d(d-1) u^{d-2} u_{2x}^2 + 6\gamma d u^{d-1} u_{2x} \partial_x^2 + 2\gamma d(d-1)(d-2) u^{d-3} u_x^2 u_{2x} \\ &\quad + 4\gamma d(d-1) u^{d-2} u_x u_{2x} \partial_x + 2\gamma d(d-1) u^{d-2} u_x^2 \partial_x^2 + 4\gamma d(d-1) u^{d-2} u_x u_{3x} \\ &\quad + 4\gamma d u^{d-1} u_{3x} \partial_x + 4\gamma d u^{d-1} u_x \partial_x^3 + 2\gamma d u^{d-1} u_{4x} + 2\gamma u^d \partial_x^4 \end{aligned} \quad (32)$$

Eq. (31) has a solution of the form

$$v(\xi, T) = e^{-iwT} \phi(\xi) + e^{iw^*T} \phi^*(\xi) \quad (33)$$

where $\phi(\xi)$ satisfies the equation

$$w\phi(\xi) = i\partial_\xi \hat{L}\phi(\xi) \quad (34)$$

Integrating the compacton equation of motion (Eq. (4)) once w.r.t. $\xi = x - Dt$, the resulting equation can be written as

$$\hat{L}\partial_\xi u_c = 0 \quad (35)$$

Similarly, integrating Eq. (4) once w.r.t. ξ and differentiating the resulting equation w.r.t D we get

$$\hat{L}\left(\frac{\partial u_c}{\partial D}\right) = -u_c \quad (36)$$

Eq. (35) shows that the $w = 0$ solution of Eq. (34) is given by $\phi(x) \propto \partial_\xi u_c$. Similarly, Eq. (34) has also the solution $(-w, \phi(-\xi))$. Thus the compacton u_c is stable if w is real and unstable if w is complex.

Since Eq. (34) contains the product of two hermitian operators, hence all w are real if one of the operators is positive definite. This implies that a sufficient condition of real eigenvalue w is [5]

$$\langle \psi, \hat{L}\psi \rangle > 0 \quad (37)$$

where ψ is a function in the the subspace orthogonal to u_c , i.e.

$$\langle \psi, u_c \rangle = \langle \psi, \partial_\xi u_c \rangle = 0 \quad (38)$$

Using Eqs.(35) and (36) and following Karpman [5], it can be shown that the condition for the existence of such function ψ satisfying Eqs. (37) and (38) is equivalent to the condition

$$\left(\frac{\partial P_c}{\partial D}\right) > 0 \quad (39)$$

From Eqs. (20), (21) and the exact compacton solution [Eq. (17)] of the fifth-order nonlinear dispersive equation [Eq. (4)], it can be shown that the condition in Eq. (39) is satisfied for arbitrary values of the nonlinear parameters $k = a = b + 1 = c + 3 = d + 1$. However, as has been mentioned above, the compacton solutions of the fifth-order nonlinear dispersive equation are allowed for the nonlinearity parameter in the range $2 \leq k \leq 5$. Since the stability condition [Eq. (39)] is satisfied for arbitrary values of the nonlinearity parameter k , this implies that all the allowed compacton solutions [Eq. (17)] are stable.

We can obtain another condition for the compacton stability from the Hamiltonian minimum condition. This is because the condition in Eq. (37) is also associated with the extremum of $H + DP$, since, using the relation $\delta(H + DP) = 0$, one can show that the second variation of $H(u)$ and $P(u)$ at $u = u_c$ can be written as

$$\delta^{(2)}(H + DP)_{u_c} = \frac{1}{2} \int_{-\infty}^{\infty} \langle v, \hat{L}v \rangle d\xi > 0 \quad (40)$$

where the operator \hat{L} is given by Eq. (32). This means that, if the condition in Eq. (37) is fulfilled, then $H(u) + DP(u)$ has a minimum at $u = u_c$. Inversely, the minimum of $H(u) + DP(u)$ at $u = u_c$ is a sufficient condition of compacton stability with respect to small perturbations. Thus from Eq. (30) we obtain the condition for the minimum of the perturbed Hamiltonian $H_c(\lambda)$ at $\lambda = 1$ as

$$2DP_c(k-1)(k-7) > 64\alpha J_2(k+1) \quad (41)$$

Using Eq. (20), (21) and the exact compacton solutions [Eq. (17)] it can be shown that the condition in Eq. (41) is satisfied for arbitrary values of nonlinearity parameter k within the range $2 \leq k \leq 5$ for which the compacton solutions are allowed. This result, that the condition for the compacton stability is satisfied for arbitrary values of the nonlinearity parameter, is unlike the soliton stability results, where it has been shown that the stability condition of the soliton solutions puts a restriction on the allowed values of the nonlinearity parameters [5,6,9,10].

We would like to mention here that the stability condition [Eq. (39)] is obtained by assuming that at sufficiently small dispersion, there is only one eigenstate with negative eigenvalue for the operator \hat{L} [Eq. (32)]. Details regarding this conjecture are explained in [5]. The validity of this conjecture has been proven from numerical experiments for many other systems, such as third and fifth-order Korteweg-de Vries equations as well as nonlinear Schrödinger equations [5]. At present we do not have any evidence to show that this conjecture is also valid for our operator \hat{L} [Eq. (32)], except for the fact that the stability result that follows from using this conjecture also agrees with the results obtained independently from the Hamiltonian minimum condition [Eqs. (40) and (41)]. Numerical experiments along the lines as referred in [5] will be required to verify the validity of this conjecture for the systems described here.

4 Conclusions

To conclude, in this paper we have shown how the nonlinear dispersion term interacts with the nonlinear convection term in the fifth-order nonlinear dispersive equations to generate exact compacton solutions free from exponential tails and many other unusual nonlinear localised solutions like peakons, cuspons etc. Using simple scaling relations and the invariance property of the equations under stretching group as well as spiral group of transformations, we have shown how these higher order nonlinear dispersive equations support self similar solutions of various patterns. Unlike the third-order nonlinear dispersive equations [2], for the fifth-order nonlinear dispersive equations case, various solitary wave solutions such as cuspons, peakons, tipons etc cannot be shown as a plot in phase diagrams as functions of the corresponding potentials. However, using an ansatz, we have been able to obtain the peakon solutions of the fifth-order nonlinear dispersive equations. We have also shown that the fifth-order nonlinear dispersive equations when expressed in the form of the $K(m, n, p)$ equations [Eq. (2)] have four conserved densities (Q 's in [Eq. (15)]) same as that for the third-order nonlinear dispersive $K(m, n)$ equations [Eq. (1)], though the corresponding flux densities (X 's in [Eq. (15)]) are obviously different. Further, even for the arbitrary odd order nonlinear dispersive equations we have proved the existence of three conservation laws and provided strong evidence for the existence of the fourth one. However, it should be noted that, for the case of the fifth-order nonlinear dispersive equations [Eq. (4)] which are derivable from a Lagrangian, there are only three conserved quantities [3]. We have used linear stability analysis to examine the stability of the compacton solutions for the fifth-order nonlinear dispersive equations. The important differences between soliton and compacton solutions that come out from the stability analysis of the corresponding solutions are that, whereas the soliton solutions are allowed for arbitrary values of the nonlinearity parameters, the stability condition of the soliton solutions puts restrictions on the range of the nonlinearity parameters for which stable soliton solutions are allowed. On the other hand, the compacton solutions are allowed only within a certain range of the nonlinearity parameters and all the allowed compacton solutions within this specific range of the nonlinearity parameters are stable. This is because, as shown above, as in the case of the third-order nonlinear dispersive equations [8], the linear stability analysis of the compacton solutions of the fifth-order nonlinear dis-

persive equations also does not put any additional constraint on the range of the nonlinear parameters.

5 References

1. P. Rosenau and J.M. Hyman, Phys. Rev. Lett. **70**, 564 (1993); P. Rosenau, Phys. Rev. Lett. **73**, 1737 (1994).
2. P. Rosenau, Phys. Lett. A **230**, 305 (1997)
3. B. Dey, Phys. Rev. E **57**, 4733 (1998)
4. P. Rosenau, Phys. Lett A, **252**, 297 (1999) and references therein.
5. V.I. Karpman, Phys. Lett. A **210**, 77 (1996) and references therein
6. B. Dey, A. Khare and C.N. Kumar, Phys. Lett. A **223**, 449 (1996)
7. F. Cooper, J.M. Hyman and A. Khare, e-print patt-sol/9704003
8. B. Dey and A. Khare, Phys. Rev. E **58**, R2741 (1998)
9. E.A. Kuznetsev, Phys. Lett. A **101**, 314 (1984)
10. V.I. Karpman, Phys. Lett. A **215**, 254 (1996) and references therein.