Continued fractions for complex numbers and values of binary quadratic forms

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Abstract

We describe various properties of continued fraction expansions of complex numbers in terms of Gaussian integers. Numerous distinct such expansions are possible for a complex number. They can be arrived at through various algorithms, as also in a more general way from what we call "iteration sequences". We consider in this broader context the analogues of the Lagrange theorem characterizing quadratic surds, the growth properties of the denominators of the convergents, and the overall relation between sequences satisfying certain conditions, in terms of nonoccurrence of certain finite blocks, and the sequences involved in continued fraction expansions. The results are also applied to describe a class of binary quadratic forms with complex coefficients whose values over the set of pairs of Gaussian integers form a dense set of complex numbers.

1 Introduction

Continued fraction expansions of real numbers have been a classical topic in Number theory, playing an important role in Diophantine approximation and various other areas. Considerable work has also been done on multi-dimensional versions of the concept. There has however been relatively little work on analogous continued fraction expansion of complex numbers in terms of Gaussian integers. A beginning was indeed made by Hurwitz in [9] as far back as 1887, where certain properties of the nearest integer algorithm were studied and an analogue of the classical Lagrange theorem for quadratic surds was proved. Some further properties of the Hurwitz continued fraction expansions have been described recently by Hensley [8]. In [10] Leveque explored another direction, focussing primarily on the aspect of relating continued fractions to best approximations in analogy with the classical case. In another work A. Schmidt [12] introduces an entirely different approach to the phenomenon of continued fraction expansions; while the results have some relation with certain classical results the overall framework is quite different from the classical case.

This article is an attempt to explore how far the spirit of classical continued fractions for real numbers can be extended to complex numbers, in terms of the Gaussian integers, viz. $x+iy$ with x and y integers. We view a sequence $\{a_n\}_{n=0}^{\infty}$, of Gaussian integers, with $a_n \neq 0$ for $n \geq 1$, as a continued fraction expansion of a complex number z if the corresponding convergents can be formed in the usual way (see below for details) from $a_0, a_1, \ldots, a_n$ for all n in analogy with the usual convergents and they converge to z. Each complex number has uncountably many distinct continued fraction expansions in this sense.

A class of continued fraction expansions can be generated through algorithms for describing the successive partial quotients, of which the Hurwitz algorithm is an example. Various examples of such algorithms are described in §2. In §3 we discuss a more general approach to continued fraction expansions of complex numbers, introducing the notion of "iteration sequences", the terms of which correspond to the tails of continued fraction expansions. We discuss the asymptotics of these sequences in a general framework and prove the convergence of the "convergents" associated with them, which is seen to give a comprehensive picture of the possibilities for continued fractions of complex numbers; see Theorem 3.7.

In §4 we consider continued fraction expansions of quadratic surds, a classical topic for the usual continued fractions. We prove the analogue of the Lagrange theorem in the general framework, and show in particular that for a large class of algorithms (satisfying only a mild condition) the continued fraction expansion of a complex number z is eventually periodic if and only if z is a quadratic surd; see Corollary 4.6.

Sections 5 and 6 are devoted to the behaviour of the convergents associated to general sequences of partial quotients. In particular we describe conditions on the sequences of partial quotients, in terms of nonoccurrence of certain blocks, which ensure that the denominators of the convergents have increasing absolute values and exponential growth. This facilitates in setting up spaces of sequences and continuous maps onto the space of irrational complex numbers such that each sequence is a continued fraction expansion of its image under the map; see Theorem 6.1. We exhibit also a space of sequences described purely in terms of nonoccurrence of finitely many blocks of length two, each of which serves as a continued fraction expansion of a complex number, and conversely every complex number admits an expansion in terms of a such a sequence; see Theorem 6.7.

In §7 we relate the complex continued fractions to fractional linear transformations of the complex plane and the projective space. Associating to each initial block of the continued fraction expansion of a number z a projective transformation in a canonical fashion, we show that the corresponding sequence of transformations asymptotically takes all points of the projective space to a point associated with z; see Theorem 7.2.

Results from §6 and §7 are applied in §8 to study the question of values of binary quadratic forms, with complex coefficients, over the set of pairs of Gaussian integers being dense in complex numbers. It has been known that for a nondegenerate quadratic form in $n \geq 3$ complex variables the set of values over the n-tuples of Gaussian integers is dense in complex numbers, whenever the form is not a multiple of a form with rational coefficients; see §8 for some details on the problem. The corresponding statement is however not true for binary quadratic forms. From general dynamical considerations the set of values can be deduced to be dense in $\mathbb{C}$ for "almost all" binary quadratic forms (see §8 for details), but no specific examples are known. We describe a set of complex numbers, called "generic", such that the values of the quadratic form $(z_1 - \varphi z_2)(z_1 - \psi z_2)$, where $\varphi, \psi \in \mathbb{C}$, over the set with z_1, z_2 Gaussian integers are dense in $\mathbb{C}$, if φ or ψ is generic; see Corollary 8.8. The set of generic complex numbers contains a dense $\mathcal{G}_\delta$ subset of $\mathbb{C}$, and using a result of D. Hensley [8] it can also be seen to have full Lebesgue measure.

2 Preliminaries on continued fraction expansions

Let $\mathfrak{G}$ denote the ring of Gaussian integers in $\mathbb{C}$, viz. $\mathfrak{G} = \{x + iy \mid x, y \in \mathbb{Z}\}$. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathfrak{G}$. We associate to it two sequences $\{p_n\}_{n=-1}^{\infty}$ and $\{q_n\}_{n=-1}^{\infty}$ defined recursively by the relations

$$p_{-1} = 1, p_0 = a_0, p_{n+1} = a_{n+1}p_n + p_{n-1}, \text{ for all } n \geq 0,$$

$$\text{and} \quad q_{-1} = 0, q_0 = 1, q_{n+1} = a_{n+1}q_n + q_{n-1}, \quad \text{for all } n \geq 0.$$

We call $\{p_n\}, \{q_n\}$ the *$\mathcal{Q}$-pair of sequences associated to* $\{a_n\}$; ($\mathcal{Q}$ signifies "quotient"). We note that $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}$ for all $n \geq 1$. If $q_n \neq 0$ for all n then we can form the "convergents" p_n/q_n and if they converge, as $n \to \infty$ to a complex number z then $\{a_n\}$ can be thought of as a continued fraction expansion of z; when this holds we say that $\{a_n\}$ *defines a continued fraction expansion of* z, and write $z = \omega(\{a_n\})$. We note that this confirms to the usual idea of continued fraction expansion as $z = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$ (see [7] for example).

Also, given a finite sequence $\{a_n\}_{n=0}^m$, where $m \in \mathbb{N}$ we associate to it sequences $\{p_n\}_{n=-1}^m$ and $\{q_n\}_{n=-1}^m$ by the recurrence relations as above, and it may also be called the corresponding $\mathcal{Q}$-pair; in this case, provided all q_n are nonzero we set $\omega(\{a_n\}) = p_m/q_m$.

Though there will be some occasions to deal with continued fraction expansions of rational complex numbers (namely $z = x + iy$ with x and y rational), by and large we shall be concerned with continued fraction expansions of complex numbers which are not rational (namely $z = x + iy$ with either x or y irrational); this set of complex numbers will be denoted by $\mathbb{C}'$.

We note here the following general fact about the convergence of the sequence of the "convergents".

Proposition 2.1. *Let $\{a_n\}$ be a sequence in $\mathfrak{G}$ defining a continued fraction expansion of a $z \in \mathbb{C}'$ and $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Suppose that there exist $r_0 \in \mathbb{N}$ and $\theta > 1$ such that $|q_{n+r}| \geq \theta |q_n|$ for all $n \geq 0$ and $r \geq r_0$. Then there exists a $C > 0$ such that $|z - \frac{p_n}{q_n}| \leq C|q_n|^{-2}$ for all $n \in \mathbb{N}$.*

Proof. We have $p_{n+1}q_n - p_n q_{n+1} = (-1)^n$ for all $n \geq 0$, so $\dfrac{p_{n+1}}{q_{n+1}} - \dfrac{p_n}{q_n} = \dfrac{(-1)^n}{q_n q_{n+1}}$, and in turn $|\dfrac{p_{n+m}}{q_{n+m}} - \dfrac{p_n}{q_n}| \leq \displaystyle\sum_{k=0}^{m-1} \frac{1}{|q_{n+k}q_{n+k+1}|}$ for all $n \geq 0$ and $m \geq 1$. Therefore

$$|z - \frac{p_n}{q_n}| \leq \sum_{k=0}^{\infty} \frac{1}{|q_{n+k}q_{n+k+1}|} \leq r_0(1-\theta^{-1})^{-1}\frac{1}{|q_n|^2},$$

which proves the proposition, with the choice $C = r_0(1-\theta^{-1})^{-1}$. □

There is considerable flexibility in writing continued fraction expansions of complex numbers, as will become apparent as we proceed. One of the simplest ways of arriving at such expansions is via algorithms for generating a continued fraction expansion starting with a given complex number. Through the rest of this section we shall discuss various possibilities in this respect, together with their significance.

We begin by introducing some notation. For any $z \in \mathbb{C}$ and $r > 0$ we shall denote by $B(z,r)$ the open disc with center at z and radius r, namely $\{\zeta \in \mathbb{C} \mid |\zeta - z| < r\}$, and by $\bar{B}(z,r)$ its closure $\{\zeta \in \mathbb{C} \mid |\zeta - z| \leq r\}$. Furthermore, we shall also write, for $z \in \mathbb{C}$, $B(z)$ for $B(z,1)$, $\bar{B}(z)$ for $\bar{B}(z,1)$, B for $B(0,1)$ and $\bar{B}$ for $\bar{B}(0,1)$. The set of all nonzero complex numbers will be denoted by $\mathbb{C}^*$.

Let $f : \mathbb{C}^* \to \mathfrak{G}$ be a map such that $f(z) \in \bar{B}(z)$ for all $z \in \mathbb{C}^*$. There are multiple choices possible for the values of f at any point. Using the map we associate to each $z \in \mathbb{C}$ two sequences $\{z_n\}$ and $\{a_n\}$ (finite or infinite) as follows. We define $z_0 = z$, $a_0 = f(z)$ and having defined $z_0, \ldots, z_n$ and $a_0, \ldots, a_n$, $n \geq 0$, we terminate the sequences if $f(z_n) = z_n$ and otherwise define $z_{n+1} = (z_n - f(z_n))^{-1}$ and $a_{n+1} = f(z_{n+1})$. The map f will be called the *choice function* of the algorithm, and $\{a_n\}$ will be called the *continued fraction expansion of* z with respect to the algorithm. When the sequences are terminated as above, say with z_m and a_m where $m \in \mathbb{N}$, then it can be verified inductively that $z = p_m/q_m$, where $\{p_n\}$, $\{q_n\}$ is the corresponding $\mathcal{Q}$-pair. The crucial case will be when the sequences are infinite, which will necessarily be the case when z is irrational. The following theorem is a particular case of Theorem 3.7.

Theorem 2.2. *Let $f : \mathbb{C}^* \to \mathfrak{G}$ be the choice function of an algorithm such that $|f(z) - z| \neq 1$ for any z with $|z| = 1$. Let $z \in \mathbb{C}'$ be given and $\{a_n\}$ be the sequence of partial quotients associated to it with respect to the algorithm corresponding to f. Let $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Then $q_n \neq 0$ for all n and $\{p_n/q_n\}$ converges to z.*

This condition in the theorem is satisfied in particular if we assume at the outset that $f(z) \in B(z)$, in place of $\bar{B}(z)$, for all z, but retaining the generality seems worthwhile.

In some contexts it is convenient to have a slightly more general notion in which the choice function may be allowed to be multi-valued over a (typically small, lower dimensional) subset of $\mathbb{C}$, and the algorithm proceeds by picking one of the values. Thus we pick a map $\varphi : \mathbb{C}' \to \mathcal{P}(\mathfrak{G})$ where $\mathcal{P}(\mathfrak{G})$ is the set of subsets of $\mathfrak{G}$, such that $\varphi(z) \subset \bar{B}(z)$ for all $z \in \mathbb{C}'$; we choose $z_0 = z$, $a_0 \in \varphi(z)$ and having chosen $z_0, \ldots, z_n$ and $a_0, \ldots, a_n$, $n \geq 0$, we terminate the sequences if $z_n \in \varphi(z_n)$ and otherwise define $z_{n+1} = (z_n - a_n)^{-1}$ and $a_{n+1} \in \varphi(z_{n+1})$. We call φ as above a *multivalued choice function* for an algorithm. One advantage of considering multivalued choice functions φ is that these can be chosen to have symmetry properties like $\varphi(\overline{z}) = \varphi(z)$ and $\varphi(-z) = \varphi(z)$ for all z. Given a multivalued choice function one can get a (single-valued) choice function by auxiliary conventions; in general it may not be possible to ensure the latter to have the symmetry properties.

For the algorithm defined by a choice function f (respectively by a multivalued choice function φ) we call the subset $\overline{\{z - f(z) \mid z \in \mathbb{C}'\}}$ (respectively $\overline{\{z - \alpha \mid \alpha \in \varphi(z), z \in \mathbb{C}'\}}$) the *fundamental set* of the algorithm, and denote it by Φ_f (respectively Φ_φ), or simply Φ. The fundamental sets play a crucial role in the properties of the associated algorithm.

Examples 2.3.

1. One of the well-known algorithms is the *nearest integer algorithm*, discussed in detail by Hurwitz [9], known also as the Hurwitz algorithm. It consists of assigning to $z \in \mathbb{C}$ the Gaussian integer nearest to it; this may be viewed as a multivalued choice function $\varphi : \mathbb{C} \to \mathcal{P}(\mathfrak{G})$ defined by $\varphi(z) = \{\alpha \in \mathfrak{G} \mid |z - \alpha| \leq |z - \gamma| \text{ for all } \gamma \in \mathfrak{G}\}$. In this case Φ is the square $\{x + iy \mid |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}$.

2. The nearest integer algorithm may be generalized to a class of algorithms as follows: Let $\psi : \mathbb{C}^* \to (0, \infty)$ be a function, and $r \in [\frac{1}{\sqrt{2}}, 1]$ be given. We define a multivalued choice function φ by choosing, for $z \in \mathbb{C}'$, $\varphi(z)$ to be the set of points x in $\bar{B}(z, r) \cap \mathfrak{G}$ where $\psi(x - z)$ is minimum. The nearest integer algorithm is a special case with $\psi(z) = |z|$ for all $z \in \mathbb{C}^*$. A class of algorithms f_d, where $d \in \mathbb{C}$, may be defined by considering the functions defined by $\psi(z) = |z - d|$, in place of $|z|$; we call φ_d the *shifted Hurwitz algorithm* with displacement d. We note that for the algorithm $\varphi_{1/2}$ the continued fraction expansions of real numbers coincide with the usual simple continued fraction expansions. It may also be seen that for d with $|d| < 1 - \frac{1}{\sqrt{2}}$, Φ_{φ_d} is contained in $\bar{B}(0, |d| + \frac{1}{\sqrt{2}})$.

3. Let $f : \mathbb{C}^* \to \mathfrak{G}$ be the map defined as follows: let $z \in \mathbb{C}^*$ and $z_0 \in \mathfrak{G}$ be such that $z - z_0 = x + iy$ with $0 \leq x < 1$ and $0 \leq y < 1$; if $x^2 + y^2 < 1$ we define $f(z)$ to be z_0 and otherwise $f(z)$ is defined to be $z_0 + 1$ or $z_0 + i$ whichever is nearer to z (choosing say the former if they are equidistant). It may be seen that with respect to this algorithm if $z \in \mathbb{C}$ and $\{a_n\}$ is the corresponding sequence of partial quotients then for all $n \geq 1$ we have $\mathrm{Re}\, a_n \geq -1$ and $\mathrm{Im}\, a_n \geq -1$; this may be compared with the simple continued fractions for real numbers where the later partial quotients are positive.

4. Let $\mu : \mathfrak{G} \to (0, \infty)$ be a function on the set of Gaussian integers. Let $\frac{1}{\sqrt{2}} \leq r \leq 1$. A multivalued choice function φ for an algorithm may be defined by setting $\varphi(z)$, for $z \in \mathbb{C}'$ to be the subset of $\bar{B}(z, r) \cap \mathfrak{G}$ consisting of points where the value of μ over the set is maximum.

5. One may also restrict the range of the choice function to a subset of $\mathfrak{G}$. If $\mathfrak{G}'$ is a subset of $\mathfrak{G}$ such that for some $r \leq 1$, $B(z, r) \cap \mathfrak{G}'$ is nonempty for all $z \in \mathbb{C}'$ then choice functions f can be constructed with values in $\mathfrak{G}'$, by adopting any of the above strategies. Here is an example. We shall say that $x + iy \in \mathfrak{G}$ is *even* if $x + y$ is even. Let $\mathfrak{G}'$ be the set of all even elements in $\mathfrak{G}$; it is the subgroup generated by $1 + i$ and $1 - i$, and one can see that $B(z, r) \cap \mathfrak{G}'$ is nonempty for all $z \in \mathbb{C}'$. In this case by analogy with the nearest integer algorithm we get the "nearest even integer" algorithm; for this algorithm Φ is the square with vertices at ± 1 and $\pm i$.

In §6 we discuss an algorithm with certain interesting properties, involving choice of nearest integers as well as nearest odd integers depending on the point.

Proposition 2.4. *Let $f : \mathbb{C} \to \mathfrak{G}$ be a choice function such that Φ_f is contained in $B(0, r)$ for some $0 < r < 1$. Then for any rational complex number the continued fraction expansion with respect to f terminates.*

Proof. Let $z = a/b$ be a rational complex number, where $a, b \in \mathfrak{G}$ and $b \neq 0$, and $\{z_n\}$ and $\{a_n\}$ be the sequences, finite or infinite, as above starting with z, with respect to the algorithm given by f. Let $\{p_n\}$, $\{q_n\}$ be the $\mathcal{Q}$-pair corresponding to $\{a_n\}$. From the recurrence relations it may be seen inductively that $q_n z - p_n = (-1)^n(z_0 - f(z_0))(z_1 - f(z_1)) \cdots (z_n - f(z_n))$ for all $n \geq 0$, until it is defined (see the proof of Proposition 3.3 below). The condition in the hypothesis then implies that $|q_n z - p_n| < r^{n+1}$ for all n as above. For n such that $r^{n+1} \leq |b|^{-1}$ the condition can hold only if $q_n z - p_n = 0$. From the equality as above this implies that $f(z_n) = z_n$ for some n, so the sequences terminate. This proves the proposition. □

The algorithms discussed above are time-independent "Markovian" in the choice strategy, where z_{n+1}, $n \geq 1$, is chosen depending only on z_n (and independent of z_m, $0 \leq m \leq n-1$). For certain purposes it is also of interest to consider "time-dependent" algorithms. An interesting example of this is seen in the work of Leveque [10], where an algorithm is introduced such that the corresponding continued fraction expansions have the best approximation property like the classical continued fractions; this is not shared by the nearest integer algorithm (and also perhaps by other time-independent algorithms). We shall not specifically deal with this aspect. Our discussion in the following sections will however include the continued fraction expansions arising through such algorithms as well.

3 Iteration sequences

In this section we discuss another, more general approach to the issue of continued fraction expansions, introducing what we call iteration sequences. In this respect for convenience we restrict to irrational complex numbers. Let $z \in \mathbb{C}'$ be given. We call a sequence $\{z_n\}_{n=0}^{\infty}$ an *iteration sequence* for z if

$$z_0 = z, z_0 - z_1^{-1} \in \mathfrak{G}, \text{and for all } n \geq 1,\ |z_n| \geq 1 \text{ and } z_n - z_{n+1}^{-1} \in \mathfrak{G}\backslash\{0\}.$$

An iteration sequence is said to be *degenerate* if there exists n_0 such that $|z_n| = 1$ for all $n \geq n_0$; it is said to be *nondegenerate* if it is not degenerate.

We note that for any $z \in \mathbb{C}'$, depending on the location of z in the unit sized square with integral vertices that contains z, there are 2, 3 or all the 4 of the vertices within distance at most 1 from z; z_1 can be chosen so that $|z_1| \geq 1$ and $z_0 - z_1^{-1} \in \mathfrak{G}$ is one of these points, and after the choice of a z_n, $n \geq 1$, is made, between 2 and 4 choices are possible for z_{n+1}, for continuing the iteration sequence. For the points on the unit circle there are at least two nonzero points of $\mathfrak{G}$ within distance 1 that satisfy the nondegeneracy condition at each n. Thus the sequence can be continued in multiple ways at each stage to get a nondegenerate iteration sequence.

Remark 3.1. Given an algorithm with the choice function $f : \mathbb{C}^* \to \mathfrak{G}$ we get an iteration sequence $\{z_n\}$ by setting $z_0 = z$ and $z_{n+1} = (z_n - f(z_n))^{-1}$ for all $n \geq 0$. The class of iteration sequences is however more general than sequences arising from the algorithms.

Remark 3.2. Motivated by the classical simple continued fractions, one may consider stipulating that $|z_n| > 1$, for $n \geq 1$ but as we shall see allowing $|z_n| \geq 1$ introduces no difficulty, provided we ensure that $z_n - z_{n+1}^{-1}$ is nonzero, and this enables somewhat greater generality. Also, since in $\mathbb{C}$ there exist irrational numbers z with $|z| = 1$ this generality seems appropriate.

For each iteration sequence $\{z_n\}$ we get a sequence $\{a_n\}$ in $\mathfrak{G}$, called the corresponding *sequence of partial quotients*, defined by

$$a_n = z_n - z_{n+1}^{-1}, \text{ for all } n \geq 0;$$

we note that $a_n \neq 0$ for all $n \geq 1$. We shall show that if z_n is nondegenerate and $\{p_n\}_{n=-1}^{\infty}$, $\{q_n\}_{n=-1}^{\infty}$ is the $\mathcal{Q}$-pair corresponding to $\{a_n\}$ then $q_n \neq 0$ for all n and p_n/q_n converge to z (Proposition 3.7). Thus corresponding to every nondegenerate iteration sequence we get a continued fraction expansion for z. We shall also obtain an estimate for the difference $|z - \frac{p_n}{q_n}|$ when the iteration sequence is bounded away from the unit circle (see Proposition 3.7).

Proposition 3.3. *Let $z \in \mathbb{C}'$, $\{z_n\}$ be an iteration sequence for z and $\{a_n\}$ be the corresponding sequence of partial quotients. Let $\{p_n\}, \{q_n\}$ be the $\mathcal{Q}$-pair of sequences associated to $\{a_n\}$. Then we have the following:*

i) $q_n z - p_n = (-1)^n (z_1 \cdots z_{n+1})^{-1}$ for all $n \geq 0$,

ii) $z = \dfrac{z_{n+1} p_n + p_{n-1}}{z_{n+1} q_n + q_{n-1}}$ for all $n \geq 0$,

iii) for all $n \geq 0$, $|z - \dfrac{p_n}{q_n}| \leq \dfrac{1}{|q_n|^2 |z_{n+1} + \frac{q_{n-1}}{q_n}|}$; in particular, if $\{|q_n|\}$ is monotonically increasing and there exists a $\gamma > 1$ such that $|z_n| \geq \gamma$ for all n then $|z - \dfrac{p_n}{q_n}| \leq \dfrac{1}{(\gamma - 1)|q_n|^2}$.

Proof. i) We shall argue by induction. Note that as $p_0 = a_0$, $q_0 = 1$ and $z - a_0 = z_1^{-1}$, the statement holds for $n = 0$. Now let $n \geq 1$ and suppose by induction that the assertion holds for $0, 1, \ldots, n-1$. Then we have $q_n z - p_n = (a_n q_{n-1} + q_{n-2})z - (a_n p_{n-1} + p_{n-2}) = a_n(q_{n-1}z - p_{n-1}) + (q_{n-2}z - p_{n-2}) = (-1)^{n-1}(z_1 \cdots z_n)^{-1} a_n +$

$(-1)^{n-2}(z_1 \cdots z_{n-1})^{-1} = (-1)^n (z_1 \cdots z_n)^{-1}(-a_n + z_n) = (-1)^n (z_1 \cdots z_{n+1})^{-1}$. This proves (i).

ii) By (i) we have, for all $n \geq 1$,

$$z_{n+1} = -\frac{q_{n-1}z - p_{n-1}}{q_n z - p_n}.$$

Solving the equation for z we get the desired assertion.

iii) Using (iii) and the fact that $|p_{n-1}q_n - p_n q_{n-1}| = 1$ we get

$$|z - \frac{p_n}{q_n}| = |\frac{z_{n+1}p_n + p_{n-1}}{z_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n}| = \frac{1}{|q_n|^2 |z_{n+1} + \frac{q_{n-1}}{q_n}|}.$$

The second assertion readily follows from the first. □

Remark 3.4. In the context of (iii) it may be noted here that we shall later see sufficient conditions on $\{a_n\}$ which ensure that $\{|q_n|\}$ is monotonically increasing. Also the condition $|z_n| \geq \gamma > 1$ arises naturally under various circumstances, and holds in particular for iteration sequences arising from any algorithm whose fundamental set is contained in $B(1)$. It may be mentioned that the estimates we get from (iii) in Proposition 3.7 are rather weak and not really significant from the point of view of Diophantine approximation; in fact even the pigeon hole principle yields better inequalities. The main point about the statement however is the generality in terms of allowed sequences $\{a_n\}$ for which the estimates hold.

Remark 3.5. Let R be the set of all 12th roots of unity in $\mathbb{C}$. We note that R is precisely the set of points on the unit circle which are also at unit distance from some nonzero Gaussian integer. (For $\rho = \pm 1$ and $\pm i$ there are three such points each, while for the other 12 th roots there is a unique one.) Thus if $\{z_n\}$ is an iteration sequence for $z \in \mathbb{C}'$ and for some $m \in \mathbb{N}$ we have $|z_m| = |z_{m+1}| = 1$ then $z_m \in R$. It follows in particular that $\{z_n\}$ is a degenerate iteration sequence if and only if $z_n \in R$ for all large n. It may also be seen, noting that $\{z_n\}$ is contained in $\mathbb{C}'$, that if, for some $m \in N$, $z_m \in R$ and $|z_{m+1}| = 1$ then $z_{m+1} = -z_m$.

Towards proving that each nondegenerate iteration sequence of any $z \in \mathbb{C}'$ yields a continued fraction expansion of z we first note the following:

Proposition 3.6. *Let $z \in \mathbb{C}'$ and $\{z_n\}$ be a nondegenerate iteration sequence for z. Then* $\limsup |z_n| > 1$.

Proof. Suppose this is not true. As $|z_n| \geq 1$ for all $n \geq 1$ this means that $|z_n| \to 1$ as $n \to \infty$. Then $|z_n - a_n| = |z_{n+1}|^{-1} \to 1$ as $n \to \infty$. As $a_n \in \mathfrak{G}\backslash\{0\}$ for all $n \geq 1$ the above two conditions imply that $d(z_n, R) \to 0$ as $n \to \infty$. Furthermore there exist $\rho_n \in R$ such that $|z_n - \rho_n| \to 0$. The conditions $|z_n - a_n| \to 1$ and $|z_n - \rho_n| \to 0$ imply in turn that $|a_n - \rho_n| \to 1$ as $n \to \infty$. Since $\mathfrak{G}$ and R are discrete we get that $|a_n - \rho_n| = 1$ for all large n, say $n \geq n_0$. We note also that for any $\rho \in R$ and $a \in \mathfrak{G}$ such that $|\rho - a| = 1$, we have $\rho - a \in R$; this can be seen from an inspection of the possibilities. In particular $\rho_n - a_n \in R$ for all $n \geq n_0$.

Now for all n let $\zeta_n = z_n - \rho_n$ and $\beta_n = \rho_n - a_n$. Then we have $\beta_n \in R$ and $z_n - a_n = \beta_n + \zeta_n$, and $\zeta_n \to 0$ as $n \to \infty$. Then

$$z_{n+1} = \frac{1}{\beta_n + \zeta_n} = \frac{1}{\beta_n} + \frac{-\zeta_n}{\beta_n(\beta_n + \zeta_n)}.$$

Note that

$$|\frac{\zeta_n}{\beta_n + \zeta_n}| \leq \frac{|\zeta_n|}{1 - |\zeta_n|} \to 0.$$

Since $\beta_n \in R$, so is β_n^{-1}, and hence the two relations above imply that, for all large n,

$$\rho_{n+1} = \frac{1}{\beta_n} \text{ and } \zeta_{n+1} = \frac{-\zeta_n}{\beta_n(\beta_n + \zeta_n)}.$$

Then, for all large n,

$$|\zeta_{n+1}| = |\frac{\zeta_n}{\beta_n + \zeta_n}| = \frac{|\zeta_n|}{|z_n - a_n|} \geq |\zeta_n|,$$

since $|z_n - a_n| \leq 1$. Since $\zeta_n \to 0$ as $n \to \infty$ this implies that $\zeta_n = 0$ for all large n. Thus $z_n = \rho_n \in R$ for all large n. But this is a contradiction, since $\{z_n\}$ is a nondegenerate iteration sequence; see Remark 3.5. Therefore the assertion as in the proposition must hold. □

Theorem 3.7. *Let $z \in \mathbb{C}'$, $\{z_n\}$ be a nondegenerate iteration sequence for z and $\{a_n\}$ be the corresponding sequence of partial quotients. Let $\{p_n\}, \{q_n\}$ be the $\mathcal{Q}$-pair of sequences associated to $\{a_n\}$. Let $n_0 \geq 0$ be such that $|z_{n_0+1}| > 1$. Then $q_n \neq 0$ for all $n \geq n_0$, and $\{p_n/q_n\}_{n_0}^{\infty}$ converges to z as n tends to infinity. In particular, if $|z_1|$ or $|z_2|$ is greater than 1 then $q_n \neq 0$ for all $n \geq 0$ and $\{a_n\}$ defines a continued fraction expansion for z.*

Proof. By Proposition 3.3(i) we have $q_n z - p_n = (-1)^n (z_1 \cdots z_{n+1})^{-1}$ for all $n \geq 0$. Under the condition in the hypothesis we get that $0 < |q_n z - p_n| < 1$ for all $n \geq n_0$. Since $p_n \in \mathfrak{G}$ this shows in particular that $q_n \neq 0$ for all $n \geq n_0$. Furthermore, for $n \geq n_0$ we have $|z - \frac{p_n}{q_n}| = |z_1 \cdots z_{n+1}|^{-1} |q_n|^{-1} \leq |z_1 \cdots z_{n+1}|^{-1} \to 0$, by Proposition 3.6. The second part readily follows from the first, noting that $q_0 = 1$. This proves the Theorem. □

4 Lagrange theorem for continued fractions

In this section we prove an analogue of the classical Lagrange theorem for the usual simple continued fraction expansions, that the expansion is eventually periodic if and only if the number is a quadratic surd. We shall continue to follow the notation as before.

A number $z \in \mathbb{C}$ is called a *quadratic surd* if it is not rational and satisfies a quadratic polynomial over $\mathfrak{G}$, the ring of Gaussian integers; namely, if there there exist $a, b, c \in \mathfrak{G}$, with $a \neq 0$, such that $az^2 + bz + c = 0$; it may be noted that since z is irrational such a polynomial is necessarily irreducible over $\mathfrak{G}$ and both its roots are irrational.

Now, through the rest of the section let $z \in \mathbb{C}'$ and $\{z_n\}_{n=0}^{\infty}$ be an iteration sequence for z. Also, as before let $a_n = z_n - z_{n+1}^{-1} \in \mathfrak{G}$, for all $n \geq 0$, and $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair.

Proposition 4.1. *If $z_m = z_n$ for some $0 \leq m < n$, then z is a quadratic surd.*

Proof. By Proposition 3.3(ii) we have

$$z_m = -\frac{zq_{m-2} - p_{m-2}}{zq_{m-1} - p_{m-1}} \text{ and } z_n = -\frac{zq_{n-2} - p_{n-2}}{zq_{n-1} - p_{n-1}}.$$

It follows in particular that if z_m is a quadratic surd then so is z. We may therefore suppose that $m = 0$, so $z_m = z$, and $n \geq 1$. The condition $z_n = z$ yields, from the above relation, that $z(zq_{n-1} - p_{n-1}) + zq_{n-2} - p_{n-2} = 0$, or $q_{n-1}z^2 + (q_{n-2} - p_{n-1})z - p_{n-2} = 0$. If $q_{n-1} \neq 0$ then this shows that z is a quadratic surd and we are through. Now suppose that $q_{n-1} = 0$. Since $q_0 = 1$, this means that $n \geq 2$. Also, as $q_{n-1} = 0$, by Proposition 3.3 we have $|p_{n-1}| = |z_1 \cdots z_n|^{-1}$. Since $p_{n-1} \in \mathfrak{G}$ and $|z_k| \geq 1$ for all k, this yields that $|z_k| = 1$ for all $k = 1, \ldots, n$. Since $n \geq 2$, by Remark 3.5 it follows that $z_1 \in R$. Furthermore, as it is irrational it must be one of the 12th roots which is a quadratic surd. Hence z is also a quadratic surd. □

We now prove the following partial converse of this.

Theorem 4.2. *Let z be a quadratic surd. Then for given $C > 0$ there exists a finite subset F of $\mathbb{C}$ such that the following holds: if $\{z_n\}$ is an iteration sequence for z, with the corresponding $\mathcal{Q}$-pair $\{p_n\}$, $\{q_n\}$, and N is an infinite subset of $\mathbb{N}$ such that for all $n \in N$,*

$$|q_{n-2}z - p_{n-2}| \leq C|q_{n-2}|^{-1} \text{ and } |q_{n-1}z - p_{n-1}| \leq C|q_{n-1}|^{-1},$$

then there exists n_0 such that $\{z_n \mid n \in N, n \geq n_0\}$ is contained in F.

Proof: Let $a, b, c \in \mathfrak{G}$, with $a \neq 0$, be such that $az^2 + bz + c = 0$. Let $C > 0$ be given. Let $\mathcal{P}$ be the set of all quadratic polynomials of the form $\alpha z^2 + \beta z + \gamma$, with $\alpha, \beta, \gamma \in \mathfrak{G}$ such that $|\alpha| \leq C|2az + b| + 1, |\gamma| \leq C|2az + b| + 1$ and $|\beta^2 - 4\alpha\gamma| = |b^2 - 4ac|$. Then $\mathcal{P}$ is a finite collection of polynomials. Let F be the set of all $\zeta \in \mathbb{C}$ such that $P(\zeta) = 0$ for some $P \in \mathcal{P}$; then F is finite. Now let $\{z_n\}$ be any iteration sequence for z and $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. By Proposition 3.7(ii) the condition implies that for all $n \geq 1$,

$$a(\frac{z_n p_{n-1} + p_{n-2}}{z_n q_{n-1} + q_{n-2}})^2 + b\frac{z_n p_{n-1} + p_{n-2}}{z_n q_{n-1} + q_{n-2}} + c = 0 \quad \text{or}$$

$$a(z_n p_{n-1} + p_{n-2})^2 + b(z_n p_{n-1} + p_{n-2})(z_n q_{n-1} + q_{n-2}) + c(z_n q_{n-1} + q_{n-2})^2 = 0.$$

We set

$$A_n = ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2, \quad C_n = A_{n-1} \quad \text{and}$$

$$B_n = 2ap_{n-1}p_{n-2} + b(p_{n-1}q_{n-2} + q_{n-1}p_{n-2}) + 2cq_{n-1}q_{n-2}.$$

Then we have

$$A_n z_n^2 + B_n z_n + C_n = 0 \text{ for all } n \geq 1.$$

We note that $A_n \neq 0$, since otherwise p_{n-1}/q_{n-1} would be a root of the equation $ax^2 + bx + c = 0$ which is not possible, since the roots of the polynomial are irrational. We rewrite each A_n as

$$\begin{aligned} A_n &= (ap_{n-1}^2 + bp_{n-1}q_{n-1} + cq_{n-1}^2) - q_{n-1}^2(az^2 + bz + c) \\ &= a(p_{n-1}^2 - q_{n-1}^2 z^2) + b(p_{n-1} - q_{n-1}z)q_{n-1} \\ &= (p_{n-1} - zq_{n-1})(a(p_{n-1} + zq_{n-1}) + bq_{n-1}) \\ &= (p_{n-1} - zq_{n-1})(a(p_{n-1} - zq_{n-1}) + (2az + b)q_{n-1}). \end{aligned}$$

As $|p_{n-1} - zq_{n-1}| = |z_1 \cdots z_n|^{-1} \to 0$ as $n \to \infty$, there exists n_0 such that for all $n \geq n_0$ in N we have

$$|A_n| \leq |2az + b||q_{n-1}||p_{n-1} - zq_{n-1}| + 1 \leq C|2az + b| + 1,$$

by the condition in the hypothesis. Since $C_n = A_{n-1}$ we get also that $|C_n| \leq C|2az + b| + 1$ for all large n in N. An easy computation shows that for all $n \geq 1$, $B_n^2 - 4A_nC_n = b^2 - 4ac$. Thus for all $n \geq n_0$ in N, $A_n z^2 + B_n z + C_n$ is a polynomial belonging to $\mathcal{P}$, and since z_n is a root of $A_n z^2 + B_n z + C_n$ we get that $\{z_n \mid n \in N, n \geq n_0\}$ is contained in F. This proves the theorem. □

Theorem 4.2 and Proposition 2.1 imply the following.

Corollary 4.3. *Let z be a quadratic surd. Let $\{z_n\}$ be an iteration sequence for z, $\{a_n\}$ be the corresponding sequence, in $\mathfrak{G}$, of partial quotients, and $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Suppose that there exist $r_0 \in \mathbb{N}$ and $\theta > 1$ such that $|q_{n+r}| \geq \theta |q_n|$ for all $n \geq 0$ and $r \geq r_0$. Then $\{z_n \mid n \in \mathbb{N}\}$ is finite. Consequently, $\{a_n \mid n \in \mathbb{N}\}$ is also finite.*

Corollary 4.3 may be viewed as the analogue Lagrange theorem in terms of iteration sequences, under the mild growth condition on the denominators of the convergents. Note that when successive choices are not made according to any algorithm the sequence $\{a_n\}$ can not be expected to be eventually periodic. The growth condition on the denominators as above is satisfied is ensured under various conditions (see § 6).

Remark 4.4. We have $|q_n z - p_n| = |q_n(\frac{z_{n+1}p_n + p_{n-1}}{z_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n})| = \frac{1}{|z_{n+1}q_n + q_{n-1}|}$ for all $n \in \mathbb{N}$ and hence the condition in the hypothesis of Theorem 4.2 may be expressed as $|z_n + \frac{q_{n-2}}{q_{n-1}}| > \delta$ and $|z_{n+1} + \frac{q_{n-1}}{q_n}| > \delta$, for some $\delta > 0$. In this respect it may be observed that $|z_n| > 1$ for all n while $|q_{n-1}/q_n|$ may typically be expected to be less than 1; as $|p_n - q_n z| = |z_1 \cdots z_{n+1}|^{-1} \to 0$, $\{|q_n|\}$ is unbounded and hence the latter holds for infinitely many n, at any rate. The condition in the hypothesis would be satisfied if $\{|z_n| \mid n \in \mathbb{N}\}$ and $\{|q_{n-1}/q_n| \mid n \in \mathbb{N}\}$ are at a positive distance from each other. This motivates the following Corollary.

Corollary 4.5. *Suppose that the z as above is a quadratic surd. Let $N = \{n \in \mathbb{N} \mid |q_{n-1}| > |q_{n-2}|\}$. Suppose that there exists $\gamma > 1$ such that $|z_n| \geq \gamma$ for all $n \in N$. Then $\{z_n \mid n \in N\}$ is finite.*

Proof. For $n \in N$ we have $|z_n + \frac{q_{n-2}}{q_{n-1}}| \geq |z_n| - |\frac{q_{n-2}}{q_{n-1}}| > \gamma - 1$. By the observation in Remark 4.4 we get $|q_{n-1}z - p_{n-1}| \leq C|q_{n-1}|^{-1}$, with $C = (\gamma - 1)^{-1}$. Thus if n and $n+1$ are both contained in N, then $|q_{n-1}z - p_{n-1}| \leq C|q_{n-1}|^{-1}$ and $|q_n z - p_n| \leq C|q_n|^{-1}$. Now suppose that $n \in N$ and $n+1 \notin N$. Then $|q_n| \leq |q_{n-1}|$. Therefore $|q_n z - p_n| = |z_1 \cdots z_{n+1}|^{-1} < |q_{n-1}z - p_{n-1}| \leq C|q_{n-1}|^{-1} \leq C|q_n|^{-1}$. Thus the condition as in the hypothesis of Theorem 4.2 is satisfied for all n in N as above. The corollary therefore follows from that theorem. □

We now apply the result to continued fraction expansions arising from time-independent algorithms, say corresponding to a choice function f. Let $z \in \mathbb{C}'$ and $\{z_n\}$ be the sequence produced from the algorithm. We note that in this case if $z_n = z_m$ for some $m, n \in \mathbb{N}$ then $z_{n+l} = z_{m+l}$ for all $l \in \mathbb{N}$. We note also that if Φ_f is contained in $B(0, r)$ for some $r < 1$ then $|z_n| \geq r^{-1}$ for all $n \geq 1$. Let $\{a_n\}$ be the corresponding continued fraction expansion. The sequence $\{a_n\}$ is said to be *eventually periodic* if there exist $m, k \in \mathbb{N}$ such that $a_{j+k} = a_j$ for all $j \geq m$.

Corollary 4.6. *Let f be a choice function such that the fundamental set Φ_f is contained in $B(0,r)$ for some $r<1$. Then $z \in \mathbb{C}'$ is a quadratic surd if and only if its continued fraction expansion $\{a_n\}$ with respect to f is eventually periodic.*

Proof. Suppose $\{a_n\}$ is eventually periodic, and let $m,k \in \mathbb{N}$ be such that $a_{j+k} = a_j$ for all $j \geq m$. Then $z_{m+k} = z_m$, and hence by Proposition 4.1 z is a quadratic surd. Now suppose that z is a quadratic surd. Then by Corollary 4.5 there exist $0 \leq m < n$ such that $z_n = z_m$. Let $k = n - m$. Then $z_{m+k} = z_m$, and as noted above this implies that $z_{j+k} = z_j$ for all $j \geq m$. Since $a_j = f(z_j)$ for all $j \geq 0$ we get $a_{j+k} = a_j$ for all $j \geq m$. This shows that $\{a_n\}$ is eventually periodic. □

Remark 4.7. Let $z \in \mathbb{C}'$ be a quadratic surd. There are at least two distinct eventually periodic continued fraction expansions for z arising from two different algorithms, since given one such expansion we can find an algorithm to which the entries do not conform; furthermore the both the algorithms may be chosen to be such that their fundamental sets are contained in the open ball $B(0)$. Secondly, there are infinitely many eventually periodic expansions for z, since given any such expansions it can be altered after any given stage to get a new one, which is also eventually periodic; not all of these may correspond to sequences arising from algorithms. Clearly, the set of these eventually periodic expansions is countable, since each of them is determined by two finite blocks of Gaussian integers (one corresponding to an initial block and one to repeat itself in succession). Recall that for any z there are uncountably many continued fraction expansions (see Remark 3.4). Therefore there are also uncountably many continued fraction expansions which are not eventually periodic.

Remark 4.8. We note that the eventually periodic continued fraction expansions of a quadratic surd corresponding to two different algorithms may differ in their eventual periodicity as also in respect of whether the expansion is purely periodic (periodic from the beginning) or not. For example for $\frac{\sqrt{5}+1}{2}$ we have the usual expansion $\{a_n\}$ with $a_n = 1$ for all n, which may also be seen to be its continued fraction expansion with respect any of the shifted Hurwitz algorithms f_d (see Example 2 in §2), for any real number d in the interval $(\frac{\sqrt{5}-1}{2} - \frac{1}{2}, 1)$; we recall that from among these for $d \in (\frac{\sqrt{5}-1}{2} - \frac{1}{2}, 1 - \frac{1}{\sqrt{2}})$ the associated set Φ_f is contained in $B(0,r)$ for some $r < 1$, thus satisfying the condition in Corollary 4.6. On the other hand, for the continued fraction expansion of $\frac{\sqrt{5}+1}{2}$ with respect to the Hurwitz algorithm may be seen to be given by $\{a_n\}$ where $a_0 = 2$ and $a_n = (-1)^n 3$ for all $n \geq 1$; it is not purely periodic, and the eventual period is 2, rather than 1.

5 Growth properties of the convergents

In this section we discuss various growth properties of the sequences $\{q_n\}$ from the $\mathcal{Q}$ pairs. We begin with the following general fact.

Proposition 5.1. *Let $\{a_n\}_{n=0}^m$, where $m \in \mathbb{N}$, be a sequence in $\mathfrak{G}$ such that $|a_n| > 1$ for all $n \geq 1$ and let $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Suppose that $q_n \neq 0$ for all $n \geq 0$. Let $\sigma \in (\sqrt{2}, \frac{\sqrt{5}+1}{2})$, and $1 \leq n \leq m-1$ be such that $|q_{n-1}| > \sigma |q_{n-2}|$. Then the following statements hold: i) if $|a_n| > 2$ then $|q_n| > \sigma |q_{n-1}|$; ii) if $|a_n| \leq 2$ and $\operatorname{Re} a_n a_{n+1} \geq \chi$, where $\chi = 2$ if $|a_{n+1}| = \sqrt{2}$ and 0 otherwise, then $|q_{n+1}| > \sigma |q_n|$.*

Proof. Let $r_k = q_{k-1}/q_k$, for all $k \geq 0$, and $r = \sigma^{-1}$. Then $r \in (\frac{\sqrt{5}-1}{2}, 1/\sqrt{2})$. We have $r_n = 1/(a_n + r_{n-1})$, so $|a_n| \leq |r_{n-1}| + |r_n|^{-1} < r + |r_n|^{-1}$. When $|a_n| > 2$ we get $r + |r_n|^{-1} > |a_n| \geq \sqrt{5} > r + r^{-1}$, since $r > \frac{\sqrt{5}-1}{2}$ and $r + r^{-1}$ is a decreasing function in $(0, 1)$. Therefore $|r_n| < r$. This proves the first statement. Now suppose that $|a_n| \leq 2$ and $\operatorname{Re} a_n a_{n+1} \geq \chi$ where χ is as above; in particular $\operatorname{Re} a_n a_{n+1} \geq 0$.

Before continuing with the proof we note that for any $r > 0$ and $z \in \mathbb{C}$ such that $|z| > r$ the set $B(z, r)^{-1} := \{\zeta^{-1} \mid \zeta \in B(z, r)\}$ of the inverses of elements from $B(z, r)$, is precisely $B(\dfrac{\bar{z}}{|z|^2 - r^2}, \dfrac{r}{|z|^2 - r^2})$.

Now, as $r_{n-1} \in B(0, r)$ and $r < |a_n|$, we get that r_n, which is $1/(a_n + r_{n-1})$, is contained in $B(a_n, r)^{-1} = B(y\bar{a}_n, yr)$, where $y = 1/(|a_n|^2 - r^2)$. We have $r < 1/\sqrt{2}$, so $1/(\sqrt{2} - r) < 1/(\sqrt{2} - 1/\sqrt{2}) = \sqrt{2}$. Since $|a_n|$ and $|a_{n+1}|$ are at least $\sqrt{2}$, this shows that $|a_{n+1}| > 1/(|a_n| - r)$, which is the same as $y(|a_n| + r)$. Therefore $|y\bar{a}_n + a_{n+1}| \geq y|a_n| - |a_{n+1}| > yr$. We have $r_{n+1} = 1/(a_{n+1} + r_n) \in B(a_{n+1} + y\bar{a}_n, yr)^{-1}$ and in the light of the observation above it follows that $r_{n+1} \in B(\frac{\bar{z}}{|z|^2 - s^2}, \frac{s}{|z|^2 - s^2})$, where $z = y\bar{a}_n + a_{n+1}$ and $s = yr$. We would like to conclude that $r_{n+1} \in B(0, r)$ and for this it suffices to show that $|\frac{\bar{z}}{|z|^2 - s^2}| + |\frac{s}{|z|^2 - s^2}| < r$, or equivalently that $r(|z| - s) > 1$. Substituting for z and s and putting $x = y^{-1}$ we see that the above condition is equivalent to $r(|\bar{a}_n + x a_{n+1}| - r) > x = |a_n|^2 - r^2$, or equivalently $r|\bar{a}_n + x a_{n+1}| > |a_n|^2$. Squaring both the sides and recalling that $r^2 = |a_n|^2 - x$ the condition may be written as

$$(|a_n|^2 - x)(\bar{a}_n + x a_{n+1})(a_n + x \bar{a}_{n+1}) > |a_n|^4.$$

The condition can be readily simplified to

$$|a_{n+1}|^2 x^2 + (-\alpha + \beta)x - |a_n|^2(\beta - 1) < 0,$$

where $\alpha = |a_n|^2|a_{n+1}|^2$ and $\beta = 2\text{Re } a_na_{n+1}$. The discriminant Δ of the quadratic polynomial is seen to be $(\alpha+\beta)^2 - 4\alpha$. Since $\beta = 2\text{Re } a_na_{n+1} \geq 0$ and $\alpha \geq 4$, we get that $\Delta \geq 0$.

The (real) roots of the quadratic are given by $\lambda_\pm = (\alpha - \beta \pm \sqrt{\Delta})|a_n|^2/2\alpha$ and the desired condition as above holds if $x \in (\lambda_-, \lambda_+)$. As $r^2 = |a_n|^2 - x$ we see that the condition holds if r^2 is in the interval $(|a_n|^2 - \lambda_+, |a_n|^2 - \lambda_-)$, or equivalently if the following inequalities hold:

$$(2\alpha)^{-1}|a_n|^2(\alpha+\beta-\sqrt{\Delta}) < r^2 < (2\alpha)^{-1}|a_n|^2(\alpha+\beta+\sqrt{\Delta}).$$

We note that the last term is at least $(2\alpha)^{-1}|a_n|^2\alpha = |a_n|^2/2 \geq 1$, and hence it is greater than r^2. It now suffices to prove that the first inequality holds, or equivalently that

$$\sigma^2 < (2\alpha)|a_n|^{-2}(\alpha+\beta-\sqrt{\Delta})^{-1} = \frac{1}{2}|a_n|^{-2}(\alpha+\beta+\sqrt{\Delta}),$$

as $(\alpha+\beta)^2 - \Delta = 4\alpha$. Now first suppose that $|a_{n+1}| \geq 2$. Then, as $\beta \geq 0$, the right hand side is at least $\frac{1}{2}|a_n|^{-2}(4|a_n|^2 + \sqrt{16|a_n|^4 - 16|a_n|^2}) = 2(1+\sqrt{1-|a_n|^{-2}}) \geq 2+\sqrt{2}$, and hence it indeed exceeds σ^2. Now suppose $|a_{n+1}| = \sqrt{2}$. Then we have $\text{Re } a_na_{n+1} \geq 2$. Under these conditions it can be explicitly verified that the right hand expression above exceeds $(\frac{\sqrt{5}+1}{2})^2 = \frac{3+\sqrt{5}}{2}$; for instance, if $a_{n+1} = 1+i$ then the only possibilities for a_n in $\mathfrak{G}$ with $|a_n| \leq 2$ and $\text{Re } a_na_{n+1} \geq 2$ are $1-i, 2$ or $-2i$ and the values of the expression may be seen to be $2+\sqrt{3}$ in the first case and $(3+\sqrt{7})/2$ in the other two cases; the corresponding assertion for other choices of a_{n+1} follows from this from symmetry considerations, or may be verified directly. Therefore r^2 is contained in the desired interval and the argument shows that $r_{n+1} \in B(0,r)$. This proves the proposition. □

The following corollary follows immediately, by application of Proposition 5.1 to values of σ smaller than $(\sqrt{5}+1)/2$.

Corollary 5.2. *Let $\{a_n\}_{n=0}^m$ be a sequence in $\mathfrak{G}$ such that $|a_n| > 1$ for all $n \geq 1$ and let $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Suppose that $q_n \neq 0$ for all $n \geq 0$. Let $1 \leq n \leq m-1$ be such that $|q_{n-1}| \geq (\frac{\sqrt{5}+1}{2})|q_{n-2}|$, and either $|a_n| > 2$ or* Re *$a_na_{n+1} \geq \chi$, where $\chi = 2$ if $|a_{n+1}| = \sqrt{2}$ and 0 otherwise. Then either $|q_n|/|q_{n-1}|$ or $|q_{n+1}|/|q_n|$ is at least $\frac{\sqrt{5}+1}{2}$.*

If $\{a_n\}$ is a sequence in $\mathfrak{G}$ corresponding to continued fraction expansion with respect to the Hurwitz algorithm then for all $n \geq 1$ such that $|a_n| \leq 2$ we have $\text{Re } a_na_{n+1} \geq \chi$, where χ is as in Proposition 5.1 (see [9] or [8]). We thus get the following Corollary.

Corollary 5.3. *Let $\{a_n\}$ be a sequence in $\mathfrak{G}$ given by the continued fraction expansion of a complex number with respect to the Hurwitz algorithm, and let $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Then for any $n \geq 1$ either $|q_n|/|q_{n-1}|$ or $|q_{n+1}|/|q_n|$ is at least $(\sqrt{5}+1)/2$.*

Proof. We note that the condition in Corollary 5.2 holds for $n = 1$, since $q_{-1} = 0$. Applying the Corollary successively we get that for any $n \geq 1$ one of the ratios $|q_n|/|q_{n-1}|$ or $|q_{n+1}|/|q_n|$ is at least $(\sqrt{5}+1)/2$. □

Remark 5.4. Corollary 5.3 strengthens Lemma 5.1 of [8], where the corresponding assertion is proved (for sequences arising from the Hurwitz algorithm) with the constant $\frac{3}{2}$ in place of $\frac{\sqrt{5}+1}{2}$.

As noted earlier, in various contexts it is useful to know whether $|q_n|$ is increasing with n. We shall next address the issue and describe conditions under which this is assured. In this respect we first note the following simple fact about certain finite sequences of complex numbers, which will also be used later.

We denote by $\sigma_y : \mathbb{C} \to \mathbb{C}$ the reflexion in the y-axis, namely the map defined by $\sigma_y(z) = -\bar{z}$ for all $z \in \mathbb{C}$.

Lemma 5.5. *Let $z_0, z_1, \ldots, z_m$ be nonzero complex numbers such that $|z_0| \leq 1$, $|z_n| > 1$ for $n = 1, \ldots, m$, and let $b_n = z_n - z_{n+1}^{-1}$ for all $n = 0, \ldots, m-1$. Suppose that $b_n \in \mathfrak{G}$ for all $n \leq m-1$, $|b_0| > 1$ and that for $1 \leq n \leq m-1$ either $|b_n - \sigma_y^n(b_0)| \geq 2$ or $b_n = 2\sigma_y^n(b_0)$. Then $|b_0| = \sqrt{2}$, $b_n = 2\sigma_y^n(b_0)$ for $n = 1, \ldots, m-1$, and $|z_{n+1}| \leq \sqrt{2}+1$ for all $n = 1, \ldots, m$.*

Proof. We have $b_0 = z_0 - z_1^{-1}$, $|z_0| \leq 1$ and $|z_1| > 1$, which yields $|b_0| < 2$, and since $b_0 \in \mathfrak{G}$ and $|b_0| > 1$ we get $|b_0| = \sqrt{2}$. We have $z_1^{-1} \in \bar{B}(-b_0) \cap B$ and hence we get that $z_1 \in \bar{B}(\sigma_y(b_0)) \backslash \bar{B}$, and in particular $|z_1| \leq \sqrt{2}+1$. Also, since $z_1 \in \bar{B}(\sigma_y(b_0))$ and $|z_1 - b_1| < 1$ we get that $|b_1 - \sigma_y(b_0)| < 2$. Hence by the condition in the hypothesis $b_1 = 2\sigma_y(b_0)$. Then $z_2^{-1} + 2\sigma_y(b_0) = z_1 \in \bar{B}(\sigma_y(b_0))$, and so $z_2^{-1} \in \bar{B}(-\sigma_y(b_0))$. Also, by hypothesis we have $z_2^{-1} \in B$. Hence $z_2 \in \bar{B}(\sigma_y^2(b_0)) \backslash \bar{B}$, and in particular $|z_2| \leq \sqrt{2}+1$. Again, since $z_2 \in \bar{B}(\sigma_y^2(b_0))$ and $|z_2 - b_2| < 1$ this implies that $|b_2 - \sigma_y^2(b_0)| < 2$. Therefore by the condition in the hypothesis we get that $b_2 = 2\sigma_y^2(b_0)$. Repeating the argument we see $b_n = 2\sigma_y^n(b_0)$ and $|z_{n+1}| \leq \sqrt{2}+1$ for all $n = 0, \ldots, m-1$. This proves the Lemma. □

Definition 5.6. A sequence $\{a_n\}_{n=0}^{\infty}$ in $\mathfrak{G}$ such that $|a_n| > 1$ for all $n \geq 1$ is said to satisfy *Condition (H)* if for all i, j such that $1 \leq i < j$, $|a_j| = \sqrt{2}$ and $a_k = 2\sigma_y^{j-k}(a_j)$ for all $k \in \{i+1, \ldots, j-1\}$ (empty set if $i = j-1$), either $a_i = 2\sigma_y^{j-i}(a_j)$ or $|a_i - \sigma_y^{j-i}(a_j)| \geq 2$.

(It may be noted that depending on the specific value of a_j and whether $j-i$ is odd or even, in the above definition, a_i is disallowed to belong to a set of five elements of $\mathfrak{G}$ with absolute value greater than 1; if $a_j = 1+i$ and $j-i$ is odd then these are $2i, -1+i, -1+2i, -2$ and $-2+i$; for other a_j with $|a_j| = \sqrt{2}$ the corresponding sets can be written down using symmetry under the maps $z \mapsto -z$ and $z \mapsto \bar{z}$).

Definition 5.7. A sequence $\{a_n\}_{n=0}^{\infty}$ in $\mathfrak{G}$ such that $|a_n| > 1$ for all $n \geq 1$ is said to satisfy *Condition (H′)* if $|a_n + \bar{a}_{n+1}| \geq 2$ for all $n \geq 1$ such that $|a_{n+1}| = \sqrt{2}$.

Remark 5.8. We note that if Condition (H′) holds, then Condition (H) also holds, trivially. Condition (H) is strictly weaker than Condition (H′); viz. it can be seen there exists $z \in \mathbb{C}'$ such that in the corresponding continued fraction sequence $\{a_n\}$ via the nearest integer algorithm $-2+2i$ may be followed by $1+i$. While Condition (H′) is simpler, and may be used with other algorithms (see Theorem 6.7), Condition (H) is of significance since the sequences occurring as continued fraction expansions with respect the Hurwitz algorithm satisfy it, as was observed in Hurwitz [9], but not Condition (H′) in general.

Proposition 5.9. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\mathfrak{G}$, with $|a_n| > 1$ for all $n \geq 1$, satisfying Condition (H) and let $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Then $|q_n| > |q_{n-1}|$ for all $n \geq 1$.*

Proof. If possible suppose $|q_{n+1}/q_n| \leq 1$ for some $n \in \mathbb{N}$ and let m be the smallest integer for which this holds. For all $0 \leq k \leq m$ let $z_k = q_{m-k+1}/q_{m-k}$, and $b_k = a_{m-k+1}$. We have $|z_0| \leq 1$, while for all $k = 1, \dots, m$, $|z_k| > 1$. Also $z_k = \dfrac{q_{m-k+1}}{q_{m-k}} = a_{m-k+1} + \dfrac{q_{m-k}}{q_{m-k-1}} = a_{m-k+1} + z_{k+1}^{-1} = b_k + z_{k+1}^{-1}$ for all $k \leq m$. As $\{a_n\}$ satisfies Condition (H) it follows that the condition in Lemma 5.5 is satisfied for $z_0, z_1, \dots, z_m$ as above, and the lemma therefore implies that $|b_0| = \sqrt{2}$, $b_m = 2\sigma_y^m(b_0)$ and $|z_m| \leq \sqrt{2}+1$. But this is a contradiction, since $z_m = q_1/q_0 = a_1 = b_m$ and $|b_m| = |2\sigma_y^m(b_0)| = 2\sqrt{2}$. Therefore $|q_n| > |q_{n-1}|$ for all $n \geq 1$. This proves the proposition. □

Definition 5.10. A sequence $\{a_n\}$ in $\mathfrak{G}$ such that $|a_n| > 1$ for all $n \geq 1$, is said to satisfy *Condition (A)* if for all $n \geq 1$ such that $|a_n| \leq 2$, Re $a_n a_{n+1} \geq \chi$, where $\chi = 2$ if $|a_{n+1}| = \sqrt{2}$ and 0 otherwise. (If $|a_n| \leq 2$ then a_{n+1} is required to be such that Re $a_n a_{n+1} \geq 0$; if moreover $|a_{n+1}| = \sqrt{2}$ then the condition further stipulates that Re $a_n a_{n+1} \geq 2$).

We note that both Condition (H) and Condition (A) hold for the sequences $\{a_n\}$ arising from the Hurwitz algorithm (see [9], [8]). Corollary 5.2 and Proposition 5.9 together imply the following.

Proposition 5.11. *Let $\{a_n\}$ be a sequence in $\mathfrak{G}$, with $|a_n| > 1$ for all $n \geq 1$, satisfying Conditions (H) and (A), and let $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Then $|q_n| > (\frac{\sqrt{5}+1}{2})|q_{n-2}|$ for all $n \geq 2$.*

An analogous conclusion with a smaller constant can also be proved under a weaker condition than Condition (A), which has also the advantage that it involves exclusion of occurrence of only finitely many pairs, as (a_n, a_{n+1}).

Definition 5.12. A sequence $\{a_n\}$ in $\mathfrak{G}$ such that $|a_n| > 1$ for all $n \geq 1$, is said to satisfy *Condition (A′)* if for all $n \geq 1$ such that $|a_n| = \sqrt{2}$ and $|a_{n+1}| \leq 2$, a_{n+1} is one of the Gaussian integers $\bar{a}_n - a_n, \bar{a}_n$ or $\bar{a}_n + a_n$.

(The condition can also be stated equivalently as follows: if $|a_n| = \sqrt{2}$ and $|a_{n+1}| \leq 2$, then $|a_n a_{n+1} + 1| \geq |a_{n+1}| + 1$; this may be explicitly verified to be equivalent to the above condition; when the latter inequality holds, in fact one has $|a_n a_{n+1} + 1| > |a_{n+1}| + \sqrt{5} - 1$.)

Proposition 5.13. *Let $\{a_n\}$ be a sequence in $\mathfrak{G}$ with $|a_n| > 1$ for all $n \geq 1$, satisfying Conditions (H) and (A′), and let $\{p_n\}$, $\{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Then $|q_n| > (\sqrt{5} - 1)|q_{n-2}|$ for all $n \geq 1$.*

Proof. Since $q_{-1} = 0$ and $q_1 = a_1$ the assertion trivially holds for $n = 1$. Now suppose that $n \geq 2$. Since $\{a_n\}$ satisfies Condition (H), by Proposition 5.9 $\{|q_n|\}$ is increasing, and in particular $q_n \neq 0$ for all n. Recall that, we have $q_n = a_n q_{n-1} + q_{n-2}$ and hence $|q_n/q_{n-1}| \geq |a_n| - |q_{n-2}/q_{n-1}| > |a_n| - 1$, as $|q_{n-2}/q_{n-1}| < 1$. Since $|q_{n-1}| > |q_{n-2}|$, when $|a_n| \geq \sqrt{5}$ this yields $|q_n/q_{n-2}| > |q_n/q_{n-1}| > \sqrt{5} - 1$, as desired. Similarly if $|a_{n-1}| \geq \sqrt{5}$ then we have $|q_n/q_{n-2}| > |q_{n-1}/q_{n-2}| > \sqrt{5} - 1$. Now suppose that both $|a_n|$ and $|a_{n-1}|$ are less than $\sqrt{5}$, namely $|a_{n-1}| \leq 2$ and $|a_n| \leq 2$. We recall that $q_n = a_n q_{n-1} + q_{n-2} = a_n(a_{n-1} q_{n-2} + q_{n-3}) + q_{n-2} = (a_n a_{n-1} + 1) q_{n-2} + a_n q_{n-3}$, and hence $|q_n/q_{n-2}| > |a_n a_{n-1} + 1| - |a_n|$. When $|a_{n-1}| = 2$ it can be explicitly verified that for all possible choices of a_n with $|a_n| \leq 2$ that are admissible under Condition (H) we have $|a_n a_{n-1} + 1| - |a_n| > \sqrt{5} - 1$. Now suppose $|a_{n-1}| = \sqrt{2}$. Then as noted above, Condition (A′) implies that for all admissible choices of a_n, $|a_n a_{n-1} + 1| - |a_n|$ exceeds $\sqrt{5} - 1$. This proves the proposition. □

Remark 5.14. If $\{a_n\}$ is a sequence with $a_n \in \mathbb{Z} \subset \mathfrak{G}$ for all $n \geq 0$ and $|a_n| \geq 2$ for all $n \geq 1$, then Condition (H) and Condition (A′) are satisfied and hence the conclusion of Proposition 5.13 holds for these sequences. If moreover a_{n+1}/a_n is positive whenever $|a_n| = 2$, then Condition (A) also holds and in this case the conclusion of Proposition 5.11 holds.

6 Sequence spaces and continued fraction expansion

In this section we introduce various spaces of sequences which serve as continued fraction expansions.

For $\theta > 1$ let $\Omega_E^{(\theta)}$ denote the set of sequences $\{a_n\}$ such that if $\{p_n\}$, $\{q_n\}$ is the corresponding $\mathcal{Q}$-pair then, for all $n \geq 1$, $|q_n| > |q_{n-1}|$ and $|q_n| \geq \theta |q_{n-2}|$; we note that for these sequences $|q_n| \geq \theta^{(n-1)/2}$ for all $n \geq 0$ (the suffix E in the notation signifies exponential growth of the $|q_n|$'s). By Proposition 5.11 the sequences $\{a_n\}$ satisfying Condition (H) and Condition (A) are contained in $\Omega_E^{(\frac{\sqrt{5}+1}{2})}$; in particular every sequence arising as the continued fraction expansion of any z in $\mathbb{C}'$ with respect to the Hurwitz algorithm is contained in $\Omega_E^{(\frac{\sqrt{5}+1}{2})}$. Similarly by Proposition 5.13 the sequences $\{a_n\}$ satisfying Condition (H) and Condition (A′) are contained in $\Omega_E^{(\sqrt{5}-1)}$.

Theorem 6.1. *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence in $\Omega_E^{(\theta)}$, for some $\theta > 1$, and let $\{p_n\}, \{q_n\}$ the corresponding $\mathcal{Q}$-pair. Then the following conditions hold:*

i) $\{p_n/q_n\}$ converges as $n \to \infty$;

ii) if z is the limit of $\{p_n/q_n\}$ then $|z - \frac{p_n}{q_n}| \leq \frac{c}{|q_n|^2}$, where $c = \frac{2\theta^2}{(\theta^2 - 1)}$;

iii) the limit z is an irrational complex number, viz. $z \in \mathbb{C}'$.

Proof. We have $p_{n+1}q_n - p_n q_{n+1} = (-1)^n$ for all $n \geq 0$, so $\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n+1}}$, and in turn $|\frac{p_{n+m}}{q_{n+m}} - \frac{p_n}{q_n}| \leq \sum_{k=0}^{m-1} \frac{1}{|q_{n+k}q_{n+k+1}|}$ for all $n \geq 0$ and $m \geq 1$. As $|q_k/q_{k-2}| \geq \theta$ for all $k \geq 1$ this shows that $\{p_n/q_n\}$ is convergent. This proves (i). Assertion (ii) then follows from Proposition 2.1. Finally, it is easy to see that such an approximation can not hold for rational complex numbers and hence the limit must be an irrational number. □

For any $\{a_n\}$ in $\Omega_E^{(\theta)}$, $\theta > 1$, we denote by $\omega(\{a_n\})$ the limit of $\{p_n/q_n\}$ where $\{p_n\}, \{q_n\}$ is the corresponding $\mathcal{Q}$-pair. The following corollary shows that for any fixed $\theta > 1$ the convergents coming from possibly different representations of a number z from $\Omega_E^{(\theta)}$ also converge to z, provided the corresponding indices go to infinity.

Corollary 6.2. *Let $\theta > 1$, $z \in \mathbb{C}'$ be given, and $\{l_j\}$ be a sequence in $\mathbb{N}$ such that $l_j \to \infty$ as $j \to \infty$. Let $\{z_j\}$ be a sequence in $\mathbb{C}$ such that the following holds: for* *each j there exists a sequence $\{a_n\}$ in $\Omega_E^{(\theta)}$ such that $z = \omega(\{a_n\})$ and $z_j = p_n/q_n$ for some $n \geq l_j$, where $\{p_n\}$, $\{q_n\}$ is the $\mathcal{Q}$-pair corresponding to $\{a_n\}$. Then $z_j \to z$ as $j \to \infty$.*

Proof. By Proposition 6.1 we have $|z - z_j| \leq \dfrac{c}{|q_n|^2} \leq \dfrac{c}{\theta^{n-1}} \leq \dfrac{c}{\theta^{l_j-1}}$, where $c = 2\theta^2/(\theta^2 - 1)$. As $l_j \to \infty$ this shows that $z_j \to z$, as $j \to \infty$. □

We equip $\Omega_E^{(\theta)}$, $\theta > 1$, with the topology induced as a subset of the space of all sequences $\{a_n\}$ with $a_n \in \mathfrak{G}$ and $|a_n| > 1$ for all $n \geq 1$, the latter being equipped with the product topology as a product of the individual copies of $\mathfrak{G}$ (for the 0'th component) and $\mathfrak{G}\backslash\{0\}$ (for the later components) with the discrete topology.

Proposition 6.3. *For all $\theta > 1$ the map $\omega : \Omega_E^{(\theta)} \to \mathbb{C}$ is continuous.*

Proof. Let $\{a_n\}$ be a sequence from $\Omega_E^{(\theta)}$ and let $\{p_n\}, \{q_n\}$ be the corresponding $\mathcal{Q}$-pair. Let $\epsilon > 0$ be given and $m \in \mathbb{N}$ be such that $|q_m|^2 > 2c\epsilon^{-1}$, where $c = 2\theta^2/(\theta^2 - 1)$. If $\{a'_n\}$ is any sequence in $\Omega_E^{(\theta)}$ such that $a'_n = a_n$ for $n = 0, 1, \ldots, m$, and $\{p'_n\}, \{q'_n\}$ is the corresponding $\mathcal{Q}$-pair then $p'_n = p_n$ and $q'_n = q_n$ for $n = 0, 1, \ldots, m$. Then by Proposition 6.1 (ii), $|\omega(\{a_n\}) - \dfrac{p_m}{q_m}| \leq \dfrac{c}{|q_m|^2} < \epsilon/2$, as well as $|\omega(\{a'_n\}) - \dfrac{p_m}{q_m}| < \epsilon/2$, so $|\omega(\{a'_n\}) - \omega(\{a_n\})| < \epsilon$. This shows that ω is continuous. □

The spaces $\Omega_E^{(\theta)}$ deal with continued fraction expansions in terms of the growth of the $|q_n|$'s, and do not seem to be amenable to a priori description, though of course as seen earlier verification of certain conditions on the sequences enables to conclude that they belong to $\Omega_E^{(\theta)}$ for certain values of θ. Our next objective is to describe a space of sequences defined in terms of nonoccurrence of certain simple blocks, related to Conditions (H′) and (A′), which yield continued fraction expansion. We begin by introducing an algorithm such that the corresponding sequences of partial quotients satisfy these conditions.

Let S be the square $\{z = x + iy \mid |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}$. Let P be the set of 8 points on the boundary of S such that one of the coordinates is $\pm\frac{1}{6}$ (the other being $\pm\frac{1}{2}$). Let R be the set defined by

$$R = \{z \in S \mid d(z, p) \geq \frac{1}{3} \text{ for all } p \in P\}.$$

It may be noted that R is a star-like subset with the origin as the center, bounded by 8 circular segments; see Figure 1.

We now define a choice function for a continued fraction algorithm as follows: let $z \in \mathbb{C}^*$ and $z_0 \in \mathfrak{G}$ be such that $z - z_0 \in S$. If $z - z_0 \notin R$ we define $f(z)$ to

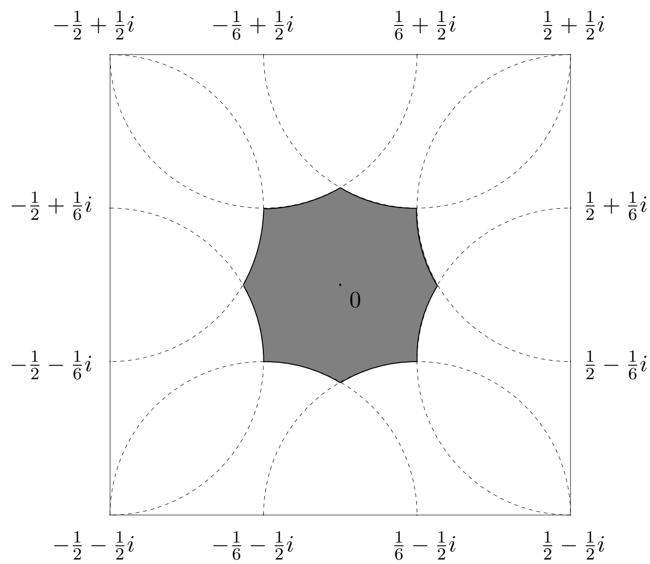

Figure 1: The set R.

be the Gaussian integer nearest to z, and if $z - z_0 \in R$ we choose $f(z)$ to be the nearest odd Gaussian integer; the choice for z_0 as above may not be unique, but the condition $z - z_0 \notin R$ does not depend on the specific choice; the choices of the nearest integer or odd integer are not unique for certain z but we shall assume that these are fixed with some convention - this will not play any role in the sequel. We shall call the algorithm corresponding to the function f as above the *PPOI algorithm*, using as name the acronym for "partially preferring odd integers".

Let C denote the set of four corner points of S, namely $\frac{1}{2}(\pm 1 \pm i)$, and for each $c \in C$ let $T(c)$ denote the subset of S bounded by the triangle formed by the other three vertices; the boundary triangle is included in $T(c)$. Also for each $c \in C$ let $\Phi(c) = c + (R \cap T(c)) = \{c + z \mid z \in R \cap T(c)\}$. It can be seen that the fundamental set of the PPOI algorithm is given by

$$\Phi = S \cup \bigcup_{c \in C} \Phi(c).$$

Furthermore, if $a \in \mathfrak{G}$ is an odd integer then $f^{-1}(\{a\})$ contains all interior points of $a + \Phi$ and if a is even $f^{-1}(\{a\})$ is disjoint from $\cup_{c \in C} \Phi(c)$ (and contains the interior of its complement in Φ).

Proposition 6.4. *Let $z \in \mathbb{C}'$ and $\{a_n\}$ be the associated sequence of partial quotients with respect to the PPOI algorithm. Then $\{a_n\}$ satisfies Condition (H′) and Condition (A′).*

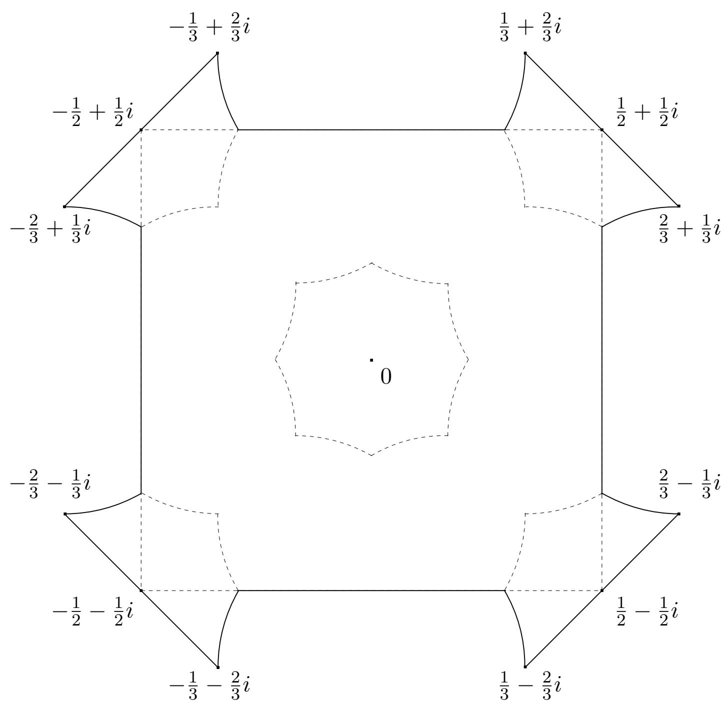

Figure 2: The fundamental set Φ.

Proof. We have to show that for $n \geq 1$ if $|a_{n+1}| = \sqrt{2}$ then $|\bar{a}_n + a_{n+1}| \geq 2$ and if $|a_{n+1}| \leq 2$ and $|a_n| = \sqrt{2}$ then $|a_n a_{n+1} + 1| \geq |a_{n+1}| + 1$. By symmetry considerations it suffices to show that if $a_{n+1} = 1+i$ then a_n is not one of $-2, -1+i, 2i, -2+i, -1+2i, -2+2i$ and also not one of $-1+i, -1-i$ (correspondingly to the two conditions respectively), if $a_{n+1} = 2$ then a_n is not one of $-2, -1+i, -1-i$ and if $a_{n+1} = 2i$ then a_n is not one of $2i, 1+i, -1+i$.

Let $\{z_n\}$ denote that iteration sequence corresponding to z, with respect to the PPOI algorithm and let $n \geq 1$ be such that $a_{n+1} = 1+i$. Then $z_{n+1} \in (1+i+S) \cap \Phi^{-1}$. It can be seen that the latter is the part of $1+i+S$ in the complement of $B(2), B(2i)$ and $B(\frac{1}{2}(1+i), \frac{1}{\sqrt{2}})$. We deduce from this that z_{n+1}^{-1}, which is the same as $z_n - a_n$, is contained in $\Phi(\frac{1}{2}(1-i))$. Since $f^{-1}(a)$ is disjoint from $\Phi(c)$ for all $c \in C$ when a is even, this implies that a_n is odd. Thus a_n does not take any of the values listed above, except possibly $-2+i$ or $-1+2i$. But we see directly that if a is one of these then $(a+S) \cap \Phi^{-1}$ is disjoint from $\Phi(\frac{1}{2}(1-i))$. Thus a_n does not take any of the values as above.

Now suppose that $a_{n+1} = 2$. We have $z_{n+1} \in (2+S) \cap \Phi^{-1}$, which may be seen to be contained in $(2+S) \backslash B(1)$. It can be readily deduced from this that z_{n+1}^{-1}, or equivalently $z_n - a_n$, is contained in $B(\frac{1}{2}(1-i), \frac{1}{\sqrt{2}})$ as also $B(\frac{1}{2}(1+i), \frac{1}{\sqrt{2}})$. On the other hand $(-1+i+S) \cap \Phi^{-1}$ and $(-1-i+S) \cap \Phi^{-1}$ are seen to be disjoint from $B(\frac{1}{2}(1-i), \frac{1}{\sqrt{2}})$ and $B(\frac{1}{2}(1+i), \frac{1}{\sqrt{2}})$ respectively. Thus a_n is not one of $-1+i$ or $-1-i$. Similarly, observing that z_{n+1}^{-1} is contained in $B(1)$ while $(-2+S) \cap \Phi^{-1}$ is disjoint from $-2+B(1) = B(-1)$ we conclude that a_n is not -2.

A similar argument also shows that if $a_{n+1} = 2i$ then a_n is not one of $2i, 1+i, -1+i$. This completes the proof. □

Remark 6.5. In general if $\{a_n\}_{n=0}^{\infty}$ is a sequence defining a continued fraction expansion of a $z \in \mathbb{C}'$, we may not have $|z - a_0| \leq 1$. For example, let $z' \in B(0,1) \cap B(1+i,1) \cap \mathbb{C}'$ and let $\{b_n\}$ be a continued fraction expansion of z' with $b_0 = 0$; if z' is such that 0 is the nearest Guassian integer then $\{b_n\}$ may be taken to be the sequence with respect to the Hurwitz algorithm - otherwise we may use an iterated sequence construciton for this. Then for $z = (1+i-z')^{-1}$ we see that while $|z| > 1$, the sequence $\{a_n\}$ with $a_0 = 0$, $a_1 = 1+i$ and $a_n = b_{n-1}$ for all $n \geq 2$ defines a continued fraction expansion of z, and we have $|z - a_0| > 1$. It may also be noted that when $\{a_n\}_{n=0}^{\infty}$ defines a continued fraction expansion of a $z \in \mathbb{C}'$, while for all $m \in \mathbb{N}$ the sequence $\{a_n\}_{n=m}^{\infty}$ defines a continued fraction expansion of a $z_m \in \mathbb{C}'$, the sequence $\{z_m\}$ may not be an iteration sequence of z, since the z_m may not be in $\bar{B}(0,1)$. We shall see below, for sequences $\{a_n\}$ satisfying Condition (H′) $|\omega(\{a_n\}) - a_0| \leq 1$ and $\omega(\{a_n\}_{n=m}^{\infty})$ is an iteration sequence of $\omega(\{a_n\})$.

As before let σ_y denote the map defined by $\sigma_y(z) = -\bar{z}$ for all $z \in \mathbb{C}$.

Lemma 6.6. *Let $\{a_n\}_{n=0}^{m}$ be a finite sequence with $a_n \in \mathfrak{G}$ for all $n \geq 0$ and $|a_n| > 1$ for $n = 1, \dots, m$. Let $\{p_n\}_{n=0}^{m}$ and $\{q_n\}_{n=0}^{m}$ be the corresponding $\mathcal{Q}$-pair. Suppose also that for all $n \geq 1$ with $|a_n| = \sqrt{2}$ the following holds: if for some k, with $1 \leq k \leq m-n$, we have $a_{n+j} = 2\sigma_y^j(a_n)$ for all $j \in \{1, \dots, k-1\}$ (empty set if $k=1$) and $|a_{n+k} - \sigma_y^k(a_n)| < 2$ then $a_{n+k} = 2\sigma_y^k(a_n)$. Then $|\frac{p_m}{q_m} - a_0| < 1$.*

Proof. Let $\zeta = p_m/q_m$ and let $\zeta_0, \zeta_1, \dots, \zeta_m$ be defined by $\zeta_0 = \zeta$, and $\zeta_n = (\zeta_{n-1} - a_{n-1})^{-1}$ for $n = 1, \dots, m$. Then we see that $\zeta_m = a_m$. Suppose that $|\zeta - a_0| \geq 1$. Then we have $|\zeta_m| = |a_m| > 1$ while on the other hand $|\zeta_1| = |\zeta - a_0|^{-1} \leq 1$. Hence there exists $1 \leq k \leq m-1$ such that $|\zeta_k| \leq 1$ and $|\zeta_{k+n}| > 1$ for $n = 1, \dots, m-k$. For all $n \leq m-k$ let $z_n = \zeta_{k+n}$ and $b_n = a_{n+k}$. Then we have $|z_0| \leq 1$ and $|z_n| > 1$ for $n = 1, \dots, m-k$. Also $z_n - z_{n+1}^{-1} = \zeta_{n+k} - \zeta_{n+k+1}^{-1} = a_{n+k} = b_n$. Applying Lemma 5.5 and using the condition as in the hypothesis we get that $|\zeta_m| = |z_{m-k}| \leq \sqrt{2}+1$ and $|a_m| = |b_{m-k}| = 2\sqrt{2}$. But this is a contradiction since $\zeta_m = a_m$. Therefore $|\frac{p_m}{q_m} - a_0| < 1$. □

We can now conclude the following.

Theorem 6.7. *Every irrational complex number z admits a continued fraction expansion $\{a_n\}_{n=0}^{\infty}$, $a_n \in \mathfrak{G}$ for all n, satisfying the following conditions:*

i) $|a_n| > 1$ *for all* $n \geq 1$,

ii) $|a_n + \bar{a}_{n+1}| \geq 2$ *for all* $n \geq 1$ *such that* $|a_{n+1}| = \sqrt{2}$, *and*

iii) $|a_n a_{n+1} + 1| \geq |a_{n+1}| + 1$ *for all* $n \geq 1$ *such that* $|a_n| = \sqrt{2}$ *and* $|a_{n+1}| \leq 2$.

Conversely, if $\{a_n\}_{n=0}^{\infty}$, $a_n \in \mathfrak{G}$, is a sequence satisfying conditions (i), (ii) and (iii), and $\{p_n\}$, $\{q_n\}$ is the corresponding $\mathcal{Q}$-pair, then the following holds:

a) $q_n \neq 0$ *for all* $n \geq 0$ *and* p_n/q_n *converges to an irrational complex number* z *as* $n \to \infty$, *(so* $\{a_n\}$ *defines a continued fraction expansion of* z*);*

b) $|z - a_0| \leq 1$ *and* $\{z_n\}$ *is an iteration sequence for* z*;*

c) z *is a quadratic surd if and only if* $\{a_n \mid n \in \mathbb{N}\}$ *is finite.*

Proof. The first assertion follows from Proposition 6.4. In the converse part, assertion (a) follows from Proposition 5.13 and Theorem 6.1. Lemma 6.6 implies by condition (ii) that $|z - a_0| \leq 1$, and applying the argument successively to $\omega(\{a_n\}_{n=k}^{\infty})$, $k = 1, 2, \ldots$ we see that $\{z_n\}$ is an iteration sequence for z, and thus (b) is proved. Assertion (c) follows from Proposition 4.1 and Corollary 4.3. $\square$

7 Associated fractional transformations

As with the usual continued fractions we can associate with the complex continued fractions transformations of the projective space, the complex projective space in this instance.

We denote by $\mathbb{C}^2$ the two-dimensional complex vector space, viewed as the space of 2-rowed columns with complex number entries. We denote by e_1 and e_2 the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, in $\mathbb{C}^2$, respectively. For all $z \in \mathbb{C}$ we denote by v_z the vector $ze_1 + e_2$.

We denote by $\mathbb{P}$ the corresponding one-dimensional complex projective space, consisting of complex lines (one-dimensional vector subspaces) in $\mathbb{C}^2$ and by $\pi : \mathbb{C} \to \mathbb{P}$ the map defined by setting, for all $z \in \mathbb{C}$, $\pi(z)$ to be the complex line containing v_z.

We denote by $SL(2, \mathbb{C})$ the group of 2×2 matrices with complex entries, with determinant 1. The natural linear action of $SL(2, \mathbb{C})$ on $\mathbb{C}^2$ induces an action of the group on $\mathbb{P}$. We shall be using this action freely in the sequel, with no further mention. We denote by $SL(2, \mathfrak{G})$ the subgroup of $SL(2, \mathbb{C})$ consisting of all its elements with entries in $\mathfrak{G}$.

When $\mathbb{C}$ is realised as a subset of $\mathbb{P}$ via the map π as above, the subset $\mathbb{C}'$ (as before, consisting of complex numbers which are not rational) is readily seen to be invariant under the action of $SL(2, \mathfrak{G})$; the action of the element $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathfrak{G})$ on $\mathbb{C}'$ is then given by $z \mapsto \dfrac{pz+q}{rz+s}$ for all $z \in \mathbb{C}'$.

We denote by ρ the matrix defined by

$$\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

For any $g \in SL(2, \mathbb{C})$ we denote by ${}^t g$ the transpose of the matrix g. It can be verified that for any $g \in SL(2, \mathbb{C})$, ${}^t g \rho g = \rho$; this identity will be used crucially below. For any $a \in \mathfrak{G}$ we denote by U_a and L_a the matrices defined by

$$U_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ and } L_a = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

By a *block* β of Gaussian integers we mean a finite l-tuple $\beta = (b_1, \dots, b_l)$, where $l \geq 1$ and $b_1, \dots, b_l \in \mathfrak{G}$; the l is called the *length* of the block and will be denoted by $l(\beta)$. For a block $\beta = (b_1, \dots, b_l)$ we define

$$g(\beta) = U_{b_1} L_{b_2} \cdots U_{b_{l-1}} L_{b_l} \quad \text{or} \quad U_{b_1} L_{b_2} \cdots L_{b_{l-1}} U_{b_l},$$

according to whether l is even or odd, respectively.

A block $\beta = (b_1, \dots, b_l)$ is said to *occur* in a sequence $\{a_n\}$ in $\mathfrak{G}$, starting at $k \geq 0$ if $a_k = b_1$, $a_{k+1} = b_2, \dots, a_{k+l-1} = b_l$, and it is said to be an *initial block* of $\{a_n\}$ if it occurs starting at 0.

Remark 7.1. Let $\{a_n\}$ be a sequence from $\Omega_E^{(\theta)}$, $\theta > 1$, and $\beta = (b_1, \dots, b_l)$ be an initial block of $\{a_n\}$. Let $\{p_n\}$, $\{q_n\}$ be the $\mathcal{Q}$-pair corresponding to $\{a_n\}$. Then it is straightforward to verify, say by induction, that for $r = 1$ and 2, $g(\beta)(e_r) = p_m e_1 + q_m e_2$, where m is the largest odd integer less than $l(\beta)$ if $r = 1$ and the largest even integer less than $l(\beta)$ if $r = 2$.

For $\theta > 1$, a sequence $\{\beta_i\}$ of blocks of Gaussian integers is said to *θ-represent* a number $z \in \mathbb{C}'$ if $l(\beta_i) \to \infty$ and for each i there exists a sequence $\{a_n\}$ (depending on i) in $\Omega_E^{(\theta)}$ such that $\omega(\{a_n\}) = z$ and β_i is an initial block of $\{a_n\}$.

Theorem 7.2. *Let $\zeta \in \mathbb{C}'$ and $\{\beta_i\}$ be a sequence of blocks of Gaussian integers θ-representing ζ. Then as $i \to \infty$,*

$$g(\beta_i)(\pi(z)) \to \pi(\zeta) \text{ for all } z \text{ with } |z| > 1,$$

uniformly over $\{z \in \mathbb{C} \mid |z| \geq 1 + \epsilon\}$, for all $\epsilon > 0$.

Proof. It would suffice to prove the assertion in the Theorem separately with all $l(\beta_i)$ being odd and all being even. We shall give the argument when all $l(\beta_i)$ are even; a similar argument works in the other case. Let $z \in \mathbb{C}$ with $|z| > 1$ be given. For each i let j_i and k_i be the largest odd and even integers less than $l(\beta_i)$, respectively; as $l(\beta_i)$ are even we have $j_i = l(\beta_i) - 1$ and $k_i = j_i - 1$ for all i. Then by Remark 7.1 we have

$$g(\beta_i)(\pi(z)) = \pi(\frac{zp_{j_i} + p_{k_i}}{zq_{j_i} + q_{k_i}}) = \pi(\frac{p_{j_i}}{q_{j_i}} \frac{z + (p_{k_i}/p_{j_i})}{z + (q_{k_i}/q_{j_i})}).$$

Since $\{\beta_i\}$ is a sequence of blocks θ-representing ζ, by Corollary 6.2 we get that $\{\frac{p_{j_i}}{q_{j_i}}\}$ and $\{\frac{p_{k_i}}{q_{k_i}}\}$ converge to ζ. Since $\{a_n\}$ belongs to $\Omega_E^{(\theta)}$ and $k_i < j_i$ we have $|q_{k_i}| < |q_{j_i}|$ for all i. As $|z| > 1$ this shows that $\{z + (q_{k_i}/q_{j_i})\}$ is a bounded sequence, bounded away from 0; moreover, whenever a subsequence of $\{z+(q_{k_i}/q_{j_i})\}$ converges (to a nonzero number), the corresponding subsequence of $\{z+(p_{k_i}/p_{j_i})\}$ also converges to the same number, since $\frac{p_{k_i}/p_{j_i}}{q_{k_i}/q_{j_i}} = \frac{p_{k_i}}{q_{k_i}} \frac{q_{j_i}}{p_{j_i}} \to \zeta \cdot \zeta^{-1} = 1$. Thus we get that $g(\beta_i)(\pi(z)) \to \pi(\zeta)$ as $i \to \infty$. Moreover, clearly, for any $\epsilon > 0$ the convergence is uniform over $\{z \in \mathbb{C} \mid |z| \geq 1 + \epsilon\}$. □

We next note also the following simple fact which can be proved along the lines of the proof of Lemma 2.4 of [6]; we omit the details.

Proposition 7.3. *Let $\{g_i\}$ be an unbounded sequence in $SL(2, \mathbb{C})$. Suppose that φ, ψ in $\mathbb{P}$ such that the sets $\{\xi \in \mathbb{P} \mid g_i(\xi) \to \varphi\}$ and $\{\xi \in \mathbb{P} \mid {}^t g_i(\xi) \to \psi\}$ have at least two elements each, and that $g_i(\rho(\psi))$ converges. Then $g_i(\xi) \to \varphi$ for all $\xi \neq \rho(\psi)$.*

Proposition 6.6 readily implies the following.

Corollary 7.4. *Let $\beta = (b_1, \dots, b_l)$ be a block of Gaussian integers, with $l(\beta)$ even, occurring in a sequence $\{a_n\}$ satisfying Condition (H), starting at $k \geq 0$. If $z_r \in \mathbb{C}$, $r = 1, 2$, be the numbers defined by $\pi(z_r) = {}^t g(\beta)(\pi(e_r))$. Then $|z_r - b_l| < 1$.*

We now describe the main technical result towards construction of "generic points", after introducing some more notation and terminology. For a block $\beta = (b_1, \dots, b_l)$ we denote by $\tilde{\beta}$ the block $(b_l, \dots, b_1)$, with the entries arranged in the reverse order. Given two blocks $\beta = (b_1, \dots, b_l)$ and $\beta' = (b'_1, \dots, b'_{l'})$, we denote by $\beta\beta'$ the block $(b_1, \dots, b_l, b'_1, \dots, b'_{l'})$. A block $\beta' = (b'_1, \dots, b'_{l'})$ will be called the *end-block* of $\beta = (b_1, \dots, b_l)$ if $b'_1 = b_{l-l'+1}, \dots, b'_{l'} = b_l$. For a block $\beta = (b_1, \dots, b_l)$, b_1 and b_l will be called the first and last entries respectively.

Proposition 7.5. *Let $\{x_n\}$ be a sequence in $\Omega_E^{(\theta)}$, $\theta > 1$, and let $\xi = \omega(\{x_n\})$. Let $\alpha, \beta \in \mathbb{C}'$ and $\{\alpha_i\}$ and $\{\beta_i\}$ be sequences of blocks of Gaussian integers*

θ-representing α and β respectively. Suppose that the following conditions are satisfied:

i) for all i, α_i and β_i are of even length and their first entries have absolute value greater than 1, and the last entry of α_i has absolute value greater than 2;

ii) there exists a sequence $\{s_i\}$ of even integers such that for all i the block $\tilde{\alpha}_i\beta_i$ occurs in $\{x_n\}$ starting at s_i, $s_{i+1} > s_i + l(\alpha_i) + l(\beta_i)$, $|x_{s_i-1}| > 2$ and $|x_{s_i+l(\alpha_i)+l(\beta_i)}| > 2$.

For all i let $\xi_i = (x_0, \dots, x_{s_i-1})$ and $g_i = g(\alpha_i)^t g(\xi_i)$. Then, as $i \to \infty$,

$$g_i(\pi(z)) \to \pi(\alpha) \text{ for all } z \neq \rho(\pi(\xi)), \text{ and } g_i(\rho(\pi(\xi))) \to \rho(\pi(\beta)).$$

Proof. Let $i \in \mathbb{N}$ and for $r = 1, 2$ let z_r be defined by $\pi(z_r) = {}^t g(\xi_i)(\pi(e_r))$. Since $|x_{s_{i-1}}| > 2$, and hence at least $\sqrt{5}$, by Corollary 7.4 it follows that $|z_r| \geq \sqrt{5} - 1 > 1$. Since this holds for all i by Proposition 7.2 we get that, as $i \to \infty$, $g_i(\pi(e_r)) = g(\alpha_i)(\pi(z_r)) \to \pi(\alpha)$, for $r = 1$ and 2. Similarly, under the conditions in the hypothesis we get ${}^t g_i(\pi(e_r)) \to \pi(\xi)$ for $r = 1, 2$.

We next show that $g_i(\rho(\pi(\xi))) \to \rho(\pi(\beta))$. Then by Proposition 7.3 this would imply the conclusion as stated in the present proposition. Let $1 \leq i < j$. Then from the conditions in the hypothesis we see that ξ_j may be written as $\xi_j = \xi_i \tilde{\alpha}_i \beta_i \varphi_{i,j}$, where $\varphi_{i,j}$ is the end-block of ξ_j of length $s_{i+1} - s_i - l(\alpha_i) - l(\beta_i) > 0$. Therefore $g(\xi_j) = g(\xi_i)^t g(\alpha_i) g(\beta_i) g(\varphi_{i,j})$, and hence

$$g_i \rho g(\xi_j) = g(\alpha_i)^t g(\xi_i) \rho g(\xi_i)^t g(\alpha_i) g(\beta_i) g(\varphi_{i,j}) = \rho g(\beta_i) g(\varphi_{i,j}),$$

using (twice) the fact that ${}^t g \rho g = \rho$ for all $g \in SL(2, \mathbb{C})$. We note that $s_i + l(\alpha_i) + l(\beta_i)$ are even and $x_{s_i+l(\alpha_i)+l(\beta_i)}$ is the first entry of $\varphi_{i,j}$. Since $|x_{s_i+l(\alpha_i)+l(\beta_i)}| \geq \sqrt{5}$, arguing as in the first part we conclude that $\rho g(\beta_i) g(\varphi_{i,j})$ converges to $\rho(\pi(\beta))$, as i and j tend to infinity, for $r = 1, 2$. Hence $g_i \rho g(\xi_j)(\pi(e_r))$, $r = 1, 2$, converge to $\rho(\pi(\beta))$, as i and j tend to infinity. Since any limit point of the sequence $\{g_i(\rho(\pi(\xi)))\}$ is a limit point of $g_i \rho g(\xi_j)(e_r)$ as i and j tend to ∞, this shows that $g_i(\rho(\pi(\xi))) \to \rho(\pi(\beta))$. This completes the proof of the Proposition. $\square$

8 Application to values of quadratic forms

In this section we apply the results on complex continued fractions to the study of values of quadratic forms with complex coefficients over the set of pairs of Gaussian integers. It was proved by G. A. Margulis in response to a long standing conjecture due to A. Oppenheim that for any nondegenerate indefinite real quadratic form Q in $n \geq 3$ variables which is not a scalar multiple of a rational quadratic form, the set of values of Q on the set of n-tuples of integers forms a dense subset of $\mathbb{R}$. This was generalised by A. Borel and G. Prasad to quadratic forms over other locally compact fields [4], and in particular it is known that for a nondegenerate complex quadratic form in $n \geq 3$ variables which is not a scalar multiple of a rational form, the values on the set of n-tuples of Gaussian integers are dense in $\mathbb{C}$. The reader is referred to [5] and [11] for a discussion on the overall theme, including Ratner's pioneering work on dynamics of unipotent flows on homogeneous spaces to which the results are related. The case of binary quadratic forms, viz. $n = 2$, does not fall in the ambit of the theme, and in fact the corresponding statement is not true for all irrational indefinite binary quadratic forms. For example if λ is a badly approximable number (here we may take a real number whose usual partial quotients are bounded) then for the values of the quadratic form $Q(z_1, z_2) = (z_1 - \lambda z_2)z_2$, it may be seen that 0 is an isolated point of the set of values of Q over the set of pairs of Gaussian integers.

In [6] we described a class of real binary quadratic forms for which the values on the set of pairs of integers is dense in $\mathbb{R}$; this is related to a classical result of E. Artin on orbits of the geodesic flow associated to the modular surface; in [6] Artin's result was strengthened, and an application was made to values of the binary quadratic forms over the set of pairs of integers, and also over the set of pairs of positive integers.

In this section, using complex continued fractions we describe an analogous class of complex binary quadratic forms for which the set of values on pairs of Gaussian integers is a dense subset of $\mathbb{C}$.

For $u = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$ and $v = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$, we denote by $Q_{u,v}$ the quadratic form on $\mathbb{C}^2$ defined by

$$Q_{u,v}(z_1e_1 + z_2e_2) = (\alpha_1 z_1 + \alpha_2 z_2)(\beta_1 z_1 + \beta_2 z_2) \text{ for all } z_1, z_2 \in \mathbb{C}.$$

We denote by $\eta : \mathbb{C}^2 \backslash \{0\} \to \mathbb{P}$ the canonical quotient map. We note the following fact which can be proved in the same way as Proposition 5.1 of [6].

Proposition 8.1. *Let $u, v, u', v' \in \mathbb{C}^2\backslash\{0\}$ and suppose that there exists a sequence $\{g_i\}$ in $SL(2, \mathbb{C})$ such that $g_i(\eta(u)) \to \eta(u')$ and $g_i(\eta(v)) \to \eta(v')$, as $i \to \infty$, in $\mathbb{P}$. Let E be a subset of $\mathbb{C}^2$ such that ${}^tg_i(E) \subseteq E$ for all i. Then the closure of the set $Q_{u,v}(E)$ in $\mathbb{C}$ contains $Q_{u',v'}(E)$.*

We denote by $\mathfrak{G}^2$ the subgroup of $\mathbb{C}^2$ consisting of pairs of Gaussian integers, namely $\{ae_1 + be_2 \mid a, b \in \mathfrak{G}\}$. We note that $\mathfrak{G}^2$ is invariant under the linear action of $SL(2, \mathfrak{G})$ on $\mathbb{C}^2$.

Remark 8.2. Consider the quadratic form $Q = Q_{u,v}$ where u and v are linearly independent. We write u and v as $g^{-1}(e_1), g^{-1}(e_2)$ where $g \in GL(2, \mathbb{C})$, so $Q =$

$Q_{g^{-1}(e_1),g^{-1}(e_2)}$. Replacing Q by a scalar multiple we may assume (in studying the question of values over $\mathfrak{G}^2$ being dense) g to be in $SL(2,\mathbb{C})$. Let D be the subgroup of $SL(2,\mathbb{C})$ consisting of diagonal matrices. Then we see that for any $d \in D$, $\eta(u)$ and $\eta(v)$ are fixed under the action of $g^{-1}dg$. If there exist sequences $\{\gamma_i\}$ in $SL(2,\mathfrak{G})$ and $\{d_i\}$ in D such that $\{\gamma_i g^{-1} d_i g\}$ is dense in $SL(2,\mathbb{C})$ then by Proposition 8.1 it follows that $Q_{u,v}(\mathfrak{G}^2)$ is dense in $\mathbb{C}$. The condition involved is equivalent to that the orbit of the coset of g in $SL(2,\mathbb{C})/SL(2,\mathfrak{G})$ under the action of D is dense in $SL(2,\mathbb{C})/SL(2,\mathfrak{G})$. It is well known that $SL(2,\mathfrak{G})$ is a lattice in $SL(2,\mathbb{C})$, viz. $SL(2,\mathbb{C})/SL(2,\mathfrak{G})$ admits a finite measure invariant under the action of $SL(2,\mathbb{C})$ (see [3], §13 for a general result in this respect). This implies in particular that the action of D on $SL(2,\mathbb{C})/SL(2,\mathfrak{G})$ is ergodic (with respect to the $SL(2,\mathbb{C})$-invariant measure); see [2], Chapter III, for instance. Hence for *almost all* g the above condition holds and consequently the values of $Q = Q_{g^{-1}(e_1),g^{-1}(e_2)}$ over $\mathfrak{G}^2$ are dense in $\mathbb{C}$. However, this does not enable describing any specific quadratic form for which the conclusion holds. The point about the application here is that we obtain a set of forms which may be explicitly described, in terms of continued fraction expansions, for which this property holds.

We next define the notion of a generic complex number. In this respect we shall restrict to sequences arising as continued fraction expansion with respect to the Hurwitz algorithm; the procedure can be extended to other algorithms satisfying suitable conditions, but in the present instance we shall content ourselves considering only the Hurwitz algorithm.

We shall assume that the algorithm is determined completely, fixing a convention for tie-breaking when there are more than one Gaussian integers at the minimum distance, so that we can talk unambiguously of the continued fraction expansion of any number; for brevity we shall refer to the corresponding sequence of partial quotients as the *Hurwitz expansion* of the number. Let Ω_H denote the set of sequences $\{a_n\}$ such that $\{a_n\}$ is the Hurwitz expansion of a $z \in \mathbb{C}'$. As before let $S = \{z = x + iy \mid |x| < \frac{1}{2}, |y| < \frac{1}{2}\}$. For a block $\beta = (b_1, \dots, b_l)$, with $b_i \in \mathfrak{G}$ and $|b_i| > 1$ for $i = 1, \dots, l$, we denote by S_β the subset of S consisting of all $z \in S$ with Hurwitz expansion $\{a_n\}$ such that $(a_1, \dots a_l) = \beta$ (note that since $z \in S$, $a_0 = 0$). We say that a block $\beta = (b_1, \dots, b_l)$ is *admissible* if S_β (is nonempty and) has nonempty interior. We note that S_β is a set with a boundary consisting of finitely many arc segments (circular or straight line) and the interior, if nonempty, is dense in S_β. (It seems plausible that S_β has nonempty interior whenever it is nonempty, but it does not seem to be easy to ascertain this - as this will not play a role in the sequel we shall leave it at that.)

We say that $\{a_n\} \in \Omega_H$ is *generic* if every admissible block occurs in $\{a_n\}$, and $z \in \mathbb{C}'$ is said to be *generic* if its Hurwitz expansion is a generic sequence.

Remark 8.3. We note the following facts which may be verified from inspection of the iteration sequences corresponding to the Hurwitz expansions. If $\beta = (b_1, \ldots, b_l)$ and $\beta' = (b'_1, \ldots, b'_{l'})$ are admissible blocks and $|b_l| \geq 2\sqrt{2}$ then $(b_1, \ldots, b_l, b'_1, \ldots, b'_l)$ is an admissible block. If $|b_l| < 2\sqrt{2}$ then in general $(b_1, \ldots, b_l, b)$ need not be admissible for $b \in \mathfrak{G}$, $|b| > 1$. However, given any admissible block $\beta = (b_1, \ldots, b_l)$ and $b' \in \mathfrak{G}$ with $|b'| \geq 2\sqrt{2}$ there exists $b \in \mathfrak{G}$ such that $|b| = |b'|$ and $(b_1, \ldots, b_l, b)$ is admissible. This shows in particular that given admissible blocks β and β' there exists an admissible block $(x_1, \ldots, x_m)$ such that $(x_1, \ldots, x_l) = \beta$ and $(x_{m-l'+1}, \ldots, x_m) = \beta'$, where l and l' are the lengths of β and β' respectively, and furthermore m may be chosen to be at most $l + l' + 1$. Since the set of admissible blocks is countable, this also shows that we can construct generic complex numbers, by building up their Hurwitz expansion in steps, so that each admissible block occurs at some stage.

Remark 8.4. For any block $\beta = (b_1, \ldots, b_l)$ let $r_\beta = \omega(\{b_n\}_{n=0}^{l})$ (notation as introduced § 2), with $b_0 = 0$. Then each r_β is a rational number from S; if all $b_i, i = 1, \ldots, l$ are of absolute value greater than 2 then r_β is an interior point of S_β -otherwise it is a boundary point of β. Let Q be the set consisting of all r_β with β any admissible block. Then Q is a dense subset of S since for any nonempty open subset of S we can find a block θ such that S_θ is contained in it.

Proposition 8.5. *The set of generic complex numbers contains a dense $\mathcal{G}_\delta$ subset of $\mathbb{C}$.*

Proof. Let β be any admissible block of length $l \geq 1$ and let E_β be the interior of the set of numbers $z \in S$ such that β occurs in the Hurwitz expansion of z. Consider any admissible block $\gamma = (c_1, \ldots, c_k)$. Let $\epsilon > 0$ be arbitrary and let $N \in \mathbb{N}$ such that $N > \epsilon^{-1}$. By Remark 8.3 there exists an admissible block $\xi = (x_1, \ldots, x_m)$ such that $(x_1, \ldots, x_k) = \gamma$, $x_{k+1} = N+2$ and $(x_{m-l+1}, \ldots, x_m) = \beta$. Then the interior of S_ξ is contained in E_β. We note that for any $z \in S_\xi$ by Proposition 3.3 we have $|z - r_\gamma| \leq |z_{k+1} - 1|^{-1}$, where $\{z_n\}$ denotes the iteration sequence for z with respect to the Hurwitz algorithm, and we have $|z_{k+1}| \geq (N+2) - |z_{k+2}|^{-1} > N+1$; see Proposition 3.3. Hence the distance of r_γ from S_ξ is at most $N^{-1} < \epsilon$. Since $\epsilon > 0$ is arbitrary this shows that r_γ is contained in the closure of S_ξ. As the interior of S_ξ is a dense subset of S_ξ contained in E_β, this shows that r_γ is contained in the closure of E_β. Since this holds for every admissible block γ it means that the set Q as defined in Remark 8.4 is contained in E_β, and since Q is dense in S we get that E_β is dense in S. Thus for every admissible block β there exists an open dense subset of S consisting of points such that β occurs in their Hurwitz expansion. As there are only countably many blocks we get that there exists a dense $\mathcal{G}_\delta$ subset of S consisting of generic complex numbers. Since translates of generic numbers by Gaussian integers are generic it follows that there exists a dense $\mathcal{G}_\delta$ subset of $\mathbb{C}$ all whose elements are generic complex numbers. $\square$

In the light of a result of [8] we can conclude also the following.

Proposition 8.6. *With respect to the Lebesgue measure on $\mathbb{C}$ almost every z in $\mathbb{C}$ is generic.*

Proof. Let $S = \{z = x + iy \mid |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}$. For any admissible block β consider the subset, say E_β, of S consisting of all z such that β occurs in the continued expansion of z. We note that E_β is invariant under the Gauss map, of S into itself, associated with the Hurwitz continued fractions (see [8]). It is known that the Gauss map is ergodic with respect to the Lebesgue measure; in fact there exists a probability measure on S equivalent to the Lebesgue measure which is invariant with respect to the Gauss map, and the map is mixing with respect to it; see [8], Theorem 5.5. Since E_β contains an nonempty open set it is of positive measure, and hence by ergodicity it is of measure 1. It follows that the set of generic points in S, which is the intersection of E_β over the countable collection of admissible blocks β, is of full measure in S. Since translates of generic points by Gaussian integers are generic, we get that almost every complex number is generic. $\square$

We now note the following.

Corollary 8.7. *Let $\xi \in \mathbb{C}$ be generic. Let $\alpha, \beta \in \mathbb{C}$ and $\{a_n\}$ and $\{b_n\}$ their respective continued fraction expansions. Suppose that $|a_n| > \sqrt{5}$ for all $n \geq 0$, $|b_0| \geq \sqrt{2}$, and all initial blocks of $\{b_n\}$ are admissible. Then there exists a sequence $\{g_i\}$ in $SL(2, \mathfrak{G})$ such that*

$$g_i(\pi(z)) \to \pi(\alpha) \text{ for all } z \neq \rho(\pi(\xi)), \text{ and } g_i(\rho(\pi(\xi))) \to \rho(\pi(\beta)).$$

Proof. We note that under the conditions on α and β in the hypothesis, if α_i and β_i are initial blocks of length $2i$ in $\{a_n\}$ and $\{b_n\}$ respectively, then $\tilde{\alpha}_i\beta_i$ are admissible blocks. As $\{x_n\}$ is generic we can find a sequence $\{s_i\}$ in $\mathbb{N}$ such that the conditions as in the hypothesis of Proposition 7.5 are satisfied. The assertion as in the Corollary then follows from the Proposition 7.5. $\square$

Corollary 8.8. *Let $\zeta_1, \zeta_2 \in \mathbb{C}$ and suppose that ζ_1 (or ζ_2) is generic. Then the values of the quadratic form $Q(x,y) = (x - \zeta_1 y)(x - \zeta_2 y)$ over the set of Gaussian integers form a dense subset of $\mathbb{C}$; namely, $\{(a - \zeta_1 b)(a - \zeta_2 b) \mid a, b \in \mathfrak{G}\}$ is dense in $\mathbb{C}$.*

Proof. Let $v_r = \begin{pmatrix} \zeta_r \\ 1 \end{pmatrix}$ for $r = 1, 2$. We shall show that for the quadratic form $Q = Q_{\rho(v_1),\rho(v_2)}$ on $\mathbb{C}^2$, $Q(\mathfrak{G}^2)$ is dense in $\mathbb{C}$. This is readily seen to hold in the case when $\zeta_1 = \zeta_2$, we shall now assume that $\zeta_1 \neq \zeta_2$. We have $\eta(v_1) = \pi(\zeta_1)$, $\eta(v_2) = \pi(\zeta_2)$ and $\eta(v_1) \neq \eta(v_2)$. Let $\alpha, \beta \in \mathbb{C}$ be such that the conditions in the statement of Corollary 8.7 are satisfied. Then by Corollary 8.7 there exists a sequence $\{g_i\}$ in $SL(2, \mathfrak{G})$ such that $g_i(\rho(\eta(v_1))) \to \rho(\pi(v_\beta))$ and $g_i(\eta(v_2)) \to \pi(v_\alpha)$. Then by Proposition 8.6 the closure of $Q(\mathfrak{G}^2)$ in $\mathbb{C}$ contains $Q_{\rho(v_\beta),v_\alpha}(\mathfrak{G}^2)$, for all $\alpha, \beta \in \mathbb{C}$ as above. We note that the condition in the corollary is satisfied for all β in a dense subset of $\{z \in \mathbb{C} \mid |z| \geq \sqrt{5/2}\}$; it may be noted in this respect that the set of points for which all initial blocks of the Hurwitz continued fraction expansion are admissible is a set of full Lebesgue measure, containing the complement of a countable union of analytic curves. We fix an α for which the condition in Corollary 8.7 is satisfied and consider values $Q_{\rho(v_\beta),v_\alpha}(\mathfrak{G}^2)$, for varying β as above. It is straightforward to see that these values contain a dense set of complex numbers. Hence $Q(\mathfrak{G}^2)$ is dense in $\mathbb{C}$. □

Remark 8.9. The set of all (complex) binary quadratic forms may be viewed in a natural way as the 3-dimensional vector space (over $\mathbb{C}$) of symmetric 2×2 matrices, and may be considered equipped with the Lebesgue measure. From the fact that the generic numbers in $\mathbb{C}$ form a set of full Lebesgue measure it may be seen that the quadratic forms as in the statement of Corollary 8.8 and their scalar multiples together form a set of full measure in the space of all binary quadratic forms; our conclusion shows that for each of these forms the set of values over the set of pairs $\{(a, b) \mid a, b \in \mathfrak{G}\}$ is a dense subset of $\mathbb{C}$.

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