

Convolution roots and embeddings of probability measures on Lie groups

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Abstract

We show that for a large class of connected Lie groups G , viz. from *class* \mathcal{C} described below, given a probability measure μ on G and a natural number n , for any sequence $\{\nu_i\}$ of n th convolution roots of μ there exists a sequence $\{z_i\}$ of elements of G , centralising the support of μ , and such that $\{z_i\nu_i z_i^{-1}\}$ is relatively compact; thus the set of roots is relatively compact 'modulo' the conjugation action of the centraliser of $\text{supp } \mu$. We also analyse the dependence of the sequence $\{z_i\}$ on n . The results yield a simpler and more transparent proof of the embedding theorem for infinitely divisible probability measures on the Lie groups as above, proved in [5].

1 Introduction

Let G be a Lie group and μ be a probability measure on G . If $G = \mathbb{R}^n$, or more generally if G is a nilpotent Lie group, then for any $n \in \mathbb{N}$ (natural numbers) the set of n -th convolution roots of μ on G form a compact set of probability measures (with respect to the usual weak* topology); moreover the set of all convolution roots, with varying n , forms a relatively compact subset (see [7], § 3.1). This is no longer true when G is a more general Lie group. One of the reasons why the set of n -th roots would be noncompact, in general, is that if λ is a n -th root of μ then $z\lambda z^{-1}$ is also a n -th root of μ for all $z \in G$ such that μ is invariant under the conjugation action of z . The set of roots $z\lambda z^{-1}$, with z as above, can be noncompact in general, though not always, when the subgroup $Z(\mu)$ consisting of all z in G centralising every element of the support of μ is noncompact. For a large class of groups G , including all real algebraic groups $Z(\mu)$ being compact ensures, conversely, that the set of n -th roots of μ is compact, and also that the set of all roots forms a relatively compact subset

(this is a consequence of “factor compactness” property discussed below, proved in [3]).

Our results here show that for a large class of groups (see class \mathcal{C} below) the set of roots being noncompact is accounted for by $Z(\mu)$ as above being noncompact, in the sense that if two roots are considered equivalent if they are conjugate by an element of $Z(\mu)$, then the set of equivalence classes is compact (with respect to the quotient topology). We analyse also the analogous question for the set of all roots (with varying n); see Theorem 1.1 and Remark 1.2 below.

One stream of study of the convolution roots of measures, as exposed in [7] for example, was largely inspired by the so called *embedding problem* for infinitely divisible probability measures, namely measures admitting roots of all orders. Addressing the problem it was shown in [5] that over a large class of Lie groups, namely groups of class \mathcal{C} below, every infinitely divisible probability measure is embeddable in a continuous one-parameter convolution semigroup $\{\mu_t\}_{t \geq 0}$, as $\mu = \mu_1$. Our present result offers a much shorter (even accounting for the fact that we make use of some of the intermediary results there) and more transparent proof of the main theorem of [5].

We now describe the results, with the technical details involved. A Lie group G is said to be *of class \mathcal{C}* if it is connected and admits a representation $\pi : G \rightarrow GL(V)$ over a (finite-dimensional, real) vector space V , such that the kernel of π is a discrete subgroup; we note that as G is connected, the discrete subgroup $\ker \pi$ is contained in the center of G . It can be seen that all connected semisimple Lie groups, and all simply connected Lie groups are of class \mathcal{C} (the latter follows from Ado’s theorem). In §2 we give a characterisation of Lie groups of class \mathcal{C} in terms of their structure; see Proposition 2.5.

We shall denote by $P(G)$ the space of all probability measures on G , equipped with the convolution product and the usual weak* topology. For any $\mu \in P(G)$ and $n \in \mathbb{N}$ we denote by $R_n(\mu, G)$ the set of all n -th (convolution) roots of μ in $P(G)$. For any subgroup H of G we denote by $Z(\mu, H)$ the subgroup of H consisting of all elements which commute with every element of $\text{supp } \mu$, the support of μ , and by $Z^0(\mu, H)$ the connected component of the identity in $Z(\mu, H)$.

Theorem 1.1. *Let G be a Lie group of class \mathcal{C} , and let $\mu \in P(G)$. Let $n \in \mathbb{N}$ and $\{\nu_i\}$ be a sequence in $R_n(\mu, G)$. Then there exists a sequence $\{z_i\}$ in $Z^0(\mu, [G, G])$ such that the following conditions are satisfied:*

- i) $\{z_i \nu_i z_i^{-1}\}$ is relatively compact;*
- ii) if $r \in \mathbb{N}$, and $\{\lambda_i\}$ is a sequence in $P(G)$ such that, for every i , $\lambda_i^r = \nu_i$ and $Z^0(\lambda_i, [G, G]) = Z^0(\nu_i, [G, G])$, then $\{z_i \lambda_i z_i^{-1}\}$ is relatively compact.*

Remark 1.2. The first conclusion in the theorem signifies that if on $R_n(\mu, G)$ we define an equivalence relation by setting $\lambda_1 \sim \lambda_2$ if there exists a $z \in$

$Z^0(\mu, [G, G])$ such that $\lambda_2 = z\lambda_1 z^{-1}$, then the quotient space $R_n(\mu, G)/\sim$ is compact with respect to the quotient topology.

A set E of roots of μ is said to be *infinitely divisible* if for every $n \in \mathbb{N}$, there exists $\nu \in E$ such that $\nu^n = \mu$.

Theorem 1.3. *Let G be a Lie group of class \mathcal{C} . Let $\mu \in P(G)$ be infinitely divisible, and let $r : \mathbb{N} \rightarrow P(G)$ be a map such that $r(m)^m = \mu$ for all $m \in \mathbb{N}$. Then there exist sequences $\{m_i\}$ in \mathbb{N} and $\{z_i\}$ in $Z^0(\mu, [G, G])$, and $n \in \mathbb{N}$, such that n divides m_i for all i and the sequence $\{z_i r(m_i)^{m_i/n} z_i^{-1}\}$ consisting of n -th roots of μ converges to a n -th root of μ which is strongly root compact and rationally embeddable on the subgroup*

$$\{g \in G \mid gx = xg \text{ for all } x \in Z^0(\nu, G)\},$$

namely, the centraliser of $Z^0(\nu, G)$ in G .

Furthermore, the limit ν of $\{z_i r(m_i)^{m_i/n} z_i^{-1}\}$ is embeddable in a one-parameter semigroup $\{\nu_t\}_{t \geq 0}$, with each ν_t contained in the closure of the subsemigroup generated by $\{zr(m)z^{-1} \mid z \in Z^0(\mu, [G, G]), m \in \mathbb{N}\}$.

The theorem in particular implies the following.

Corollary 1.4. *Let G be a Lie group of class \mathcal{C} , and $\mu \in P(G)$. Let S be a closed subsemigroup of G such that $zSz^{-1} = S$ for all $z \in Z^0(\mu, [G, G])$. If μ is infinitely divisible on S then it is embeddable on S .*

Remark 1.5. For possible future reference we note that the proof of Theorem 1.3 shows that the sequence $\{m_i\}$ as in the theorem can be chosen such that the following also holds: for all i , $i!n$ divides m_i , $r(m_i) \in R_{m_i}(\mu, G)$, and for every $k \in \mathbb{N}$ the sequence $\{z_i r(m_i)^{m_i/k!n} z_i^{-1}\}_{i=k}^\infty$ converges.

2 Preliminaries

For any locally compact group G we shall denote by $P(G)$ the space of all probability measures on G equipped with the usual weak* topology and the convolution product (see [7], for various generalities). For any closed subgroup H of G the set of μ in $P(G)$ whose support is contained in H will (also) be denoted by $P(H)$.

Let G be a locally compact group and $\mu \in P(G)$. For any $n \in \mathbb{N}$ we denote by $R_n(\mu, G)$ the set of all n th roots of μ on G , namely $R_n(\mu, G) = \{\rho \in P(G) \mid \rho^n = \mu\}$ and by $R(\mu, G)$ the set of all roots of μ on G , namely $R(\mu, G) = \cup_{n \in \mathbb{N}} R_n(\mu, G)$. Also, we denote by $\bar{R}(\mu, G)$ the set $\{\rho^k \mid \rho \in R_n(\mu, G) \text{ for some } n, 1 \leq k \leq n\}$. The measure μ is said to be *root compact* if

$R(\mu, G)$ is relatively compact in $P(G)$, and it is said to be *strongly root compact* if $\overline{R}(\mu, G)$ is relatively compact in $P(G)$.

We denote by $\text{supp } \mu$ the support of μ , and by $G(\mu)$ the smallest closed subgroup containing $\text{supp } \mu$. We denote by $N(\mu, G)$ the normaliser of $G(\mu)$ in G . For every subgroup H of G we denote by $Z(\mu, H)$ the centraliser of $G(\mu)$ in H , i.e. $Z(\mu, H) = \{h \in H \mid gh = hg \text{ for all } g \in G(\mu)\}$. We note that $Z(\mu, G)$ is a normal subgroup of $N(\mu, G)$.

A measure $\lambda \in P(G)$ is called a (*two sided*) *factor* of μ if there exists a $\nu \in P(G)$ such that $\mu = \lambda\nu = \nu\lambda$. Clearly, every root of μ is a factor of μ . We recall that every factor of μ , and in particular every root of μ , has its support contained in $N(\mu)$, (see [3], Proposition 1.1). Furthermore, it can be seen easily that if $\rho \in R_n(\mu, G)$ for $n \in \mathbb{N}$, then ρ can be expressed as λx with $\lambda \in P(G(\mu))$ and $x \in N(\mu)$ such that $x^n \in G(\mu)$ (see, for instance, [5], Proposition 2.7). We note also the following, in this respect; Proposition 2.1 below corresponds to Proposition 5.3 in [5], where a similar relation is also stated for left translates, in the place of right translates ρx below; we may note here however that in the statement of Proposition 5.3 in [5], in the hypothesis it should actually be “and $g \in N(\mu)$ ” in the place of “and $g \in G$ ”.

Proposition 2.1. *Let $\rho \in P(G)$ be of the form λx , with $\lambda \in P(G(\mu))$ and $x \in N(\mu, G)$. Let σ_x be the automorphism of $N(\mu, G)$ given by inner conjugation by x , namely, $\sigma_x(g) = xgx^{-1}$ for all $g \in N(\mu, G)$. Then for all $z \in Z(\mu, G)$ and $r \geq 1$,*

$$(\rho z)^r = \sigma_x(z)\sigma_x^2(z) \cdots \sigma_x^r(z)\rho^r.$$

Proof. We have $\rho z = \lambda x z = \lambda(x z x^{-1})x = \lambda\sigma_x(z)x = \sigma_x(z)\lambda x = \sigma_x(z)\rho$, for all $z \in Z(\mu, G)$, which proves the desired equality for $r = 1$. Now suppose, by induction, that it holds for $1, 2, \dots, r-1$. Then we have $(\rho z)^r = (\rho z)(\rho z)^{r-1} = (\sigma_x(z)\rho)(\sigma_x(z)\sigma_x^2(z) \cdots \sigma_x^{r-1}(z))\rho^{r-1}$, and, as $\sigma_x(z)\sigma_x^2(z) \cdots \sigma_x^{r-1}(z) \in Z(\mu, G)$, we have $(\rho z)^r = \sigma_x(z)\sigma_x(\sigma_x(z) \cdots \sigma_x^{r-1}(z))\rho^r = \sigma_x(z)\sigma_x^2(z) \cdots \sigma_x^r(z)\rho^r$, for all $z \in Z(\mu, G)$. This proves the proposition.

We recall also the following general fact.

Proposition 2.2. ([5], Proposition 3.4) *Let G and H be locally compact groups and $\eta : G \rightarrow H$ be a continuous homomorphism of G onto H . Suppose that the kernel of η is a compactly generated subgroup contained in the center of G . Let $\mu \in P(G)$ and let X be a subset of $\overline{R}(\mu, G)$. Then*

i) if $\eta(X)$ is relatively compact in $P(H)$, X is relatively compact in $P(G)$, and

ii) if X is closed in $P(G)$ then $\eta(X)$ is closed in $P(H)$.

We next recall the following “almost factor compactness theorem” for almost algebraic groups, which will be needed in the sequel; see [3], [6], [11] for this, and some more general results on the theme.

Theorem 2.3. *Let G be an almost algebraic group and let $\mu \in P(G)$. If $\{\rho_i\}$ is a sequence of factors of μ then there exists a sequence $\{z_i\}$ in $Z(\mu, G)$ such that $\{\nu_i z_i\}$ is relatively compact in $P(G)$.*

Theorem 2.3 together with Proposition 2.2 (i) implies the following:

Corollary 2.4. *Let G be an almost algebraic group and $\mu \in P(G)$. If $Z(\mu, G)$ is contained in the center of G , then μ is strongly root compact on G .*

Proof. Let Z be the center of G and $\eta : G \rightarrow G/Z$ be the quotient homomorphism. Let $\{\nu_i\}$ be any sequence in $\overline{R}(\mu, G)$. Then $\{\nu_i\}$ is a sequence of factors of μ and hence by Theorem 2.3 there exists a sequence $\{z_i\}$ in $Z(\mu, G)$ such that $\{\nu_i z_i\}$ is relatively compact. Since by hypothesis $Z(\mu, G)$ is contained in Z this implies that $\{\eta(\nu_i)\}$ is relatively compact. Hence by Proposition 2.2 (i) $\{\nu_i\}$ is relatively compact. This shows that $\overline{R}(\mu, G)$ is relatively compact, which proves the corollary.

We conclude this section with a characterisation of the class \mathcal{C} in terms of Lie group theoretic structure of the groups, based on a characterisation of Nahlus [13] of connected Lie groups admitting faithful finite dimensional representations.

Proposition 2.5. *Let G be a connected Lie group, R be the radical of G and Z be the center of G . Then the following are equivalent:*

- i) G is of class \mathcal{C} ;*
- ii) $[R, R]$ is a closed simply connected nilpotent Lie subgroup;*
- iii) $\overline{[R, R]} \cap Z$ contains no compact subgroup of positive dimension.*

Proof. Suppose G is of class \mathcal{C} and let $\pi : G \rightarrow GL(V)$ be a finite-dimensional representation of G , with discrete kernel. By complexification we may assume the representation to be over a complex vector space. Then by Lie's theorem $\pi(R)$ can be realised as a group of upper triangular matrices with respect to a suitable basis. Then $\pi([R, R])$ is a connected Lie subgroup consisting of unipotent upper triangular matrices. Hence it is a closed simply connected nilpotent Lie subgroup (see [15], Theorem 3.6.3). Therefore $[R, R]$ is also closed. Then the restriction of π to $[R, R]$ is a covering map onto $\pi([R, R])$ and since the latter is simply connected the restriction map is a (topological) isomorphism. Therefore $[R, R]$ is also a simply connected nilpotent Lie group. This shows that (i) implies (ii).

It is well-known that a simply connected nilpotent Lie group has no non-trivial compact subgroup (see [15], Theorem 3.6.2 for instance), and hence (ii) implies (iii).

We now prove that (iii) implies (i). Suppose that $\overline{[R, R]} \cap Z$ has no compact subgroup of positive dimension. Let S be a semisimple Levi subgroup of G . Let $D = S \cap Z$, which is a discrete central subgroup of G . Let $G' = G/D$.

It suffices to prove that G' admits a faithful finite-dimensional representation, which we shall do by applying the criterion in [13]. We note that $S' = S/D$ is a semisimple Levi subgroup of G' , and that S' admits a faithful finite-dimensional representation; the restriction of the adjoint representation of G over its Lie algebra, to the subgroup S , factors to a faithful representation of S' . This verifies condition (ii) of [13]. Condition (i) of [13] follows immediately from condition (iii) in the hypothesis. Therefore G' admits a faithful finite-dimensional representation, and so G is of class \mathcal{C} . This completes the proof of the proposition.

3 Some results on nilpotent Lie groups

In this section we prove certain results, involving properties of nilpotent Lie groups, that will be used in the sequel; the results may also be of independent interest.

We begin by recalling the following simple fact.

Lemma 3.1. *Let N be a simply connected nilpotent Lie group. Let $n \in \mathbb{N}$ and α be a (continuous) automorphism of N , such that $\alpha^n = I$, the identity automorphism. Let $z \in N$ and suppose that $\alpha(z)\alpha^2(z) \cdots \alpha^n(z) = e$, the identity element in N . Then there exists $y \in N$ such that $\alpha(z) = y^{-1}\alpha(y)$.*

We note that in the group $\text{Aff}(N)$ of affine automorphisms of N the conditions $\alpha(z)\alpha^2(z) \cdots \alpha^n(z) = e$ and $\alpha(z) = y^{-1}\alpha(y)$ as above correspond to $(\alpha z)^n = I$ and $\alpha z = y^{-1}\alpha y$. An elementary proof of the lemma may be found in [12] (Lemma 5.4), where it is formulated as the equivalent statement on $\text{Aff}(N)$. The result may also be seen to follow from the conjugacy of maximal compact subgroups in the Lie group formed as the semidirect product of the finite cyclic subgroup generated by α (as in the hypothesis) with N ; (see [8], Chapter XV, Theorem 3.1; for the case at hand Lemma 3.2 of [8] together with an inductive argument would also suffice).

Theorem 3.2. *Let G be a locally compact group. Let L be a closed subgroup topologically isomorphic to a simply connected nilpotent Lie group, contained in $Z^0(\mu, G)$ and normalized by $N(\mu, G)$. Let $n \in \mathbb{N}$, $\rho \in P(G)$, and $z \in L$ be such that $(\rho z)^n = \rho^n = \mu$. Then there exists $y \in L$ such that $\rho z = \rho y y^{-1}$.*

Proof. As $\rho^n = \mu$ there exist $\lambda \in P(G(\mu))$ and $x \in N(\mu, G)$ such that $\rho = \lambda x$ (see §2). Let $\sigma_x : L \rightarrow L$ be the automorphisms of L defined by $\sigma_x(g) = xgx^{-1}$ for all $g \in L$. By Proposition 2.1 we have $(\rho z)^n = \sigma_x(z)\sigma_x^2(z) \cdots \sigma_x^n(z)\rho^n$. Let $H(\mu) = \{g \in G \mid g\mu = \mu\}$. Then $H(\mu)$ is a compact subgroup of G (see [7], Theorem 1.2.4). Since $(\rho z)^n = \mu = \rho^n$ the preceding conclusion implies

that $\sigma_x(z)\sigma_x^2(z)\cdots\sigma_x^n(z) \in H(\mu) \cap L$. As L is a simply connected nilpotent Lie group, it has no nontrivial compact subgroups, so we conclude that $\sigma_x(z)\sigma_x^2(z)\cdots\sigma_x^n(z) = e$, the identity element. Hence by Lemma 3.1, there exists $y \in L$ such that $\sigma_x(z) = y\sigma_x(y^{-1})$. Thus $xzx^{-1} = yxy^{-1}x^{-1}$ and so $xz = yxy^{-1}$. Therefore $y\rho y^{-1} = (y\lambda y^{-1})(yxy^{-1}) = \lambda xz = \rho z$, and the proof is complete.

Another property of nilpotent Lie groups needed in the sequel is the following.

Proposition 3.3. *Let L be a simply connected nilpotent Lie group. Let $n \in \mathbb{N}$ and $\{\alpha_i\}$ be a relatively compact sequence of automorphisms of L such that $\alpha_i^n = I$, the identity automorphism of L , for all i . Let $\{x_i\}$ be a sequence in L such that $\{\alpha_i(x_i)x_i^{-1}\}$ is relatively compact. Then there exist sequences $\{y_i\}$ and $\{z_i\}$ in L such that $x_i = y_i z_i$ for all i , $\{y_i\}$ is relatively compact, and $\alpha_i(z_i) = z_i$ for all i .*

Proof. First suppose that $L = V$, a vector space. We fix a norm $\|\cdot\|$ on V . Also, we use additive notation for the group operation in V . Since α_i^n is the identity automorphism for all i , the characteristic polynomials of $\{\alpha_i\}$ come from a finite set of polynomials, and, as we can consider finitely many sequences separately, without loss of generality we may assume that the characteristic polynomials of all α_i are the same. For each i let F_i be the subspace of V consisting of all fixed points of α_i , and W_i be the largest α_i -invariant subspace of V on which all eigenvalues of α_i are different from 1 (they are necessarily n -th roots of 1). Then we have $V = W_i \oplus F_i$ for all i . Since the characteristic polynomials of all α_i are the same, it follows that all W_i are of the same dimension. Let $\{y_i\}$ and $\{z_i\}$ be the sequences in V , such that for all i , $y_i \in W_i$, $z_i \in F_i$ and $x_i = y_i + z_i$. Then we have $\alpha_i(x_i) - x_i = \alpha_i(y_i) - y_i$ for all i . For $i \in \mathbb{N}$ let $v_i = \|y_i\|^{-1}y_i$, if $y_i \neq 0$, and 0 otherwise. Then $\alpha_i(x_i) - x_i = \|y_i\|(\alpha_i(v_i) - v_i)$ for all i .

Suppose, if possible, that $\{y_i\}$ is not relatively compact. Then there exists a sequence $\{i_j\}$ in \mathbb{N} such that $\|y_{i_j}\| \rightarrow \infty$, as $j \rightarrow \infty$. Since $\{\alpha_i\}$ is relatively compact, and $\|v_i\| = 1$ for all i such that $y_i \neq 0$, passing to a subsequence of $\{i_j\}$ and modifying notation we may assume that, as $j \rightarrow \infty$, $\{\alpha_{i_j}\}$ converges to an automorphism, say α , and $\{v_{i_j}\}$ converges to a vector $v \in V$ with $\|v\| = 1$. Also, since the Grassmannian manifolds, consisting of subspaces of V of a given dimension, are compact, we may further assume that the subspaces $\{W_{i_j}\}$ and $\{F_{i_j}\}$ converge to subspaces of the corresponding dimensions, say W and F , of V , respectively. Then W and F are α -invariant, and furthermore, $\alpha(z) = z$ for all $z \in F$. As all α_i have the same characteristic polynomial, the latter is also the characteristic polynomial of α , and together with the preceding observation this implies that W does not contain any non-zero vector fixed by α .

Now, since $\{\alpha_i(x_i) - x_i\}$ is relatively compact and $\|y_{i_j}\| \rightarrow \infty$ as $j \rightarrow \infty$, it follows that $\alpha_i(v_{i_j}) - v_{i_j} \rightarrow 0$, as $j \rightarrow \infty$. Since $\alpha_{i_j} \rightarrow \alpha$ and $v_{i_j} \rightarrow v$, this implies that $\alpha(v) = v$. Also, since $v_{i_j} \in W_{i_j}$ for all j and $v_{i_j} \rightarrow v \neq 0$, it follows that $v \in W$. But this is a contradiction, since W contains no non-zero vector fixed by α . Hence $\{y_i\}$ is relatively compact. This proves the proposition in the case at hand.

Now consider the general case of a simply connected nilpotent Lie group L as in the hypothesis. We shall prove the proposition by induction on the dimension of L . In low dimensions L is a vector space and in this case we are through. Now suppose that the assertion holds in all cases when the dimension is less than that of the group L under consideration. Let $\{\alpha_i\}$ and $\{x_i\}$ be as in the hypothesis. Let Z be the center of L , and consider L/Z . For each i let $\bar{\alpha}_i$ be the factor automorphism of L/Z induced by α_i , and \bar{x}_i be the coset of x_i , viz. $\bar{x}_i = x_i Z \in L/Z$. Then by the induction hypothesis there exist sequences $\{\varphi_i\}$ and $\{\psi_i\}$ in L/Z such that $\bar{x}_i = \varphi_i \psi_i$ for all i , $\{\varphi_i\}$ is relatively compact in L/Z , and $\bar{\alpha}_i(\psi_i) = \psi_i$ for all i . Let $\{y'_i\}$ be a relatively compact sequence in L such that $\varphi_i = y'_i Z$ for all i . For each i let $\xi_i = (y'_i)^{-1} x_i$. Then $\{\xi_i\}$ is a sequence in L such that $\{\alpha_i(\xi_i) \xi_i^{-1}\}$ is a relatively compact sequence contained in Z .

Dividing the sequence $\{\xi_i\}$ (along with the sequence $\{\alpha_i\}$) into two subsequences it suffices to consider two separate cases with either $\xi_i \in Z$ for all i or $\xi_i \notin Z$ for every i . Since L is a simply connected Lie group, the center Z is a vector group, and hence in the case $\xi_i \in Z$ for all i we are through by the special case of the proposition for vector groups, proved above.

Now suppose that $\xi_i \notin Z$ for every i . We decompose ξ_i , for each i , as follows. Let $i \in \mathbb{N}$. As L is a simply connected nilpotent Lie group, each ξ_i is contained in a unique one-parameter subgroup. Let Z_i be the subgroup generated by Z and the one-parameter subgroup containing ξ_i . Since $\alpha_i(\xi_i) \in \xi_i Z$, in view of the uniqueness of the one-parameter subgroup containing ξ_i it follows that Z_i is α_i -invariant. Now, Z_i is a vector space, and the restriction of α_i to Z_i is a linear transformation. Since α_i is of finite order, it is a semisimple element (namely diagonalisable over the field of complex numbers). Since Z is α_i -invariant and Z_i/Z is a one-dimensional vector space on which the factor action of α_i is trivial, it follows that there exists a one-dimensional subspace Φ_i of Z_i such that Φ_i is pointwise fixed by α_i and $Z_i = \Phi_i Z$, a direct product. Let $z'_i \in \Phi_i$ and $v_i \in Z$ be such that $\xi_i = z'_i v_i$.

Now consider the sequence $\{v_i\}$ in Z , with each v_i as defined above. For each i we have $\alpha_i(\xi_i) \xi_i^{-1} = \alpha_i(z'_i) \alpha_i(v_i) v_i^{-1} z'^{-1}_i = \alpha_i(v_i) v_i^{-1}$, since $\alpha_i(z'_i) = z'_i$ and Z_i is abelian. Therefore $\{\alpha_i(v_i) v_i^{-1}\}$ is relatively compact. From the special case of abelian groups proved above, we get that there exist sequences $\{a_i\}$ and $\{b_i\}$ in Z such that $v_i = a_i b_i$ for all i , $\{a_i\}$ is relatively compact,

and $\alpha_i(b_i) = b_i$ for all i . Now for each i let $y_i = y'_i a_i$, and $z_i = z'_i b_i$. We have $x_i = y'_i \xi_i = y'_i z'_i v_i = y'_i z'_i a_i b_i = y_i z_i$, for each i , as a_i is contained in the center of L . Clearly $\{y_i\}$ is relatively compact, as $\{y_i\}$ and $\{a_i\}$ are, and $\alpha_i(z_i) = \alpha_i(z'_i) \alpha_i(b_i) = z'_i b_i = z_i$ for every i . This proves the proposition.

We need also the following property of nilpotent Lie groups, proved in [5], called “affine root rigidity”.

Theorem 3.4. *Let L be a connected nilpotent Lie group. Let $n \in \mathbb{N}$, and $\{\alpha_i\}$ be a sequence of automorphisms of L such that $\alpha_i^n = I$, the identity automorphism, for all $i \in \mathbb{N}$. Let $\{x_i\}$ be a sequence in L such that $\{\alpha_i(x_i) \alpha_i^2(x_i) \cdots \alpha_i^n(x_i)\}$ is relatively compact. Then there exists a sequence $\{y_i\}$ in L , such that $\{x_i^{-1} y_i\}$ is relatively compact, and $\alpha_i(y_i) \alpha_i^2(y_i) \cdots \alpha_i^n(y_i) = e$, the identity element in L , for all i .*

4 A reduction theorem

In this section we consider measures μ on almost algebraic groups and show that roots of a measure can be conjugated into a subgroup with a special property; see below.

Theorem 4.1. *Let G be an almost algebraic group, and $\mu \in P(G)$. Let \tilde{N} be the smallest almost algebraic subgroup of G containing $N(\mu, G)$ and let U be the unipotent radical of \tilde{N}^0 . Then there exists an almost algebraic subgroup H of G containing $G(\mu)$ and U , such that the following conditions are satisfied:*

- i) $Z^0(\mu, H)/(Z^0(\mu, \tilde{G}(\mu))Z(\mu, U))$ is compact.*
- ii) for every $\nu \in R(\mu, G)$ there exists $z \in Z^0(\mu, [G, G])$ such that $z\nu z^{-1} \in P(H)$.*

Proof. Let $P = Z^0(\mu, \tilde{G}(\mu))U$. Then P is a normal almost algebraic subgroup of \tilde{N} . Consider the quotient group $\tilde{N}/P = N$, say. It is a Lie group with finitely many connected components, and the identity component N^0 is a reductive Lie group. Let $N_0 = Z^0(\mu, G)U/P$, and $N_1 = \tilde{G}^0(\mu)U/P$. Then N_0 and N_1 are closed connected normal subgroups of N , and $N_0 \cap N_1$ is finite. As N^0 is reductive there exists a closed connected normal subgroup N_2 of N , such that $N^0 = N_0 N_1 N_2$, and $(N_0 N_1) \cap N_2$ is finite. Let M and Q be the subgroups of \tilde{N} containing P , and such that $N_0 = M/P$ and $N_1 N_2 = Q/P$. Now consider the group \tilde{N}/Q . It is a Lie group with finitely many connected components and hence admits a maximal compact subgroup, say K (see [8], Chapter 15, Theorem 3.1); the theorem also shows that K intersects all connected components of \tilde{N}/Q and its intersection with \tilde{N}^0/Q is a maximal compact subgroup of the latter. Let H be the subgroup of \tilde{N} containing Q , such that $K = H/Q$. Then H is an almost algebraic subgroup of \tilde{N} intersecting all connected components

of \tilde{N} , and $(\tilde{N}^0 \cap H)/P$ has the form $K_0 N_1 N_2$, with K_0 a maximal compact subgroup of N_0 .

We now show that the assertions as in the theorem hold for the above choice of H . The subgroup $Z(\mu, H)P$ is almost algebraic and hence closed. Since $Z(\mu, H)P/P$ is contained in N_0 as well as in H/P it is contained in the subgroup K_0 as above, and hence it is compact. Hence $Z(\mu, H)/(Z(\mu, H) \cap P)$ is compact. Since $P = Z^0(\mu, \tilde{G}(\mu))U$ and $Z^0(\mu, \tilde{G}(\mu))$ is contained in $Z(\mu, H)$ it follows that $Z(\mu, H) \cap P = Z^0(\mu, \tilde{G}(\mu))Z(\mu, U)$. This proves (i).

Let $A = Z^0(\mu, [\tilde{N}, \tilde{N}])H$ and $B = Z^0(\mu, [\tilde{N}, \tilde{N}])Q$. Then A and B are almost algebraic subgroups of \tilde{N} . Also, clearly B is normal in \tilde{N} . We shall show that A is also normal in \tilde{N} and \tilde{N}/A is a vector group (topologically isomorphic to a real vector space). Consider the action of \tilde{N} on \tilde{N}/B induced by the conjugation action. Since B contains $Z^0(\mu, [\tilde{N}, \tilde{N}])$ it follows that the \tilde{N} -action on the subgroup $Z^0(\mu, G)B/B$ of \tilde{N}/B is trivial. Thus $Z^0(\mu, G)B/B$ is contained in the center of \tilde{N}/B . On the other hand, from the choice above, and the fact that H intersects all connected components of \tilde{N} , we have that $\tilde{N} = Z^0(\mu, G)H$. We deduce from this that A/B which is the same as HB/B , is normal in \tilde{N}/B . Hence A is normal in \tilde{N} . Clearly \tilde{N}/A is then a connected abelian Lie group. Also, using the fact that H/Q is a maximal compact subgroup of \tilde{N}/Q , we see that \tilde{N}/A has no nontrivial compact subgroups. Therefore \tilde{N}/A is a vector group.

Now let $\nu \in R(\mu, G)$. Recall that then there exist $\lambda \in P(G(\mu))$ and $x \in \tilde{N}$ such that $\nu = \lambda x$. Since $G(\mu)$ is contained in A , the image of μ on \tilde{N}/A is the point mass at the identity element, and since the latter is a vector group, it follows that the same holds for ν . So $\text{supp } \nu$ is contained in A , and therefore $x \in A$. Clearly H/Q is a maximal compact subgroup of A/Q . Also, as $G(\mu)$ is contained in Q , xQ is an element of finite order in A/Q . Hence by conjugacy of maximal compact subgroup in A/Q it follows that there exists a $z \in A$ such that $zxz^{-1} \in H$. Since $A = Z^0(\mu, [\tilde{N}, \tilde{N}])H = HZ^0(\mu, [\tilde{N}, \tilde{N}])$ we may further choose z to be contained in $Z^0(\mu, [\tilde{N}, \tilde{N}])$. Then $z\nu z^{-1} = z\lambda x z^{-1} = \lambda(zxz^{-1}) \in P(H)$. This proves (ii), and completes the proof of the theorem. \square

Remark 4.2. Let the notation be as in Theorem 4.1. We note that $Z^0(\mu, \tilde{G}(\mu))$ is a connected abelian almost algebraic subgroup normalised by \tilde{N} . Hence it can be written as a direct product CV , where C and V are almost algebraic subgroups, with C compact and V a vector group. Furthermore, by the uniqueness of the decomposition with these properties it follows that C and V are normal in \tilde{N} . Now let $L = VZ^0(\mu, U)$. Then L is an almost algebraic subgroup of $Z^0(\mu, H)$ and, as it is normal in \tilde{N} , it is normalised by $N(\mu, G)$. Since V is contained in $\tilde{G}(\mu)$, it is contained in the center of L . As $Z^0(\mu, U)$ is a unipotent almost algebraic subgroup, it follows that L is a simply connected nilpotent subgroup. This shows that for the subgroup H as in Theorem 4.1 $Z^0(\mu, H)$

contains a simply connected nilpotent subgroup L normalised by $N(\mu, H)$ such that $Z^0(\mu, H)/L$ is compact.

5 Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. We shall actually prove the statements as in the theorem for a somewhat larger class of Lie groups with finitely many connected components, including all (not necessarily connected) almost algebraic groups (see class \mathcal{C}' below). We first consider the case of almost algebraic groups.

Let G be an almost algebraic group and μ be as in the hypothesis of Theorem 1.1. Let H be the almost algebraic subgroup of G such that the conclusion of Theorem 4.1 holds. Then for the sequence $\{\nu_i\}$ as in the hypothesis of Theorem 1.1 there exists a sequence $\{\zeta_i\}$ in $Z^0(\mu, [G, G])$ such that $\zeta_i \nu_i \zeta_i^{-1} \in P(H)$ for all i . Therefore it suffices to prove the statement of Theorem 1.1 for $\{\zeta_i \nu_i \zeta_i^{-1}\}$ in the place of $\{\nu_i\}$, or in other words, to prove the theorem for G it suffices to prove it for H as above. Hence, in view of Theorem 4.1 and Remark 4.2, modifying notation we may assume that $Z(\mu, G)$ contains a simply connected nilpotent almost algebraic subgroup L such that L is normalised by $N(\mu, G)$ and $Z(\mu, G)/L$ is compact. Let $L' = L \cap [G, G]$. Since L and $[G, G]$ are almost algebraic subgroups, so is L' . In particular L' has only finitely many connected components, and since it is contained in the simply connected nilpotent Lie group L , it follows that L' is in fact connected. Also, as a connected Lie subgroup of the simply connected nilpotent Lie group L , it is simply connected. We shall show that there exists a sequence $\{z_i\}$ in L' such that statements (i) and (ii) of the theorem are satisfied.

Let $n \in \mathbb{N}$ and $\{\nu_i\}$ be a sequence of n -th roots of μ be as in the hypothesis. By the factor compactness theorem (Theorem 2.3) there exists a sequence $\{x_i\}$ in $Z(\mu, G)$ such that $\{\nu_i x_i\}$ is relatively compact, and since $Z(\mu, G)/L$ is compact, we may further choose $\{x_i\}$ to be contained in L . Let $\eta : G \rightarrow G/[G, G]$ be the natural quotient homomorphism. Then for each i , $\eta(\nu_i)$ is a root of $\eta(\mu)$, and $\{\eta(\nu_i x_i)\}$ is relatively compact. Since $G/[G, G]$ is abelian, $\eta(\mu)$ is strongly root compact (see [7], Theorem 3.1.13 and Examples 3.1.12), and hence the preceding condition implies that $\{\eta(x_i)\}$ is relatively compact. Also, as L and $[G, G]$ are almost algebraic, $\eta(L)$ is almost algebraic, and hence a closed subgroup. As $\{\eta(x_i)\}$ is relatively compact, this implies that by replacing the sequence $\{x_i\}$ suitably, we may assume it to be contained in $L \cap \ker \eta = L'$.

Since $\{\nu_i x_i\}$ is relatively compact, so is the sequence $\{(\nu_i x_i)^n\}$. For each i let $y_i \in N(\mu, G)$ be such that $\nu_i y_i^{-1} \in P(G(\mu))$, and α_i be the automorphism of L' defined by $\alpha_i(\xi) = y_i \xi y_i^{-1}$ for all $\xi \in L'$. Then by Proposition 2.1 $(\nu_i x_i)^n = \alpha_i(x_i) \alpha_i^2(x_i) \cdots \alpha_i^n(x_i) \nu_i^n = \alpha_i(x_i) \alpha_i^2(x_i) \cdots \alpha_i^n(x_i) \mu$, for all i . As

$\{(\nu_i x_i)^n\}$ is relatively compact, this shows that $\{\alpha_i(x_i)\alpha_i^2(x_i)\cdots\alpha_i^n(x_i)\}$ is relatively compact. Since L' is a connected nilpotent Lie group, by affine root rigidity (Theorem 3.4) this implies that there exists a sequence $\{x'_i\}$ in L' such that $\alpha_i(x'_i)\alpha_i^2(x'_i)\cdots\alpha_i^n(x'_i) = e$, the identity element, for all i , and $\{x_i^{-1}x'_i\}$ is relatively compact. Then we have $(\nu_i x'_i)^n = \alpha_i(x'_i)\alpha_i^2(x'_i)\cdots\alpha_i^n(x'_i)\nu_i^n = \mu$ for all i , and as $\nu_i x'_i = \nu_i x_i(x_i^{-1}x'_i)$ for all i , and $\{\nu_i x_i\}$ and $\{x_i^{-1}x'_i\}$ are relatively compact, we get that $\{\nu_i x'_i\}$ is relatively compact. As $(\nu_i x'_i)^n = \mu = \nu_i^n$ for all i , by Theorem 3.2 there exists a sequence $\{z_i\}$ in L' such that $\nu_i x'_i = z_i \nu_i z_i^{-1}$, for all i . Since $\{\nu_i x'_i\}$ is relatively compact, we also get that $\{z_i \nu_i z_i^{-1}\}$ is relatively compact; this proves (i), for the case at hand.

Now let $r \geq 2$ and $\{\lambda_i\}$ be a sequence in $P(G)$ such that $\lambda_i^r = \nu_i$ and $Z^0(\lambda_i, [G, G]) = Z^0(\nu_i, [G, G])$, for all i . Using the assertion as above with nr in the place of n we get that there exists a sequence $\{\zeta_i\}$ in L' such that $\{\zeta_i \lambda_i \zeta_i^{-1}\}$ is relatively compact. Hence $\{\zeta_i \nu_i \zeta_i^{-1}\}$, which is the same as $\{\zeta_i \lambda_i^r \zeta_i^{-1}\}$, is relatively compact.

Since $\{z_i \nu_i z_i^{-1}\}$ is relatively compact, there exists a compact subset K such that, for all i , $\nu_i(z_i^{-1}Kz_i) = z_i \nu_i z_i^{-1}(K) > 1/2$, and in particular $z_i^{-1}Kz_i \cap \text{supp } \nu_i$ is nonempty. For all i let $y_i \in z_i^{-1}Kz_i \cap \text{supp } \nu_i$; thus $y_i \in \text{supp } \nu_i$, for all i , and $\{z_i y_i z_i^{-1}\}$ is relatively compact. We note that $\nu_i y_i^{-1} \in P(G(\mu))$, and hence $z_i \nu_i z_i^{-1} = (z_i \nu_i y_i^{-1} z_i^{-1})(z_i y_i z_i^{-1}) = \nu_i y_i^{-1}(z_i y_i z_i^{-1})$, and similarly $\zeta_i \nu_i \zeta_i^{-1} = \nu_i y_i^{-1}(\zeta_i y_i \zeta_i^{-1})$, for all i . Hence $\zeta_i \nu_i \zeta_i^{-1} = (z_i \nu_i z_i^{-1})(z_i y_i z_i^{-1})^{-1}(\zeta_i y_i \zeta_i^{-1})$ for all i . Since $\{z_i \nu_i z_i^{-1}\}$ and $\{\zeta_i \nu_i \zeta_i^{-1}\}$ are relatively compact, this shows that the sequence $\{(z_i y_i z_i^{-1})^{-1}(\zeta_i y_i \zeta_i^{-1})\}$ is relatively compact.

Now, for each i let $\delta_i = \zeta_i z_i^{-1}$, and α_i be the automorphism of L' defined by $\alpha_i(\xi) = (z_i y_i z_i^{-1})^{-1} \xi (z_i y_i z_i^{-1})$, for all $\xi \in L'$. As $\{z_i y_i z_i^{-1}\}$ is relatively compact, it follows that $\{\alpha_i\}$ is a relatively compact sequence of automorphisms of L' . Furthermore, we see that $\alpha_i^n = I$, the identity automorphism, for every i . Also, for all i , we have $\alpha_i(\delta_i)\delta_i^{-1} = (z_i y_i z_i^{-1})^{-1} \zeta_i z_i^{-1} (z_i y_i z_i^{-1})(\zeta_i z_i^{-1})^{-1} = (z_i y_i z_i^{-1})^{-1}(\zeta_i y_i \zeta_i^{-1})$. We have seen that the latter form a relatively compact sequence, and hence we get that $\{\alpha_i(\delta_i)\delta_i^{-1}\}$ is relatively compact. Therefore by Proposition 3.3 each δ_i can be written as $a_i b_i$, with $a_i, b_i \in L'$, $\{a_i\}$ relatively compact, and $\alpha_i(b_i) = b_i$ for all i . Since L' is a simply connected nilpotent Lie group, each b_i is contained in a unique one-parameter subgroup, say $\{\varphi_i(t)\}_{t \in \mathbb{R}}$, and hence the condition $\alpha_i(b_i) = b_i$ implies also that $\alpha_i(\varphi_i(t)) = \varphi_i(t)$ for all $t \in \mathbb{R}$. Thus, for each $i \geq 1$ and $t \in \mathbb{R}$, $\varphi_i(t)$ commutes with $z_i y_i z_i^{-1}$, and hence $z_i^{-1} \varphi_i(t) z_i$ commutes with y_i . Since $z_i^{-1} \varphi_i(t) z_i \in L \subset Z(\mu, G)$ and $\nu_i y_i^{-1} \in P(G(\mu))$, $z_i^{-1} \varphi_i(t) z_i$ commutes with $\text{supp } (\nu_i y_i^{-1})$ for all i . Therefore $z_i^{-1} \varphi_i(t) z_i \in Z(\nu_i, G)$ for all $t \in \mathbb{R}$ and all $i \in \mathbb{N}$. In particular, for each i , we have $z_i^{-1} b_i z_i \in Z^0(\nu_i, G)$, and since the element is also contained in $L' \subset [G, G]$, we get that $z_i^{-1} b_i z_i \in Z^0(\nu_i, [G, G])$. Since by hypothesis $Z^0(\lambda_i, [G, G]) = Z^0(\nu_i, [G, G])$, it follows that $z_i^{-1} b_i z_i \in Z(\lambda_i, G)$. Now, for

any i , $\zeta_i \lambda_i \zeta_i^{-1} = \delta_i z_i \lambda_i z_i^{-1} \delta_i^{-1} = a_i b_i z_i \lambda_i z_i^{-1} b_i^{-1} a_i^{-1} = a_i z_i \lambda_i z_i^{-1} a_i^{-1}$, since $z_i^{-1} b_i z_i \in Z(\lambda_i, G)$. Since $\{\zeta_i \lambda_i \zeta_i^{-1}\}$ and $\{a_i\}$ are relatively compact, the preceding conclusion shows that $\{z_i \lambda_i z_i^{-1}\}$ is relatively compact; this proves (ii) for the case of almost algebraic groups.

We next deduce the assertions as in Theorem 1.1 for a class of Lie groups with finitely many connected components, including all connected Lie groups of class \mathcal{C} . Let G be Lie group with finitely many connected components. We say that G is of class \mathcal{C}' if it has a representation $\pi : G \rightarrow GL(V)$ over a (finite-dimensional, real) vector space V such that the kernel of π is a discrete subgroup contained in the center of G and, if \tilde{G} is the smallest almost algebraic subgroup of $GL(V)$ containing $\pi(G)$ then $\pi(G)$ is normal in \tilde{G} and $[\pi(G), \pi(G)] = [\tilde{G}, \tilde{G}]$. Clearly, all almost algebraic groups are of class \mathcal{C}' . We note that if H is a connected Lie subgroup of $GL(V)$, for a vector space V , and \tilde{H} is the smallest almost algebraic subgroup of $GL(V)$ containing H , then $[H, H] = [\tilde{H}, \tilde{H}]$ (see [1], Chapter II, Theorem 13); in other words, every (connected) Lie group of class \mathcal{C} is of class \mathcal{C}' . We shall show that the statements as in Theorem 1.1 hold for measures on all Lie groups of class \mathcal{C}' .

Let G be a Lie group of class \mathcal{C}' , with the related notations π and \tilde{G} as above. Now let $\mu \in P(G)$, $n \in \mathbb{N}$, and $\{\nu_i\}$ be a sequence of n -th roots of μ , as in the hypothesis of the theorem. Then $\pi(\mu) \in P(\tilde{G})$ and $\pi(\nu_i)^n = \pi(\mu)$ for all i . By the special case proved above, there exists a sequence $\{\tilde{z}_i\}$ in $Z^0(\pi(\mu), [\tilde{G}, \tilde{G}])$ such that $\{\tilde{z}_i \pi(\nu_i) \tilde{z}_i^{-1}\}$ is relatively compact in $P(\tilde{G})$. Recall that G is of class \mathcal{C}' and $\pi : G \rightarrow GL(V)$ is such that $[\tilde{G}, \tilde{G}] = [\pi(G), \pi(G)]$. Thus $\tilde{z}_i \in Z^0(\pi(\mu), [\pi(G), \pi(G)])$ for all i . As the restriction of π to $[G, G]$ is a covering homomorphism onto $[\pi(G), \pi(G)]$, it follows that $Z^0(\pi(\mu), [\pi(G), \pi(G)]) = \pi(Z^0(\mu, [G, G]))$. As $\tilde{z}_i \in Z^0(\pi(\mu), [\pi(G), \pi(G)])$ this shows that there exists a sequence $\{z_i\}$ in $Z^0(\mu, [G, G])$ such that $\pi(z_i) = \tilde{z}_i$ for all i . Now, we have $\pi(z_i \nu_i z_i^{-1}) = \tilde{z}_i \pi(\nu_i) \tilde{z}_i^{-1}$ for all i , and since the latter form a relatively compact sequence, by Proposition 2.2 it follows that $\{z_i \nu_i z_i^{-1}\}$ is relatively compact. This proves (i).

Now let $r \geq 2$ and $\{\lambda_i\}$ be a sequence in $P(G)$ such that for every i , $\lambda_i^r = \nu_i$ and $Z^0(\lambda_i, [G, G]) = Z^0(\nu_i, [G, G])$. Since $[\tilde{G}, \tilde{G}] = [\pi(G), \pi(G)]$ and $\pi : [G, G] \rightarrow [\tilde{G}, \tilde{G}]$ is a covering homomorphism, the latter condition implies that $Z^0(\pi(\lambda_i), [\tilde{G}, \tilde{G}]) = Z^0(\pi(\nu_i), [\tilde{G}, \tilde{G}])$ for all i . Hence by the special case of the theorem proved above $\{\tilde{z}_i \pi(\lambda_i) \tilde{z}_i^{-1}\}$ is relatively compact, for the sequence $\{\tilde{z}_i\}$ as above. As $\tilde{z}_i = \pi(z_i)$ for all i , Proposition 2.2 implies that $\{z_i \lambda_i z_i^{-1}\}$ is relatively compact. This proves the theorem for groups in the class \mathcal{C}' .

6 Proof of the embedding theorem

In this section we prove the embedding theorem for class \mathcal{C} groups, viz. Theorem 1.3.

Let G be a Lie group of class \mathcal{C} , with $\pi : G \rightarrow GL(V)$ a representation of G , over a (finite-dimensional, real) vector space V , such that $\ker \pi$ is discrete, and let $\mu \in P(G)$. Given such a representation there exists a representation $\pi' : G \rightarrow GL(V')$ over a suitable vector space V' , such that $\ker \pi' = \ker \pi$ and $\pi'(G)$ is closed (see [2], the second paragraph of the proof of Proposition 2.1). Therefore, without loss of generality we may assume that $\pi(G)$ is closed.

Let \mathcal{Z} be the Lie algebra of $Z^0(\mu, G)$. The latter is a normal subgroup of $N(\mu, G)$ and the conjugation action induces an action of $N(\mu, G)$ on \mathcal{Z} . The restriction of the action to $G(\mu)$ is trivial, and hence the $N(\mu, G)$ -action on \mathcal{Z} factors to an action of $N(\mu, G)/G(\mu)$. Let $q : N(\mu, G)/G(\mu) \rightarrow GL(\mathcal{Z})$ be the representation corresponding to the action. Let Q be the smallest almost algebraic subgroup of $GL(\mathcal{Z})$ containing the image of q . Then Q is a Lie group with finitely many connected components.

We shall say that a subset M of \mathbb{N} is infinitely divisible, if for every $n \in \mathbb{N}$ there exists $m \in M$ such that $n|m$ (n divides m). It is easy to see that when an infinitely divisible subset of \mathbb{N} is expressed as a union of finitely many subsets M_1, \dots, M_l for some $l \in \mathbb{N}$, then M_k is infinitely divisible for at least one k , $1 \leq k \leq l$.

Now let $r : \mathbb{N} \rightarrow P(G)$ be a map as in the hypothesis, namely such that $r(m)^m = \mu$ for all $m \in \mathbb{N}$. For any infinitely divisible subset M of \mathbb{N} and $k \in \mathbb{N}$ such that $k|m$ for all $m \in M$, let $M(k) = \{r(m)^{m/k} \mid m \in M\}$.

Lemma 6.1. *There exists a decreasing sequence $\{M_j\}$ of subsets of \mathbb{N} such that for every $j \in \mathbb{N}$, $j!$ divides every element of M_j , and for any $\nu_1, \nu_2 \in M_j(j!)$, $q(\nu_1)$ and $q(\nu_2)$ are conjugate to each other in Q .*

Proof. We shall construct the sequence inductively, starting with $M_1 = \mathbb{N}$, the desired condition being satisfied in this case, as $M_1(1) = \{\mu\}$. Having chosen infinitely divisible subsets M_1, \dots, M_{j-1} for some $j \in \mathbb{N}$, such that the condition as in the lemma holds (up to $j-1$), we choose the next subset M_j as follows. Let $L_j = \{m \in M_{j-1} \mid j! \text{ divides } m\}$. As M_{j-1} is infinitely divisible, so is L_j . To each m in L_j we associate the conjugacy class of $q(r(m)^{m/j!})$ in Q . We note that $(q(r(m)^{m/j!}))^{j!} = I$, the identity element. As Q is a Lie group with finitely many connected components there are only finitely many conjugacy classes containing elements x such that $x^{j!} = I$; see [9], Corollary 2, on page 221. Hence the association as above partitions L_j into finitely many subsets, say $L_j^{(1)}, \dots, L_j^{(t)}$ for some $t \in \mathbb{N}$, such that for the numbers m belonging to the same subset the associated conjugacy classes are the same. As L_j is infinitely

divisible it follows that for at least one of the subsets, say $L_j^{(s)}$ where $1 \leq s \leq t$, is infinitely divisible. We choose M_j to be the subset $L_j^{(s)}$. Then M_j is contained in M_{j-1} , and from the choice we see that for any $\nu_1, \nu_2 \in M_j(j!)$, $q(\nu_1)$ and $q(\nu_2)$ are conjugate to each other in Q . This completes the inductive construction of the sequence, and proves the lemma.

Proof of Theorem 1.3. We now proceed with the proof of the theorem, using the notation as above. For each $\rho \in R(\mu, G)$ let $d(\rho)$ denote the dimension of the subspace of \mathcal{Z} consisting of all points fixed by $q(\rho)$. Let $j \in \mathbb{N}$ and consider any $\lambda \in M_j(j!)$. Since by Lemma 6.1 all these elements are conjugate, $d(\lambda)$ is the same for all of them; we shall denote the common value by d_j . If $\lambda \in M_{j+1}((j+1)!)$ then $\lambda^{j+1} \in M_j(j!)$, and hence we get that $d_{j+1} \leq d_j$ for all j . Therefore there exists $l \in \mathbb{N}$ such that $d_j = d_l$ for all $j \geq l$. Let $n = l!$.

Let $\{m_j\}$ be a sequence from \mathbb{N} such that $m_j \in M_{l+j}$ for all j . Let $\nu_j = r(m_j)^{m_j/n}$ for all j . Then $\{\nu_j\}$ is a sequence of n -th roots of μ . Hence by Theorem 1.1 there exists a sequence $\{z_j\}$ in $Z^0(\mu, [G, G])$ such that $\{z_j \nu_j z_j^{-1}\}$ is relatively compact. Let ν be any limit point of $\{z_j \nu_j z_j^{-1}\}$. Let $H = Z(Z^0(\nu, G))$, the centraliser of $Z^0(\nu, G)$ in G . Then clearly $\nu \in P(H)$. To prove the theorem it suffices to show that ν is strongly root compact and rationally embeddable on H .

Now, to begin with fix any $a \in \mathbb{N}$ consider the sequence defined by $\lambda_j = r(m_j)^{m_j/an}$, for all $j \geq a$; we note that as $m_j \in M_{l+j}$, $an | m_j$. Clearly $\lambda_j^a = \nu_j$ for all $j \geq a$. Let $j \in \mathbb{N}$ and consider the subgroups $Z^0(\nu_j, G)$ and $Z^0(\lambda_j, G)$. They are connected Lie subgroups of $Z^0(\mu, G)$, and since $\nu_j = \lambda_j^a$ we have $Z^0(\lambda_j, G) \subset Z^0(\nu_j, G)$. Also their corresponding Lie subalgebras are the subspaces of \mathcal{Z} fixed pointwise by $q(\nu_j)$ and $q(\lambda_j)$ respectively. Since $\nu_j \in M_{l+j}(n)$, $d(\nu_j) = d_l$. Also, since $\lambda_j \in M_{l+j}(an)$ and $\lambda_j^a = \nu_j$, we have $d_{l+j} \leq d(\lambda_j) \leq d(\nu_j) = d_l$. By the choice of l we have $d_{l+j} = d_l$, and hence this implies that $d(\lambda_j) = d(\nu_j)$. Therefore the subgroups $Z^0(\nu_j, G)$ and $Z^0(\lambda_j, G)$ are the same, and hence $Z^0(\nu_j, [G, G]) = Z^0(\lambda_j, [G, G])$ for all $j \geq r$ as above. Hence by assertion (ii) of Theorem 1.1 $\{z_j \lambda_j z_j^{-1}\}$ is relatively compact, for the sequence $\{z_j\}$ as above. Taking a limit along a suitable subsequence (one such that the corresponding subsequence of $\{z_j \nu_j z_j^{-1}\}$ converges to ν) we get a limit point θ of $\{z_j \lambda_j z_j^{-1}\}$ such that $\theta^a = \nu$. We note also that since $q(\theta_j)$ and $q(\nu_j)$ are elements of finite order converging to $q(\theta)$ and $q(\nu)$ respectively, $d(\theta) = d(\lambda_j)$ and $d(\nu) = d(\nu_j)$ for all large enough j . Therefore we get that $d(\theta) = d(\nu)$, and hence $Z^0(\lambda, G) = Z^0(\nu, G)$. Thus λ is an a -th root of ν on H .

Now for any $k \in \mathbb{N}$ let $\lambda_j^{(k)} = r(m_j)^{m_j/k!n}$ for all $j \geq k!$. Then $\{z_j \lambda_j^{(k)} z_j^{-1} \mid j \geq k!\}$ is relatively compact for every k . Thus if X_k , $k \geq 1$ is the closure of $\{z_j \lambda_j^{(k)} z_j^{-1} \mid j \geq k!\}$ then X_k is a compact metric space for all k and hence so is their cartesian product $\prod_{k=1}^{\infty} X_k$. It follows that there exists an increasing

sequence $\{j_i\}$ in \mathbb{N} such that for every k the sequence $\{z_{j_i}\lambda_{j_i}^{(k)}z_{j_i}^{-1}\}$ (defined for all j such that $j_i \geq k!$) converges as $i \rightarrow \infty$. Thus we see that the statement as in Remark 1.5 holds (with $\{m_{j_i}\}$ and $\{z_{j_i}\}$ as above in the place of $\{m_i\}$, and $\{z_i\}$ in the statement there, respectively).

Now let θ_k denote the limit of $\{z_{j_i}\lambda_{j_i}^{(k)}z_{j_i}^{-1}\}$ as $i \rightarrow \infty$. Then $\theta_k \in P(H)$ for all $k \in \mathbb{N}$. Also, clearly, $\theta_k^k = \theta_{k-1}$ for all $k \geq 2$. Then the map $(p/q!) \mapsto \theta_q^p$ for all $p, q \in \mathbb{N}$, is well-defined, and defines a rational embedding of ν on H .

We next show that ν is strongly root compact on H . We note that H is a closed subgroup containing $\ker \pi$. Since $\pi(G)$ is closed in $GL(V)$, this implies that $\pi(H)$ is a closed subgroup. Hence by Proposition 2.2, to prove that ν is strongly root compact on H it suffices to show that $\pi(\nu)$ is strongly root compact on $\pi(H)$. Clearly $\pi(H)$ is contained in the centraliser of $\pi(Z^0(\nu, G))$. Since $\ker \pi$ is discrete it follows that $\pi(Z^0(\nu, G)) = Z^0(\pi(\nu), \pi(G))$. Together with the preceding observation this implies that $\pi(H)$ centralises $Z^0(\pi(\nu), \pi(G))$. Let \tilde{H} be the centraliser of $Z^0(\pi(\nu), \pi(G))$ in \tilde{G} . Then \tilde{H} is an almost algebraic subgroup of $GL(V)$ containing $\pi(H)$. Furthermore, $Z(\pi(\nu), \tilde{H})$ is contained in the center of \tilde{H} . Hence by Corollary 2.4 $\pi(\nu)$ is strongly root compact on \tilde{H} . Since $\pi(H)$ is closed in \tilde{H} this implies that $\pi(\nu)$ is strongly root compact on $\pi(H)$. Therefore by Proposition 2.2 ν is strongly root compact on H . This completes the proof of the first assertion in the theorem.

Clearly every measure from the rational embedding of ν as above is contained in the closure of the subsemigroup generated by $\{z_{j_i}r(m_{j_i})z_{j_i}^{-1} \mid i \in \mathbb{N}\}$, where $\{m_{j_i}\}$ and $\{z_{j_i}\}$ are as above. Since ν is strongly root compact, the rational embedding extends to an embedding $\{\nu_t\}_{t \geq 0}$ of ν ; this can be deduced from Theorem 3.5.1 of [7] in the same way as Theorem 3.5.9 is deduced from Theorem 3.5.4 there. This completes the proof of the theorem.

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