# EQUATIONS OF SKY COMPONENTS WITH A "C.I.E. STANDARD OVERCAST SKY"*

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## Introduction

The sky component due to a given unobstructed window opening, at any point inside a room, in any plane, is defined as the ratio of the intensity of daylight illumination, in that plane, due to the patch of sky visible from that point through the opening, to the simultaneous intensity in a horizontal plane at an external point open to the entire sky vault. This component is independent of the absolute brightness of the sky, but is dependent on its surface distribution.

The general practice for design purposes is to take the worst conditions: and a heavily overcast sky defines the most adverse condition for daylighting. Prior to 1955, a heavily overcast sky was generally assumed to have a uniform distribution of illumination over its surface, and was accordingly called a 'uniform sky'. But measurements carried out by Moon and Spencer (1942) indicated that for a sky entirely covered over with thick dark clouds, the intensity of illumination at any point of the sky was independent of the azimuth and the position of the sun at the time, and varied only with the altitude in accordance with an empirical approximate formula

$$B_\theta = B_z \frac{1 + 2 \sin \theta}{3}$$

where $B_z$ and $B_\theta$ are the intensities respectively at the zenith and at an altitude $\theta$. Subsequently, the International Commission on Illumination (C.I.E.) in its sessional meeting at Zurich (1955) defined and recommended for general adoption a "C.I.E. Standard Overcast Sky" in which the intensity varied in strict accordance with the Moon-Spencer formula. Since then, the daylight illumination inside a room is expressed in terms of the intensities, both inside and outside, arising from this Standard Sky only.

As early as 1924 Yamauti derived equations to give the sky components with a "Uniform sky" for vertical rectangular openings. A geometrical

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method was employed to obtain the components at any interior point, in three planes, (*a*) horizontal, (*b*) vertical and perpendicular to the window plane and (*c*) vertical and parallel to the window plane. A need has arisen, since 1955, to derive similar equations for a C.I.E. Standard Overcast Sky. In the absence of such equations, Waldram's diagram (1950), originally constructed for a uniform sky as early as 1923, was modified by Walsh (1959) to suit the standard sky. Rivero (1958) used Yamauti's equations in drawing up his tables for a uniform sky and later modified those values to give tables of sky components for a non-uniform standard sky. Hopkinson, Longmore and Murray Graham (1958) prepared certain Simplified Daylight Tables for glazed openings with a standard sky. But in both an indirect method of computation alone was used to allow for the non-uniform distribution of intensity.

The present paper gives exact trigonometrical equations for direct calculation of sky components due to rectangular windows, both vertical and horizontal, for the C.I.E. Standard Overcast Sky. The same equations will enable direct calculation of sky components due to any rectangular sloping window also, as it can be resolved into a horizontal and a vertical opening. The method adopted is quite general. Yamauti's equations can be derived by the same method. It could also be employed to derive sky component equations arising from an intensity distribution other than that of the standard sky, provided that intensity is a known function of the altitude and independent of the azimuth or *vice-versa*.

## GENERAL METHOD OF DERIVATION

In Fig. 1 ABCD is the rectangular vertical window of height $h$ and length $l$. P is a point on the normal to the window through A and distant $d$ from it. A sphere is described with P as centre and radius $d$. A projection of ABCD on the spherical surface is AB′C′D′ which is proportional to the patch of the sky visible from P through the window. The intensity at P due that portion of the sky is the same as from AB′C′D′ if the same luminous intensities are assumed at corresponding points.

The values of $\beta$, $\beta'$, $\gamma$, $\gamma'$ the angles subtended at P by the sides AB, DC, AD, BC respectively are given by the following equations:

$$(a)\ \tan\beta = \frac{l}{d}$$

$$(b)\ \tan\gamma = \frac{h}{d}$$

$$(c)\ \tan \beta' = \tan \beta \cos \gamma$$

$$(d)\ \tan \gamma' = \tan \gamma \cos \beta \qquad (1)$$

The following additional relationships between them can also be easily derived:

$$(a)\ \sin \beta' = \sin \beta \cos \gamma'$$

$$(b)\ \sin \gamma' = \sin \gamma \cos \beta'$$

$$(c)\ \cos \beta' \cos \gamma = \cos \gamma' \cos \beta \qquad (2)$$

In Fig. 1, M is a point at azimuth $\psi$ and altitude $\theta$. $N_1$, $N_2$, $N_3$ are the feet of perpendiculars from M on the three mutually perpendicular planes, PAY, PZA, and PZY, respectively. A small element of area of the spherical surface at M between azimuths $\psi$ and $\psi + d\psi$ and altitudes $\theta$ and $\theta + d\theta$ is considered. The area of the element is $d^2 \cos \theta \, d\theta \, d\psi$. If B is the luminous intensity at M, the intensity of illumination at P due to the element, on a plane perpendicular to PM, is $B \cos \theta \, d\theta \, d\psi$. This expression when multiplied by $\sin \theta$ gives the component of the intensity in the horizontal plane PAY, when multiplied by $\cos N_2MP = \sin MPN_2 = \sin \psi \cos \theta$ (*vide* equation 2 *a*) gives the component on the plane PZA, and when multiplied by $\cos N_3MP = \sin MPN_3 = \cos \psi \cos \theta$ gives the component on the vertical plane PZY. B is a constant for the uniform sky and

$$B = B_\theta = B_z \frac{1 + 2 \sin \theta}{3}$$

for the standard sky. The above expressions are integrated over the spherical area AB′C′D′ first with respect to $\theta$ and then with respect to $\psi$, between appropriate limits, and later expressed as ratios of the corresponding horizontal intensity due to the total unobstructed hemisphere of the sky, to give the respective sky components.

## Horizontal Intensities Due to the Entire Sky Vault

Total horizontal intensity ($T_u$) at P due to the uniform sky hemisphere is given by

$$T_u = B \int_0^{\pi/2} \int_0^{2\pi} \sin \theta \cos \theta \, d\theta \, d\psi$$

$$= \pi B \text{ (on integration)} \qquad (3)$$

Total horizontal intensity ($T_s$) at P due to the standard sky hemisphere is given by

$$T_s = \frac{B_z}{3} \int_0^{\pi/2} \int_0^{2\pi} (1 + 2\sin\theta) \sin\theta \cos\theta \, d\theta \, d\psi$$

$$= \frac{7}{9} \pi B_z \qquad \text{(on integration)} \qquad (4)$$

## SKY COMPONENTS FOR VERTICAL RECTANGULAR WINDOWS

Sky component ($f_{V.H}$) at P on a horizontal plane with a uniform sky is given by

$$f_{V.H} = \frac{B \int_0^{\theta'} \int_0^{\beta} \sin\theta \cos\theta \, d\theta \, d\psi}{T_u} \qquad [T_u \text{ is given by (3)}]$$

where $\theta'$ (ref. Fig. 1) is given by $\tan\theta' = \tan\gamma \cos\psi$ (*vide* Equation 1 *d*) where $\tan\gamma$ is a constant. The above equation when integrated over the area AB′C′D′ and simplified gives

$$f_{V.H} = \frac{\beta - \beta' \cos\gamma}{2\pi} \qquad (5)$$

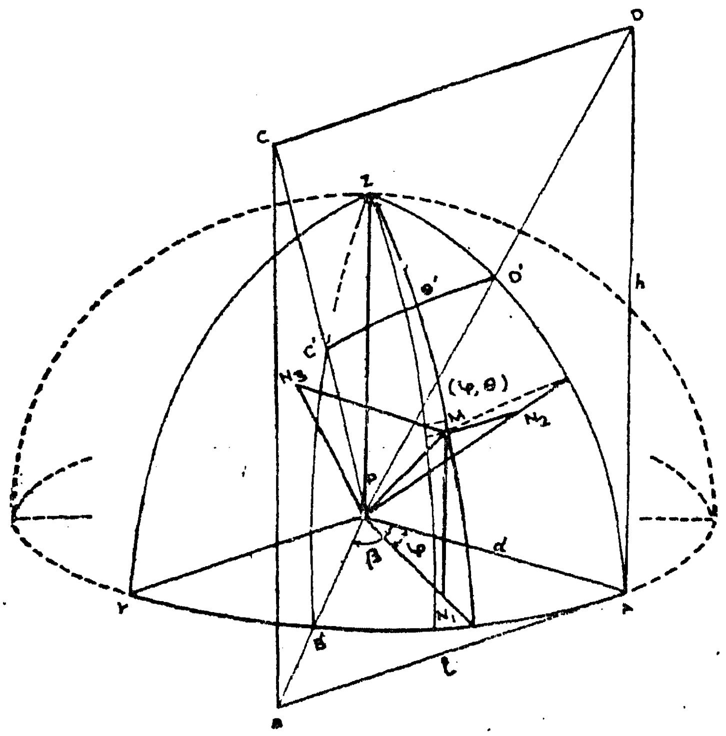

FIG. 1

Sky component ($F_{V.H}$) at P on a horizontal plane with a standard sky is given by

$$F_{V.H} = \frac{\frac{B_Z}{3}\int_0^{\theta'}\int_0^{\beta}(1+2\sin\theta)\sin\theta\cos\theta\,d\theta\,d\psi}{T_S} \qquad [T_S \text{ is given by (4)}]$$

which on integration and simplification gives

$$F_{V.H} = \frac{3}{14\pi}(\beta - \beta'\cos\gamma) + \frac{2}{7\pi}\sin^{-1}(\sin\beta\sin\gamma) - \frac{1}{7\pi}(\sin 2\gamma\sin\beta'). \tag{6}$$

Similarly expressions for $f_{V.\perp}$ and $F_{V.\perp}$ the sky components on a vertical plane through P perpendicular to the window, for uniform and standard skies respectively, and corresponding expression for $f_{V.\parallel}$ and $F_{V.\parallel}$ the sky components on a vertical plane through P parallel to the window are derived and given below.

$$f_{V.\perp} = \frac{\gamma - \gamma'\cos\beta}{2\pi} \tag{7}$$

$$F_{V.\perp} = \frac{3}{14\pi}(\gamma - \gamma'\cos\beta) + \frac{2}{7\pi}(1 - \cos\beta - \cos\gamma + \cos\beta\cos\gamma') \tag{8}$$

$$f_{V.\parallel} = \frac{\beta'\sin\gamma + \gamma'\sin\beta}{2\pi} \tag{9}$$

$$F_{V.\parallel} = \frac{3}{14\pi}(\beta'\sin\gamma + \gamma'\sin\beta) + \frac{2}{7\pi}(\sin\beta - \sin\beta'\cos^2\gamma) \tag{10}$$

Equations 5, 7 and 9 have been given by Yamauti (1924).

## Sky Components for Horizontal Rectangular Windows

Figure 2 gives the diagram of a horizontal rectangular opening ABCD. AB′C′D′ is its projection on the surface of a sphere described with P as centre and radius $d$, and is proportional to the patch of the sky visible through the window from P. $\beta$, $\beta'$, $\gamma$, $\gamma'$ are again taken as the angles subtended by the sides AB, DC, AD and BC respectively. Their values will be given by the same equations (1) and (2). The value of the angle $\alpha$ (ref. to Fig. 2) is given by the equation

$$\tan\alpha = \frac{\tan\gamma}{\tan\beta}. \tag{11}$$

For any element of area at M included between azimuths $\psi$ and $\psi + d\psi$ and altitudes $\theta$ and $\theta + d\theta$ the expression $B_\theta\cos\theta\,d\theta\,d\psi$ where $B_\theta$ is a

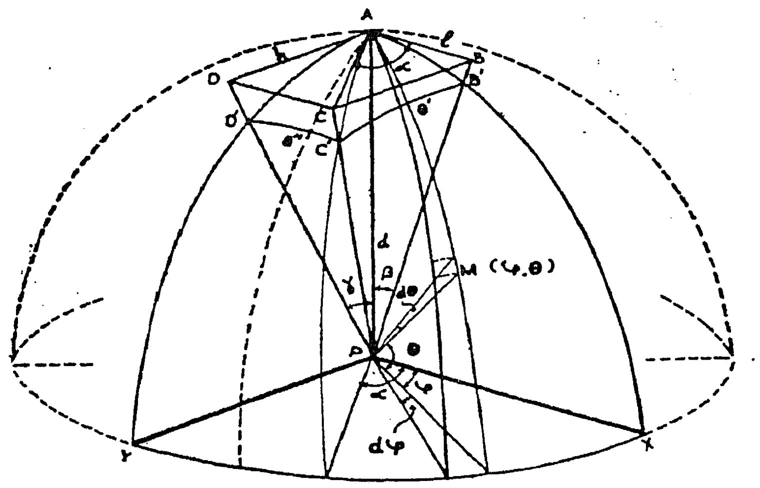

FIG. 2

constant for the uniform sky or equal to $B_z(1 + 2\sin\theta)/3$ for the standard sky gives the intensity at P in a plane normal to the line joining P with the element M. This expression is multiplied by the factors $\sin\theta$, $\sin\psi\cos\theta$, or $\cos\psi\cos\theta$ as before to give the components of intensity at P in the horizontal plane, $\beta$ plane (APX), and $\gamma$ plane (APY) respectively. The integrations however are carried out in two stages, between limits $\pi/2$ and $\theta'$ for $\theta$, and $\alpha$ and 0 for $\psi$ in the first stage, and between limits $\pi/2$ and $\theta''$ for $\theta$, and $\pi/2$ and $\alpha$ for $\psi$ in the second stage, where $\theta'$ and $\theta''$ are functions of $\psi$ given by the equations

$$\tan\theta' = \cot\beta\cos\psi \tag{12}$$

$$\tan\theta'' = \cot\gamma\sin\psi \tag{13}$$

The two integrated values are added together and expressed as a ratio of the horizontal intensity due to the total hemisphere of the sky $T_u$ for the uniform sky or $T_s$ for the standard sky (Equations 3 and 4) to give the respective sky components. The following equations can then be derived, where $f_{H.H}$, $f_{H.\beta}$ and $f_{H.\gamma}$ are the sky components with an uniform sky and $F_{H.H}$, $F_{H.\beta}$ and $F_{H.\gamma}$ are the sky components with a standard overcast sky, in the horizontal, $\beta$ and $\gamma$ planes respectively.

*With uniform sky*

$$f_{H.H} = \frac{1}{2\pi}(\beta'\sin\gamma + \gamma'\sin\beta) \tag{14}$$

$$f_{H.\beta} = \frac{1}{2\pi}(\beta - \beta'\cos\gamma) \tag{15}$$

$$f_{H.\gamma} = \frac{1}{2\pi}(\gamma - \gamma'\cos\beta). \tag{16}$$

*With C.I.E. Standard overcast sky*

$$F_{H.H} = \frac{3}{14\pi}(\beta' \sin\gamma + \gamma' \sin\beta) + \frac{1}{7} + \frac{1}{7\pi}(\sin 2\beta \sin\gamma' + \sin 2\gamma \sin\beta') - \frac{2}{7\pi}\sin^{-1}(\sin\alpha\cos\beta) - \frac{2}{7\pi}\sin^{-1}(\cos\alpha\cos\gamma) \quad (17)$$

$$F_{H.\beta} = \frac{3}{14\pi}(\beta - \beta' \cos\gamma) + \frac{2}{7\pi}(\sin\beta - \sin\beta' \cos^2\gamma) \quad (18)$$

$$F_{H.\gamma} = \frac{3}{14\pi}(\gamma - \gamma' \cos\beta) + \frac{2}{7\pi}(\sin\gamma - \sin\gamma' \cos^2\beta). \quad (19)$$

Equations (14), (15) and (16) can be surmised from Yamauti's equations for vertical windows. One can also understand how equation (19) can be readily obtained from equation (18) by merely interchanging $\beta$ with $\gamma$ and $\beta'$ with $\gamma'$.

In the equations derived above the angles are obviously to be expressed in radian measure. It however facilitates calculations to express the angles in degrees. Moreover it is usual to express the sky components as percentages. The equations modified accordingly are given below for sky components with the C.I.E. standard overcast sky.

Vertical windows:

$$\%F_{V.H} = \frac{\beta - \beta' \cos\gamma}{8 \cdot 4} + \frac{\sin^{-1}(\sin\beta\sin\gamma)}{6 \cdot 3} - 4 \cdot 547 \sin 2\gamma \sin\beta' \quad (20)$$

$$\%F_{V.\perp} = \frac{\gamma - \gamma' \cos\beta}{8 \cdot 4} + 9 \cdot 095\,(1 - \cos\beta - \cos\gamma + \cos\beta\cos\gamma') \quad (21)$$

$$\%F_{V.\parallel} = \frac{\beta' \sin\gamma + \gamma' \sin\beta}{8 \cdot 4} + 9 \cdot 095\,(\sin\beta - \sin\beta' \cos^2\gamma). \quad (22)$$

Horizontal windows:

$$\%F_{H.H} = \frac{\beta' \sin\gamma + \gamma' \sin\beta}{8 \cdot 4} + 14 \cdot 286 + 4 \cdot 547\,(\sin 2\beta \sin\gamma' + \sin 2\gamma \sin\beta') - \frac{\sin^{-1}(\sin\alpha\cos\beta) + \sin^{-1}(\cos\alpha\cos\gamma)}{6 \cdot 3} \quad (23)$$

$$\%F_{H.\beta} = \frac{\beta - \beta' \cos\gamma}{8\cdot 4} + 9\cdot 095\,(\sin\beta - \sin\beta' \cos^2\gamma) \tag{24}$$

$$\%F_{H.\gamma} = \frac{\gamma - \gamma' \cos\beta}{8\cdot 4} + 9\cdot 095\,(\sin\gamma - \sin\gamma' \cos^2\beta). \tag{25}$$

[The angles in equations (20) to (25) are expressed in degrees.]

The first terms in the above equations, with $3\cdot 6$ replacing $8\cdot 4$, gives the corresponding sky components with a uniform sky.

The above equations for a point P at infinity get reduced each to a value zero. When the point P coincides with A, *i.e.*, when

$$\beta = \gamma = \beta' + \gamma' = \frac{\pi}{2} \quad \text{or} \quad 90°$$

the horizontal components get reduced each to a value 25%. The components in the vertical planes get reduced each to a value

$$\left(\frac{3}{28} + \frac{2}{7\pi}\right) 100\% = 19\cdot 8\% \text{ (about)}.$$

This later value can be easily derived as the sky component over either of the two mutually perpendicular vertical planes for a quarter hemisphere of a standard sky enclosed between them. For a uniform sky all values will be 25% when P coincides with A.

The values calculated from these equations tally approximately with those given in Rivero's tables, indicating thereby the care and accuracy with which the latter values have been computed though employing an indirect method.

The equations given above are derived for a point P on the perpendicular through one corner of the window. If the point is not so situated and when the sill is above the point, the rectangular window should first be considered extended up to the foot of the perpendicular from the point on the plane of the window. The additional area can be considered as an algebraic sum of three rectangular areas each with one of their corners at the foot of the perpendicular from the point. The total sky component for the additional area, which can now be calculated, is then deducted from the component for the extended window, to give the component for the actual window. When the sill of the window is below the horizontal plane through the point, the portion of the window below that plane is neglected. It is thus possible to enlarge the utility of the equations to the general case of any point inside.

## Glazing Correction

The loss in transmission through glass at various angles of incidence can be calculated from the Fresnel's equation. The loss is about 4% at normal incidence from each surface of the pane. At other angles of incidence the loss is more. Rivero (1958) has given the following empirical formula for the transmission factors at different angles of incidence.

$$\tau = c\,(1 + \sin^3 \theta) \cos \theta, \quad \text{where} \quad c = 0{\cdot}81 \ \text{(about)}.$$

This formula agrees with the theoretical curve fairly well, and is of a form convenient to adopt as an additional factor in the equations to be integrated. Sky components for glazed windows can then be easily derived using the general method adopted in this paper. Such equations are however not given here, as they are not likely to be of much practical significance. The glazing losses increase rapidly with the degree of dirtiness of the glass panes, depending on the period they are left uncleaned, the density of atmospheric dust at the locality, the inclination and orientation of the windows, etc. Such losses can be estimated only approximately and may amount to as much as 50% and more under bad conditions. There does not appear therefore to be any justification to calculate it with precision taking into consideration only the relatively minor variation of transmission factor with angle of incidence. Quite satisfactory results would be obtained for all practical purposes if the sky components for unglazed openings are reduced by a value 10 to 20% to allow for glazing losses, provided the glass panes are tolerably clean.

## Summary

Rigorous trigonometrical equations for calculating directly the sky components at any point due to a C.I.E. Standard Overcast Sky for both vertical and horizontal rectangular unglazed windows are derived and presented. The equations, when suitably combined, can also be used for any sloping window. The general method adopted can be used to derive similar equations for a sky with any intensity distribution, provided that intensity is a function of altitude alone and independent of azimuth or *vice-versa*.

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