# Vibrational Spectrum of Spider-Web Networks

**Ajit C Balram**[1], **Deepak Dhar**[2]

[1]Department of Physics, Indian Institute of Science Education and Research, Sai Trinity Building, Garware Circle, Sutarwadi, Pashan, Pune 411 021, India
[2]Department of Theoretical Physics, Tata Institute of Fundamental Research, 1 Homi Bhabha Road, Mumbai 400 005, India

E-mail: ddhar@theory.tifr.res.in

**Abstract.** We consider the vibrational spectrum of masses connected by springs forming a spider-web network. These graphs were first introduced in connection with the design of telephone switching networks. We consider a graph consisting of $M$ levels, each of which has $q^N$ vertices. Each vertex in the $m$-th level is connected to $q$ vertices each in the $(m \pm 1)$th levels. We use the large symmetry group of the graph to determine all normal models and their frequencies. In the limit of large $M$ and $N$, the spectrum is a set of $\delta$-functions, and almost all the modes are localized. The fractional number of modes with frequency less than $\omega$ varies as $\exp(-C/\omega)$ for $\omega$ tending to zero, where $C$ is a constant. This implies that for an unbiased random walk on the vertices of this graph, the probability of return to the origin at time $t$ varies as $\exp(-C't^{1/3})$, for large $t$, where $C'$ is a constant.

PACS numbers: 63.70.+h,05.50.+q,05.40.Fb

Submitted to: *J. Phys. A: Math. Gen.*

## 1. Introduction

Spider-web graphs were introduced by Ikeno in 1959, for efficient design of crossbar exchanges for telephone switching networks [1]. The name spider-webs to describe them seems to have been first used by Feiner and Kappel in 1970 [2]. With the advent of digital switching and mobile communication networks, crossbar exchanges have become obsolete, and their optimal design is no longer of technological interest. But the spider-web graphs are very interesting structures for statistical physics.

These are finite graphs where all vertices are equivalent. In the thermodynamic limit when the number of vertices in the graph tends to infinity, the number of sites within a distance $R$ of a given site increases exponentially with $R$, and the effective dimension of these graphs is infinite. These are thus similar to the Bethe lattice, which has been studied a lot in statistical physics. However, for the Bethe lattice, realized as the limit of tree graphs, separating the bulk properties from the surface properties requires care as most of its sites are near the surface. One can realize a Bethe lattice without surface effects as the limit of a random graph with uniform coordination number, but this involves averaging over disorder, which may be quenched [3], or annealed [4]. Spider-web graphs avoid the difficulties associated with this averaging. They also have additional symmetries, in addition to translational invariance, which makes the study of statistical physics models on these graphs analytically more tractable.

Given its origins, it is not surprising that most of the earlier studies dealt with the percolation properties of these networks [1]. Takagi [5] showed that in a certain class of graphs, the spider-web network has the highest linking probability. But, Chung and Hwang [6] showed that, outside this class, there are other graphs that are better. For subsequent work on the percolation properties on these networks, see [7, 8, 9, 10]. Other models on these graphs have not been studied much so far.

In this paper, we look at normal modes of the spring problem on this network. We obtain the entire spectrum for finite values of $M$ and $N$, and also describe the qualitative features of the spectrum in the thermodynamic limit of $M \to \infty$, with $N$ finite. In the double limit of both $M, N \to \infty$, almost all the eigenmodes of the spring problem are localized on this network. We study the return to origin for the related problem of a random walker on this network. We show that the probability that a random walker returns to origin after $t$ steps varies as $\exp(-t^{1/3})$. This should be compared with a power-law decay $t^{-d/2}$ on $d$-dimensional hypercubic lattices, and decays exponentially with $t$ on the Bethe lattice. This shows that the loops in the spider-web graph a non-trivial effect on its large-scale properties.

The plan of this paper is as follows. In section II, we define spider-web graphs, and the spring Hamiltonian on these graphs. In section III, we discuss the sector decomposition of the Hilbert space using the symmetries of the spider-web graphs. In Section IV, we determine the spectrum of spring Hamiltonian, and determine the eigenmodes in different sectors. In Section V, we use the spectrum of spring Hamiltonian to determine the exact generating function for returns to origin of a walker doing a

random walk on the vertices of the spider web graph.

## 2. Definition of the model

A spider-web network is a graph consisting of $Mq^N$ vertices, divided into $M$ levels, each level with $q^N$ vertices. We label the levels by integers 1 to $M$, and vertices within a level are labeled by $N$-digit integers $r$, base $q$, with $0 \leq r \leq (q^N - 1)$. In the following we discuss only the case $q = 2$. Generalization to other values of $q$ is straight-forward.

On a binary string of length $N$, we define an operation $R_0$ of adding a zero from the right, making space for it by shifting all bits by one space to the left. The left-most (first) bit of the original string is lost. So, for example, for $N = 6$, action of $R_0$ on the binary string 111001 yields a string 110010, and we write $R_0(111001) = 110010$. The operator $R_1$ adds a 1 to the right. We similarly define operators $L_0$ and $L_1$, which correspond to adding 0 and 1 respectively at the left end, and shifting the bits one space to the right. In this case, the rightmost bit is lost. Clearly, $R_0(r) = 2r(mod\ 2^N)$, and $R_1(r) = 2r + 1(mod\ 2^N)$, and we can write similar arithmetic formulas for $L_0(r)$ and $L_1(r)$.

Now consider a vertex labeled by a binary integer $r$ in the level $m$. We denote this vertex by $(m, r)$. The spider-web graph is defined by connecting this vertex to the two vertices $(m + 1, R_0(r))$ and $(m + 1, R_1(r))$ in the level $(m + 1)$. We assume periodic boundary conditions, and the vertices in the level $m = M$ are connected to level $m = 1$. It is convenient to assume that the index $m$ is defined only modulo $M$, so that $m = 1$ is the same as $m = M + 1$. We will take these connecting edges to be undirected. Then, each vertex $(m, r)$ is also connected to the vertices $(m - 1, L_0(r))$ and $(m - 1, L_1(r))$. We will assume that $M \geq N + 1$, as otherwise the graph is not fully connected. Fig. 1 shows a graph with $M = 4, N = 3$.

Given this structure of connections, it is easy to see that there is unique directed path of length $N$ from any vertex $(m, r)$ to any of the $2^N$ vertices in the level $(m+N)$. If the binary label of the target vertex is $i_1 i_2 i_3 ... i_N$, one just starts at the vertex $(m, r)$, and takes steps to the next level using edges corresponding to $R_{i_1}, R_{i_2}, R_{i_3}...$ respectively.

In the context of crossbar exchanges, it is natural to work with open boundary conditions, with no direct connections between level $M$ and level 1. We note that the connections can be achieved by employing $2^{N-1}$ copies of $2 \times 2$ crossbars between levels $m$ and $m + 1$ for all $m$. The $2 \times 2$ crossbar allows us to connect any of the two input lines to any of the two output lines. Then, using only $N + 1$ levels, with only $N2^{N+1}$ crosspoints, we can connect any of $2^N$ vertices at level $m = 1$, (these correspond to the callers requesting telephone connections), with any of the $2^N$ vertices in the level $m = N + 1$ (any of these is a possible destination of the requested connection). Also, there is a simple algorithm to determine the path from the input node to the output node. This is much less than than the number $2^N \times 2^N$, which would have been required, if a single stage crossbar was used. If the number of levels is larger than this, we get multiple possible paths from the input to output. This was the original reason for

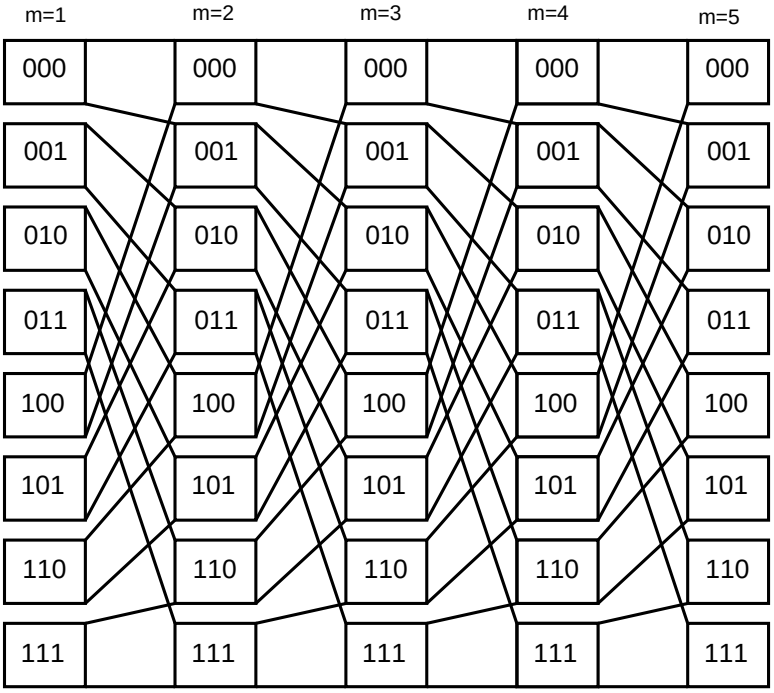


**Figure 1.** $N = 3, M = 4$ spider network with periodic boundary conditions, the vertices in the column $m = 5$ are identified with those of $m = 1$. A vertex with 3-bit label $\ell$ in the level $m$ is connected to the vertices with labels $R_0(\ell)$, and $R_1(\ell)$ in the level $(m + 1)$.

interest in these networks.

We consider a unit mass attached to each vertex of the spider-web network, coupled to its four neighbors by harmonic springs of unit spring constant. We denote a scalar variable $x(v)$ for each vertex $\mathbf{v} \equiv (m, r)$ of the graph. Let the corresponding momentum variable be denoted by $p(v)$. Then, the Hamiltonian is given by

$$H = \frac{1}{2}\sum_{\mathbf{v}} p(\mathbf{v})^2 + \frac{1}{2}\sum_{\mathbf{v},\mathbf{v}'} x(\mathbf{v})\mathbb{K}(\mathbf{v},\mathbf{v}')x(\mathbf{v}') \tag{1}$$

where $\mathbb{K}$ is the matrix of coupling constants defined by

$$\begin{aligned}\mathbb{K}(\mathbf{v},\mathbf{v}') &= 4, \text{if } \mathbf{v} = \mathbf{v}',\\ &= -1, \text{ if } \mathbf{v},\mathbf{v}' \text{ are nearest neighbors},\\ &= 0, \text{ otherwise}\end{aligned} \tag{2}$$

Let $\psi_\alpha(\mathbf{v})$ be a normal mode of this system with frequency $\omega_\alpha$. Then, $\psi_\alpha$ is a solution of the eigenvalue equation

$$\mathbb{K}\psi_\alpha(\mathbf{v}) = \omega_\alpha^2\psi_\alpha(\mathbf{v}). \tag{3}$$

We will denote by $F(\omega^2)$, the fractional number of modes of an eigenvalue less than or equal to $\omega^2$. Hence, $0 \le F(\omega^2) \le 1$. This is called the cumulative spectral density. The spectral density $D(\omega^2)$ is defined as $dF(\omega^2)/d\omega^2$. In the limit $M \to \infty$, with $N$

finite, we will show that $D(\omega^2)$ has both a discrete part, and a continuous part. The fraction of modes that give the continuous part is $2^{-N}$. Therefore, in the limit $N \to \infty$, we get only a discrete spectrum.

It is well-known that the diagonalization of the spring Hamiltonian is also related to problem of random walks. Consider a continuous-time random walk on the vertices of this graph, where the walker can jump to any one of the neighboring vertices with rate 1. Thus, in a short time $dt$, it can jump to any of the four neighbors with probability $dt$, and stay at the same vertex with probability $1 - 4dt$. Suppose the walker starts at vertex $\mathbf{v}_0$. Then, we denote the probability that the walker is at vertex $\mathbf{v}$ at time $t$, by $\text{Prob}(\mathbf{v}, t)$. We define a vector $|P(t)\rangle$, whose $v$-th component is $\text{Prob}(\mathbf{v}, t)$. Then the evolution of $|P(t)\rangle$ satisfies the Markov equation

$$\frac{d}{dt}|P(t)\rangle = -\mathbb{K}|P(t)\rangle \tag{4}$$

Then, the probability of being at vertex $\mathbf{v}'$ at time $t$ is given by

$$\text{Prob}(\mathbf{v}', t|\mathbf{v}_0, 0) = \langle \mathbf{v}'|e^{-\mathbb{K}t}|\mathbf{v}_0\rangle. \tag{5}$$

The probability of being at the starting point at time $t$ is given by $\text{Prob}(\mathbf{v}_0, t|\mathbf{v}_0, 0)$. From the translational symmetry of the graph, this does not depend on $\mathbf{v}_0$. We define the Laplace -transform of return probabilities $R(s)$ by

$$R(s) = \int_0^\infty dt\ \text{Prob}(\mathbf{v}_0, t|\mathbf{v}_0, 0)e^{-st} \tag{6}$$

Then it easy to see that

$$R(s) = \frac{1}{M2^N} Tr[s + \mathbb{K}]^{-1} = \int_0^\infty d\omega^2 D(\omega^2)[s + \omega^2]^{-1} \tag{7}$$

## 3. Symmetries and sector decomposition

The spider-web network has a high degree of symmetry. Firstly, all vertices are equivalent. We define a translation operator $\mathbb{T}$ which takes a vertex $(m, r)$ to $(m+1, r)$. Clearly, $\mathbb{T}$ is a symmetry of the graph, and we have

$$\mathbb{T}^M = \mathbb{I} \tag{8}$$

where $\mathbb{I}$ is the identity operator.

There are additional symmetries. Consider any integer $i$ lying between 1 and $M$. Define $f_k(r)$ as the string obtained from $r$ by flipping the $k$-th bit (reading from left to right), in its binary representation. If the bit is 0, it is replaced by 1, and vice versa. We define the operator $\mathbb{F}_i$ which, for $k = 1$ to $N$, takes the site $(m+k-1, r)$ to another site $(m + k - 1, f_k(r))$ in the same level, for all $r$. Sites in levels $j$ where $m - j + 1(mod\ M)$ is outside the range 1 to $N$ are left unchanged. Clearly, we have $\mathbb{F}_i^2 = \mathbb{I}$. Also, we see that this mapping preserves the adjacency strucure of the network: if vertices $\mathbf{v}$ and $\mathbf{v}'$ are nearest neighbors, so are $\mathbb{F}_i(\mathbf{v})$ and $\mathbb{F}_i(\mathbf{v}')$. Hence we have

$$[\mathbb{K}, \mathbb{F}_i] = 0. \tag{9}$$

Acting on the space of wavefunctions, the action of the operators $\mathbb{F}$ is given by

$$\mathbb{F}\psi(\mathbf{v}) = \psi(\mathbb{F}\mathbf{v}) \tag{10}$$

Now, clearly, the operators $\mathbb{T}$ and $\mathbb{F}_i, i = 1$ to $M$ commute with $\mathbb{K}$:

$$[\mathbb{K}, \mathbb{F}_i] = [\mathbb{K}, \mathbb{T}] = 0. \tag{11}$$

Also, different $\mathbb{F}_i$ commute with each other:

$$[\mathbb{F}_i, \mathbb{F}_j] = 0. \text{ for all } i, j. \tag{12}$$

Then, we can find eigenvectors $\psi$ that are simultaneous eigenvectors of $\mathbb{K}$ and $\mathbb{F}_i$, for all $i, 1 \leq i \leq M$.

Now, we denote the eigenvalues of $\mathbb{F}_i$ by $f_i$. These are $\pm 1$. Consider an eigenfunction $\psi$, with

$$\mathbb{F}_i\psi = f_i\psi, \text{ for } i = 1 \text{ to } M. \tag{13}$$

The Hilbert space of all functions $\psi(m, r)$ breaks into $2^M$ disjoint sectors, each corresponding to a subspace spanned by basis vectors that are eigenvectors corresponding to a specified set $\{f_i\}$. For $M \gg N$, the number of sectors can be substantially greater than the $M2^N$, which is the total dimension of the Hilbert space. It follows that for $M \gg N$, a large number of sectors must be empty.

It is easy to see that if $\psi(m, r)$ is an eigenfunction of $\mathbb{F}_i$ with eigenvalue $-1$, then $\psi(m, r)$ must be zero, whenever $m \notin [i - N + 1, i]$. If more than one of the $f_i$'s are negative, the corresponding eigenfunction $\psi(m, r)$ can be nonzero only on the intersection of the corresponding intervals. This severely constrains the allowed values of $\{f_i\}$ for a non-null sector. In particular, this implies that any sector with $F_i = F_j = -1$, and distance between $i$ and $j$ greater than $N - 1$ must be null.

Only in the sector with all $f_i = +1$, the sector decomposition does not break translational invariance. The sector with all $f_i = -1$ is empty for $M > N$. In this case, we can break the sector further into $M$ one-dimensional subsectors using the eigenvalues of $\mathbb{T}$.

It is straight forward to work out the sector decomposition and the spectrum, for small values of $N$ explicitly.

### *3.1. The case* $N = 1$

For $N = 1$, the graph is shown in Fig. 2. There is one sector with all $f_i = +1$. This must have $\psi(m, 0) = \psi(m, 1)$. Thus there are $M$ linearly independent variables in this sector, and dimensionality of this sector is $M$. These are further divided into one-dimensional subsectors using the eigenvalues of $\mathbb{T}$.

Then, there are $M$ sectors with exactly one $f_i = -1$, and rest positive. In the sector with $f_1 = -1$, we must have $\psi(1, 0) = -\psi(1, 1)$, and $\psi(j, 0) = \psi(j, 1) = 0$ for $j \neq 1$. So, each of these sectors is one-dimensional. The corresponding eigenfunction is fully localized within one level, and is shown in Fig. 2. All sectors with more than one $f_i$ negative are empty. The total dimension of the space is thus $= 2M$, equal to total number of modes in the problem.

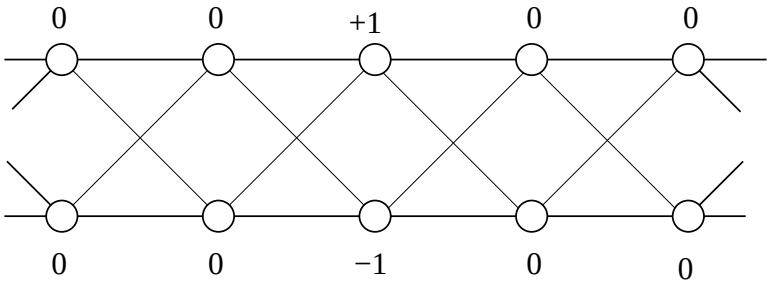


**Figure 2.** Figure shows a part of the $N = 1$ network, with a localized eigenfunction corresponding to exactly one of the $\mathbb{F}_i$'s negative.

### *3.2. The case $N = 2$*

For $N = 2$, the sector with all $f_i$'s positive has dimension $M$. It is divided into $M$ one-dimensional subsectors using the eigenvalues of $\mathbb{T}$.

There are $M$ sectors where only one $f$ is negative. Consider the sector with $f_1 = -1$, and all other $f$'s positive. In this case, let the eigenfunction be $\psi(m, r)$. We have $\mathbb{F}_1\psi(m, r) = -\psi(m, r)$. If $m$ is not equal to 0 or 1, we have $\mathbb{F}_1(m, r) = (m, r)$. This implies that $\psi(m, r)$ must be zero, for $m \neq 0, 1$. Hence the only non-zero elements are $\psi(0, 0), \psi(0, 1), \psi(0, 2)$, $\psi(0, 3)$ and $\psi(1, 0), \psi(1, 1), \psi(1, 2)$, $\psi(1, 3)$ . Using the symmetries of $\mathbb{F}_M$ and $\mathbb{F}_2$ we also have $\psi(0, 0) = \psi(0, 2) = -\psi(0, 1) = -\psi(0, 3)$ and $\psi(1, 0) = \psi(1, 1) = -\psi(1, 2) = -\psi(1, 3)$. Thus there are only two independent elements and hence this sector is of dimension 2.

The number of sectors with exactly one $f_{i,\alpha}$ negative is clearly $M$. Each of these is two dimensional.

Now consider the sectors with exactly two of the $f$'s negative. Say, we have $f_i = f_j = -1$, and the rest of the $f$ are positive. Clearly, we can assume $i \neq j$. Using the result proved above, if $|i - j| > 1$, then the sector is empty.

Hence, the non-trivial cases with two of the f's negative are only with $f_i = f_{i+1} = -1$, for some $i$. Clearly, there are $M$ choices of $i$. Say $i = 1$. then the corresponding eigenfunctions can be non-zero only for sites in the level $m = 1$. The values of $\psi(m = 1, r)$ for different $r$ can be related to each other using the the symmetries of $\mathbb{F}_1$ and $\mathbb{F}_2$. Thus this sector is one-dimensional.

If at least three of the $f$'s are negative, then one can find a pair of values $f_i = f_j = -1$ , with $|i - j| > 1$, and the sector must be empty.

Thus, we find that for $N = 2$, the only non-empty sectors are those with total number of negative eigenvalues of $\mathbb{F}_i$ at most 2. The total number of sectors $M + 2M + M = 4M$, which is equal to the number of different vertices in the graph.

### *3.3. Higher N*

It is straightforward to extend the calculation of sector decomposition for higher $N$. We will assume $M \gg N$, for convenience.

Given all the eigenvalues $\{f_i\}$, we can relate the value of the eigenfunction $\psi(m,r)$ for any $r$ to the value at the vertex $(m,0)$ by applying the operators $\mathbb{F}_i$. Clearly, it differs from $\psi(m,0)$ by at most a sign, and $\psi(m,r) = \pm\phi_m$. The dimension of the sector is the number of nonzero $\phi_m$'s that are allowed.

Consider first the sector with all $f_i = +1$. The corresponding eigenfunction $\psi(m,r)$ must be of the form $\psi(m,r) = g_m$, independent of $r$. There are $M$ possible choices of $m$. Hence this sector is $M$-dimensional. In this sector, $\mathbb{T}$ is a good symmetry, and we can use the the eigenvalues of $\mathbb{T} = t_j = \exp(i\frac{2\pi j}{M})$, with $j = 1$ to $M$, to further divide this sector into $M$ one-dimensional sectors.

Now consider the sectors with exactly one $f_i$ negative. Say $f_1 = -1$. Then the only nonzero values of $\phi_m$ are for $m = 1, 0, -1, -2, \ldots, 2-N$. This sector is of dimension $N$. There are $M$ such sectors.

For more than one negative $f_i$'s, if the largest distance between two negative $f$'s is $k$, then $k \leq N-1$. The number of non-zero $g_m$'s can be only $N-k$. Hence this is the dimension of each such sector. There are $k-1$ $f$'s between the outermost two negative $f$'s in this band, which can be $\pm 1$, and so the number of sectors with a given position of leftmost and rightmost negative $f$'s is $2^{k-1}$. also the left most negative $f$ has $M$ possible positions. Hence the number of sectors with a given value of $k$ is $M2^{k-1}$, and the dimension of each sector is $N-k$.

As a check, this gives the dimension of the full Hilbert space of eigenfunctions as

$$M + MN + M\sum_{k=1}^{N-1}(N-k)2^{k-1} = M2^N \tag{14}$$

## 4. Spectrum of the spring Hamiltonian

The problem of diagonalization of $\mathbb{K}$ is considerably simplified using its block-diagonalization induced by the symmetries $\{\mathbb{F}_i, \mathbb{T}\}$. It is instructive to consider the cases of small $N$ first.

### *4.1. The spectrum for N=1*

In this case, $\mathbb{K}$ is fully diagonalized using the symmetries alone. There are $2M$ eigenstates, and the $2M$ nontrivial sectors are all one-dimensional. We can thus write down the eigenvectors by use of symmetries alone, and then determine their eigenvalues.

There are $M$ states in the sector with all $f_i = +1$. These are further labeled by eigenvalues of $\mathbb{T}$. The vector corresponding to the eigenvalue $e^{ik}$ of $\mathbb{T}$ is given by

$$\psi(m,0) = \psi(m,1) = e^{ikm}, \tag{15}$$

with $k = (2\pi j)/M$, and $j$ taking integer values 0 to $M-1$. The corresponding eigenvalue of $\mathbb{K}$ is easily seen to be

$$\omega^2 = 4 - 4\cos k \tag{16}$$

Other non-null sectors have exactly one $f_i = -1$, and rest positive. The eigenvector corresponding to the $f_i = -1$ is easily written down

$$\begin{aligned} \psi(i,0) &= -\psi(i,1) = 1; \\ \psi(m,r) &= 0, \text{ for } m \neq i. \end{aligned} \tag{17}$$

This mode is fully localized on the sites in the level $i$. The corresponding eigenvalue of $\mathbb{K}$ is seen to be 4.

In the limit $M \to \infty$, we see that the localized modes are degenerate, and give a $\delta$-function peak in the spectrum $D(\omega^2)$ at $\omega^2 = 4$. The total weight of this peak is 1/2. The remaining modes are seen to be extended, and give rise to a continuous spectrum. Adding the delta-function peak to the continuous part given by (16), the total density of states is given by

$$D(\omega^2) = \frac{1}{2}\delta(\omega^2 - 4) + \frac{1}{4\pi[\omega^2(8-\omega^2)]^{1/2}}; \text{ for } N = 1 \tag{18}$$

*4.2. The spectrum for $N = 2$*

For the sector with $f_i = +1$ for all $i$, again we have $M$ modes. These are given by

$$\psi(m,r) = \exp(ikm), \text{ independent of } r, \tag{19}$$

and the corresponding $k$ values , and the eigenvalue of $\mathbb{K}$ is as given by (16).

Now consider the sector with only one negative $f$, say $f_i = -1$, and the rest of $f$'s are all positive. This sector is two-dimensional. From the symmetries alone, we see that the corresponding eigenfunction $\psi(m,r)$ is non-zero only for $m = i$ and $m = i-1$. Using $f_{i-1} = f_{i+1} = +1$, we get

$$\begin{aligned} &\psi(i,00) = \psi(i,01) = -\psi(i,10) = -\psi(i,11) = g(i) \\ &\psi(i-1,00) = -\psi(i-1,01) = \psi(i-1,10) = -\psi(i-1,11) = g(i-1) \end{aligned} \tag{20}$$

The eigenvalue equation relating $g(M)$ and $g(1)$ is easily seen to simplify to

$$\begin{aligned} -\omega^2 g(i-1) &= -4g(i-1) + 2g(i), \\ -\omega^2 g(i) &= -4g(i) + 2g(i-1) \end{aligned} \tag{21}$$

This is a coupled set of two linear equations. It is easily seen that allowed solutions are $g(i) = \pm g(i-1)$, and the corresponding values of $\omega^2$ are 2 and 6. The eigenmode corresponding to $\omega^2 = 2$ is shown in Fig. 3.

For the sector with $f_i = f_{i+1} = -1$, there is only one eigenvector, which is fully localized in the level $m = i$. The corresponding eigenvalue is 4, and the eigenvector is given by

$$\psi(i,00) = -\psi(i,01) = -\psi(i,10) = \psi(i,11) \tag{22}$$

In the limit $M \to \infty$, the density of states is seen to be

$$D(\omega^2) = \frac{1}{4}[\delta(\omega^2-2)+\delta(\omega^2-4)+\delta(\omega^2-6)] \quad + \frac{1}{8\pi[\omega^2(8-\omega^2)]^{1/2}}; \text{ for } N = 2 \tag{23}$$

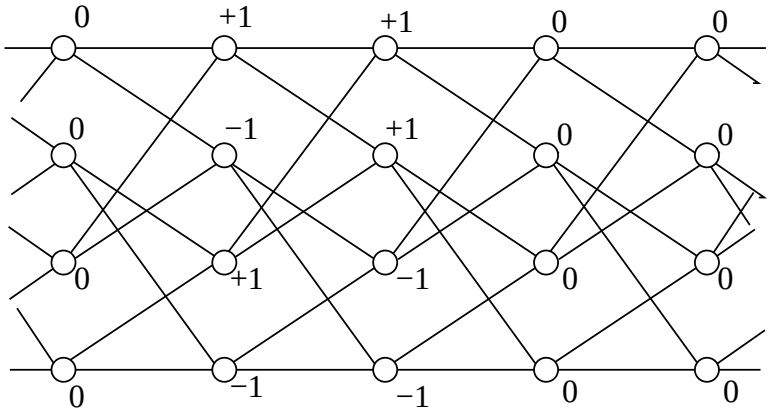


**Figure 3.** Figure shows a part of the $N = 2$ network, with a localized eigenfunction corresponding to exactly one of the $\mathbb{F}_i$'s negative.

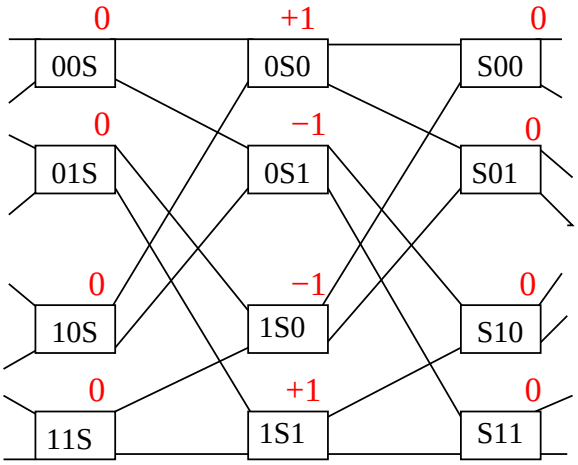


**Figure 4.** An eigenmode in the network with $N > 2$ with eigenvalue 4. The amplitude, shown in red, is nonzero only at four sites

### *4.3. The spectrum for higher N*

The treatment given above is easily extended to higher $N$. Firstly, given the set of values $\{f_i\}$, the corresponding eigenfunction $\psi(m, r)$ is of the form $\psi(m, r) = \pm g(m)$, and the sign is easily determined in terms of $\{f_i\}$ and the binary representation of $r$. We choose the convention that the sign is positive for $r = 0$.

A nontrivial eigenfunction is possible, only if the indices $i$ corresponding to negative $f$'s occur in a narrow band of width at most $N$. If the width of the band is $\ell$, the number of non-zero $g(m)$ is $u = N - \ell + 1$.

Consider, for simplicity, the sector in which the non-zero $g(m)$'s are for $m \in [i, i + u - 1]$. It is easily seen that the eigenvalue equation (3) in terms of the function $g(m)$ becomes

$$-\omega^2 g(j) = 2g(j + 1) + 2g(j - 1) - 4g(j), \text{ for } j \in [i, i + u - 1], \qquad (24)$$

$$= 0, \text{ otherwise.}$$

This equation is easily solved. The eigenvalues are

$$\omega^2 = 4 - 4\cos(\frac{\pi r}{u+1}), \text{ for } j = 1 \text{ to } u. \tag{25}$$

and the corresponding eigenfunction is

$$g(m) = A\sin[\frac{\pi(m-i+1)}{u+1}], \text{ for } m \in [i, i+u-1], \tag{26}$$
$$= 0, \quad \text{otherwise.}$$

As a simple example, if $f_1 = f_N = -1$, and all $f_j = +1$, for $j > N$, then the eigenfunction $g(m)$ is non-zero only for $m = 1$, and the eigenvalue is $\omega^2 = 4$. There are $2^{N-2}$ such sectors, corresponding to different possible choices of $f_i$ for $i \in [2, N-1]$. All these different modes are localized in the level $m = 1$. It is then possible to construct linear combinations of these modes, which are still eigenfunctions of $\mathbb{K}$, but no longer eigenfunctions of $\mathbb{F}_i$. We find that these can be chosen so that each eigenfunction is highly localized, and nonzero only at four sites.

This mode is shown in Fig. 4. Let $S$ be a binary string of length $N-2$. We consider the eigenfunction which is nonzero only at the four sites $(1, 0S0), (1, 1S0), (1, 0S1)$ and $(1, 1S1)$, defined by

$$\psi(1, 0S0) = -\psi(1, 0S1) = -\psi(1, 1S0) = \psi(1, 1S1) \tag{27}$$

Clearly, this is an eigenfunction of $\mathbb{K}$ with eigenvalue 4.

A similar construction works for all other localized eigenmodes. An eigenmode which extends over $u$ levels will have a degeneracy of $2^{N-u-1}$, and one can choose a linear combination that localizes the mode maximally. Such a mode will have non-zero amplitude only at $2^{u+1}$ sites in each of the $u$ layers.

It is easily seen that the spectral density function for a general $N$ is:

$$D(\omega^2) = \sum_{u=1}^{N-1} \frac{1}{2^{u+1}} \sum_{s=1}^{u} \delta(\omega^2 - (4 - 4\cos(\frac{\pi s}{u+1})))$$
$$+\frac{1}{2^N} \sum_{s=1}^{N} \delta(\omega^2 - (4 - 4\cos(\frac{\pi s}{N+1})))$$
$$+\frac{2^{-N}}{\pi} \sqrt{\frac{1}{\omega^2(8-\omega^2)}} \tag{28}$$

The spectral density for large $N$ is shown in Fig. 5.

## 5. Returns to the origin

From the expression for $D(\omega^2)$, it is straightforward to determine how the probability of the random walker being found at the starting site at time $t$ varies with $t$ for large

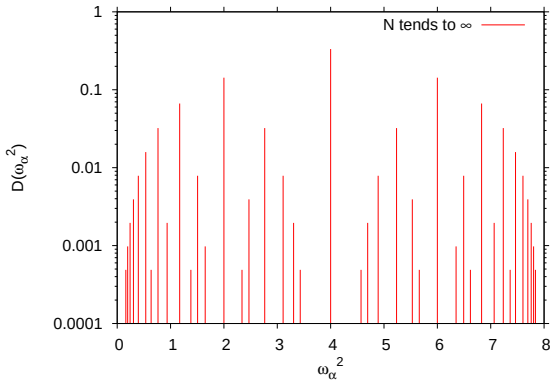


**Figure 5.** The spectral density in the limit of large $N$. Only the peaks with weight greater than $10^{-4}$ have been shown.

$t$. Using (5), and the fact that the return probability $\mathrm{Prob}(\mathbf{v}_0, t; \mathbf{v}_0, 0)$ does not depend on $\mathbf{v}_0$, it may be written as

$$\mathrm{Prob}(\mathbf{v}_0, t; \mathbf{v}_0, 0) = \int_0^\infty d\omega^2 D(\omega^2) \exp(-\omega^2 t) \tag{29}$$

For finite $N$, with $M$ large, the network is effectively one-dimensional. The integral above is dominated by by the continuous spectral density for small $\omega^2$. As $D(\omega^2) \sim 2^{-N}/\omega$ for small $\omega$, it is seen that $\mathrm{Prob}(\mathbf{v}_0, t; \mathbf{v}_0, 0)$ varies as $2^{-N} t^{-1/2}$ for large $t$.

For $N$ large, the spectrum is purely discrete. In (25), For a given value of the localization width $u$, the mode with lowest frequency corresponds to $r = 1$, and then for large $u$, the frequency may be approximated by $\omega^2 \approx 2\pi^2 u^{-2}$. Then, we can write

$$\mathrm{Prob}(\mathbf{v}_0, t; \mathbf{v}_0, 0) \sim \sum_u 2^{-u} \exp(-\frac{2\pi^2 t}{u^2}) \tag{30}$$

For large $t$, the leading behavior of this summation can be determined by steepest descent. The term corresponding to $u = u^* = \left[\frac{4\pi^2 t}{\ln 2}\right]^{1/3}$, and we get

$$\mathrm{Prob}(\mathbf{v}_0, t; \mathbf{v}_0, 0) \sim \exp\left[-\frac{3}{2}(4\pi^2)^{1/3} t^{1/3}\right], \text{ for large } t. \tag{31}$$

We note that this is qualitatively different from the behavior for the Bethe lattice, where this probability decays exponentially with time.

The function $R(s)$ in the limit $N \to \infty$, using (7) and (28), is given by the formula

$$R(s) = \sum_{u=1}^{\infty} \sum_{r=1}^{u} \frac{1}{2^{u+1}(s + 4 - 4\cos\frac{\pi r}{u+1})} \tag{32}$$

We note that $R(s)$ can be written as

$$T_r(s) = (1/2) Q_r(s) - \frac{(s+4)}{s(s+8)} \tag{33}$$

where

$$Q(s) = \sum_{u=1}^{\infty} \sum_{r=1}^{2u+1} \frac{1}{2^{u+1}(s + 4 - 4\cos\frac{\pi r}{u+1})} \tag{34}$$

The summation over $r$ in the above equation can be done explicitly using the identity

$$\sum_{r=0}^{2u+1} \frac{1}{1 - x\cos\frac{\pi r}{u+1}} = \frac{2(u+1)}{\sqrt{1-x^2}} \left[\frac{(\alpha_+^{u+1} + \alpha_-^{u+1})}{(\alpha_+^{u+1} - \alpha_-^{u+1})}\right] \tag{35}$$

where $\alpha_\pm = (1 \pm \sqrt{1-x^2})/x$. We see that $Q(s)$ is a rational function of terms purely in integer powers of $s$ and rational coefficients and radicals in $s$.

## Acknowledgments

ACB thanks Kishore Vaigyanik Protsahan Yojana for financial support, and TIFR for hospitality during his visit. DD would like to acknowledge the financial support from the Department of Science and Technology, Government of India, through a JC Bose Fellowship.

## References

[1] N. Ikeno, *A limit on crosspoint numbers*, IRE Trans. Inform. Theory, Volume IT-**5**, Issue 2, Pages 187 – 196, Summer 1959.

[2] A. Feiner and J.G. Kappel, *A method of deriving efficient switching network configurations*, Proc. Nat. Electron. Conf. **26**, 1970, 818 – 823.

[3] C.F. Baillie, D.A. Johnston, 1, J.P. Kownacki, Ising spins on thin graphs, Nucl. Phys. B432, pp 551 – 570 1994, .

[4] D. Dhar, R. Rajesh, and J.F. Stilck, *Hard rigid rods on a Bethe-like lattice*, Phys. Rev. E **84**, 011140 2011.

[5] K. Takagi, *Design of multi-stage link systems by means of optimal channel graphs*, Electron. Comm. Japan **51A**, 1968 37 – 46.

[6] F.R.K. Chung and F.K. Hwang, *The Connection Pattern of Two Binary Trees*, SIAM Journal on Algebraic and Discrete Methods, Volume **1**, Pages 322 – 335, 1980.

[7] F.K. Hwang, *Computing the Blocking Probabilities of Spiderweb Channel Graphs*, IEEE Transactions on Communications, Volume COM-**30**, No. 8, Pages 1916 – 1918, August, 1982.

[8] N. Pippenger, *The blocking probability of spider-web networks*, Random Structure and Algorithms, Volume **2**, Issue 2, Pages 121 – 149, Summer 1991.

[9] N. Pippenger, *The asymptotic optimality of spider-web networks*, Discrete Applied Mathematics - Special double volume: interconnection network, Volume **37 – 38**, July15, 1992.

[10] N. Pippenger, *The Linking Probability of Deep Spider-Web Networks*, SIAM Journal on Discrete Mathematics, Volume **20**, Issue 1, 2006.