# Exact Ground State of several $N$-body Problems with an $N$-body Potential

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**Abstract**

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# 1 Introduction

Over the years, the exact solution of $N$-body problems has attracted considerable attention because of its possible relevance in statistical mechanics as well as in atomic, nuclear and gravitational many-body problems. Whereas exact solution of several $N$-body problems in one dimension is known by now [1,2], a class of exact solutions including the bosonic ground state have been obtained for several $N$-body problems in higher dimensions when they are interacting via a harmonic oscillator potential [3,4,5,6,7,8]. Clearly it is of considerable interest to discover other exactly solvable $N$-body problems in one as well as in higher dimensions.

The purpose of this paper is to show that a class of exact solutions including the bosonic ground state of all these $N$-body problems in higher dimensions can also be obtained in case they are interacting via an $N$-body potential of the form

$$V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = -\frac{e^2}{\sqrt{\sum_i \mathbf{r}_i^2}} \tag{1}$$

In this context, I may add that recently I have already obtained a class of exact solutions of the $N$-anyon problem (in two dimensions) in case they are interacting via this $N$-body potential [9]. Further, last year, I also obtained the complete bound state spectrum of the $N$-particle problem in one dimension in case they are interacting via a variant of the above potential as given by [10,11,12]

$$V(x_1, x_2, ..., x_N) = -\frac{e^2}{\sqrt{\sum_{i,j}(x_i - x_j)^2}}\,. \tag{2}$$

Subsequently, Gurappa *et al.* [13] showed that the complete bound state spectrum can also be obtained in case an $N$-body potential of the form $\beta^2/\sum_{i<j}(x_i - x_j)^2$ is added either to the oscillator or to the $N$-body potential (2).

Based on all these results, I conjecture that whenever an $N$-body problem is solvable in case these $N$ particles are interacting via an external one body ( or pairwise) oscillator potential, the same $N$-body problem is also solvable in case they interact via the $N$-body potential as given by eq. (1) or by

$$V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = -\frac{e^2}{\sqrt{\sum_{i<j}(\mathbf{r}_i - \mathbf{r}_j)^2}}\,. \tag{3}$$

I further conjecture that in either case, one can also add an $N$-body potential of the form

$$V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = -\frac{\delta^2}{\sum_{i<j}(\mathbf{r}_i - \mathbf{r}_j)^2} \tag{4}$$

or its variant

$$V(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = -\frac{\delta^2}{\sum_i \mathbf{r}_i^2} \tag{5}$$

and the problem is still solvable except that the degeneracy in the spectrum is now much reduced. Clearly, it is necessary to examine the known solvable $N$-body problems with the oscillator potential and check if these conjectures are valid or not.This is precisely what I propose to do in this paper.

The plan of the paper is as follows. In Sec.II, I discuss the $N$-body Calogero-Marchioro [3] model in $D$-dimensions [4] and show that *a la* the oscillator case, even for the $N$-body potential as given by eq. (3), one can obtain some exact eigenstates including the (bosonic) ground state. I also consider the Sutherland variant of the problem and as in the oscillator case, I obtain the exact (bosonic) ground state when the $N$-bodies are interacting via the $N$-body potential as given by eq. (1). In Sec.III, I discuss the two-dimensional model of Murthy *et al.* [5] in which they have obtained some exact eigenstates including the (bosonic) ground state all of which show novel correlations. I show that similar exact solutions with novel correlations can also be obtained in case one replaces the one-body oscillator potential by the potential as given in eq. (1). I also discuss the two-body problem in great detail and show that as in [6], in this case too it is completely solvable. In Sec.IV, I consider the $D$-dimensional generalization [7] of Murthy *et al's* model [5,6] and show that exact solutions including the (bosonic) ground state can again be obtained in case they are interacting via the $N$-body potential (1). In Sec.V, I consider a Calogero type model in $D$-dimensions [8] which has only two-body interactions and show that one can obtain some exact eigenstates including the (bosonic) ground state in case the $N$-bodies are interacting via the $N$-body potential (1). Finally, in Sec.VI, I obtain the entire discrete bosonic spectrum in case the $N$-particles are interacting in

$D$-dimensions purely via the $N$-body potential as given by (3). Further, in all the above cases I show that the discrete spectrum can still be obtained even if one adds the $N$-body potential as given by eq. (5) (or (4)) to the oscillator or our $N$-body potential except that the degeneracy in the discrete spectrum is now much reduced.

# 2 Calogero-Marchioro Model With $N$-body Potential

Long time ago, in an effort to generalize the original Calogero model [2] to dimensions higher than one, Calogero and Marchioro [3] considered a model in three dimensions in which the N-particles are interacting via the two-body and the three-body inverse square interactions as well as by the pairwise harmonic oscillator potential. They were able to obtain some exact eigenstates including the bosonic ground state of the system. Recently, we [4] have been able to generalize their work to arbitrary space dimensions. In particular we considered the following $N$-body Hamiltonian in $D$-dimensions

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\nabla_i^2 + \frac{g\hbar^2}{2m}\sum_{i<j}^{N}\frac{1}{\mathbf{r}_{ij}^2} + \frac{G\hbar^2}{2m}\sum_{i<j}\frac{\mathbf{r}_{ki}\cdot\mathbf{r}_{kj}}{\mathbf{r}_{ki}^2\mathbf{r}_{kj}^2} + \frac{m\omega^2}{4}\sum_{i<j}^{N}\mathbf{r}_{ij}^2 \tag{6}$$

and showed that some eigenstates including the bosonic ground state for the system are given by

$$\psi_{n_r} = (\prod_{i<j}\mathbf{r}_{ij}^2)^{\Lambda_D/2} exp(-\frac{1}{2}\sqrt{\frac{1}{2N}}\sum_{i<j}\mathbf{r}_{ij}^2)L_{n_r}^{\Gamma_D}(\sqrt{\frac{1}{2N}}\sum_{i<j}\mathbf{r}_{ij}^2) \tag{7}$$

with energy

$$E_{n_r} = \sqrt{\frac{N}{2}}(2n_r + \Gamma_D + 1)\,,\ n_r = 0, 1, 2, \ldots . \tag{8}$$

Here $\mathbf{r}_i$ is the $D$-dimensional position vector of the $i$'th particle and $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ denotes the relative separation of the $i$'th and $j$'th particles while $r_{ij}$ denotes its magnitude. In writing this exact solution, we have scaled all distances $\mathbf{r}_i \to \sqrt{\frac{\hbar}{m\omega}}\mathbf{r}_i$ and the energy is measured in units of $\hbar\omega$. Throughout this paper, whenever we discuss the exact solutions for the oscillator potential, we shall always be working with these scaled coordinates and the energy will always be measured in units of $\hbar\omega$. The parameters $g$ and $G$ characterize the (dimension-less) two-body and three-body coupling constants respectively while $L_a^b$ denotes the Laguerre polynomial. Here $g > -1/2$ to stop the *fall to the origin*. Further, $\Lambda_D$ and $\Gamma_D$ are two parameters determined in terms of the parameters of the Hamiltonian by

$$\Lambda_D \equiv \sqrt{G} = \frac{1}{2}\Big[\sqrt{(D-2)^2 + 4g} - (D-2)\Big] \tag{9}$$

$$\Gamma_D = \frac{1}{2}\Big[D(N-1) - 2 + \Lambda_D N(N-1)\Big]\,. \tag{10}$$

Let us now consider the same many body problem as given by eq. (6) but with the oscillator potential being replaced by the $N$-body potential as given by eq. (3). Throughout this paper, whenever we talk of the exact solutions with the $N$-body potential as given by eqs. (1) or (3), we shall rescale all distances $\mathbf{r}_i \to \hbar^2\mathbf{r}_i/me^2$ and measure energy in units of $me^4/\hbar^2$ so that $m, e, \hbar$ are all scaled away.

On substituting the ansatz

$$\psi = (\prod_{i<j} \mathbf{r}_{ij}^2)^{\Lambda_D/2}\phi(\rho) \tag{11}$$

in the Schrödinger equation for the potential (3) one obtains

$$\rho\phi''(\rho) + (2\Gamma_D + 1)\phi'(\rho) + (\frac{2}{\sqrt{N}} - 2\rho \mid E \mid)\phi(\rho) = 0 \tag{12}$$

where $\Gamma_D$ is as given by eq. (10) while

$$\rho^2 = \frac{1}{N}\sum_{i<j}\mathbf{r}_{ij}^2\,. \tag{13}$$

On further substituting

$$\phi(\rho) = \ exp(-\sqrt{2 \mid E \mid}\rho)\chi(\rho) \tag{14}$$

it is easily shown that $\chi(\rho)$ satisfies the equation

$$\rho\chi''(\rho) + (2\Gamma_D + 1 - 2\sqrt{2 \mid E \mid}\rho)\chi'(\rho) + [\frac{2}{\sqrt{N}} - (2\Gamma_D + 1)\sqrt{2 \mid E \mid}]\chi(\rho) = 0 \tag{15}$$

whose solution is a Laguerre polynomial

$$\chi(\rho) = L_{n_r}^{2\Gamma_D}(2\sqrt{2 \mid E \mid}\rho) \tag{16}$$

with the corresponding energy eigenvalues being (in units of $me^4/\hbar^2$)

$$E_{n_r} = -\frac{1}{2N(n_r + \Gamma_D + \frac{1}{2})^2} \, . \tag{17}$$

Several comments are in order at this stage.

1. For $D = 1$ this expression agrees with the one derived by us earlier [10].

2. For $n_r = 0$, the eigenfunction

$$\psi_0 = (\prod_{i<j} \mathbf{r}_{ij}^2)^{\Lambda_D/2} \; exp(-\sqrt{2 \mid E_0 \mid}\rho) \tag{18}$$

   has no nodes besides those coming from the singular centrifugal potential which forces the eigenfunction to vanish whenever the coordinates of any two particles coincide. Thus $\psi_0$ corresponds to the ground state of the system with the corresponding ground state energy being as given by eq. (17) with $n_r = 0$. For $n_r > 0$, we have radial excitations over the ground state.

3. As $g \to 0$, we see from eq. (9) that also $G \to 0$ and the wave function (18) becomes the ground state eigenfunction of the hyperspherical "Coulomb problem" in $D$-dimensions without centrifugal barrier and with Bose statistics. Thus the situation is different from the one dimensional problem [10] where as $g \to 0$, the eigenfunction is the ground state of the "Coulomb problem" but with Fermi statistics. We shall in fact see that in all the higher dimensional many body problems ($D > 1$), unlike the one dimensional case, as the coupling is switched off, the eigenfunction corresponds to that of Bose statistics. It is not clear whether this difference in statistics between the one and the higher dimensions has any deeper physical significance.

4. The $N$-body problem is still solvable if apart from our $N$-body potential (3) we also add the potential (4) to the Hamiltonian (6). In this case,

on substituting the ansatz (11) in the Schrödinger equation for the combined potential yields

$$\phi''(\rho) + \frac{(2\Gamma_D + 1)}{\rho}\phi'(\rho) + (\frac{2}{\rho\sqrt{N}} - \frac{\delta^2}{N\rho^2} - 2 \mid E \mid)\phi(\rho) = 0 \,. \quad (19)$$

On following the steps as given by eqs. (13) to (17) it is easily shown that the exact eigenstates are given by (in units of $me^4/\hbar^2$)

$$E_{n_r} = -\frac{1}{2N(n_r + \gamma + \frac{1}{2})^2} \quad (20)$$

$$\phi(\rho) = \rho^{(\gamma - \Gamma_D)} exp(-\sqrt{2 \mid E \mid}\rho) L_{n_r}^{2\gamma}(2\sqrt{2 \mid E \mid}\rho) \quad (21)$$

where

$$\gamma = \sqrt{\Gamma_D^2 + \delta^2/N} \,. \quad (22)$$

5. In the same way, the $N$-body problem with the oscillator potential as given by eq. (6) is also solvable in case we also add the $N$-body potential (4) to it. In particular, it is easily shown that the corresponding exact eigenstates are

$$E_{n_r} = \sqrt{\frac{N}{2}}[2n_r + 1 + \gamma] \quad (23)$$

$$\psi_{n_r} = (\prod_{i<j} \mathbf{r}_{ij}^2)^{\Lambda_D/2} exp(-\frac{1}{2}\sqrt{\frac{1}{2N}}\sum_{i<j}\mathbf{r}_{ij}^2)\rho^{(\gamma - \Gamma_D)} L_{n_r}^{\gamma}(\sqrt{\frac{1}{2N}}\sum_{i<j}\mathbf{r}_{ij}^2) \,. \quad (24)$$

It may be added here that following Calogero and Marchioro [3], we can also obtain a subset of the (spin-less) fermionic eigenfunctions in the special case of $N = 3$ and 4 when these particles are interacting via the $N$-body potential as given by eq. (3). In particular, it is easily shown that for $N = 3$, a set of completely antisymmetric eigenstates in three space dimensions are

$$\psi_{n_r} = (\mathbf{r}_{12} \times \mathbf{r}_{23})(\prod_{i<j} \mathbf{r}_{ij}^2)^{f/2} exp(-\sqrt{2 \mid E \mid}\rho) L_{n_r}^{2F}(2\sqrt{2 \mid E \mid}\rho) \quad (25)$$

$$E_{n_r} = -\frac{1}{6(n_r + F + \frac{1}{2})^2} \quad (26)$$

where

$$f = \sqrt{G} = \frac{3}{2}\Big[\sqrt{1+\frac{4g}{9}}-1\Big],\ F = 3f+3\,. \qquad (27)$$

Generalization to $D$ dimensions is straight forward.

On the other hand for $N = 4$, a set of completely anti-symmetrical eigenstates in three space dimensions are given by

$$\psi_{n_r} = [(\mathbf{r}_{12}\times\mathbf{r}_{23})\cdot\mathbf{r}_{34}](\prod_{i<j}\mathbf{r}_{ij}^2)^{f/2} exp(-\sqrt{2\mid E\mid}\rho) L_{n_r}^{2F'}(2\sqrt{2\mid E\mid}\rho) \qquad (28)$$

$$E_{n_r} = -\frac{1}{8(n_r+F'+\frac{1}{2})^2} \qquad (29)$$

where f is as given by eq. (27) while $F' = 6f+5$. Again, generalization to $D$ dimensions is straight forward.

I might add here that a la the oscillator case [4], we can also obtain the exact ground state of the corresponding Sutherland variant of the problem. In particular, consider the Hamiltonian as given by eq. (6) but with the oscillator potential being replaced by the potential as given by eq. (1). It is easily shown that the exact ground state of the system is given by

$$\psi_0 = (\prod_{i<j}\mathbf{r}_{ij}^2)^{\Lambda_D/2}\ exp(-\sqrt{2\mid E\mid\sum_i\mathbf{r}_i^2}) \qquad (30)$$

where

$$E_0 = -\frac{2}{(N-1)^2[1+N\Lambda_D]^2}\,. \qquad (31)$$

One of the unsolved problem is whether one can map this problem (at least in some specific dimension $D$) to some random matrix problem and obtain exact results for the corresponding many-body theory. In this context it may be noted that for $g = 2$, the corresponding oscillator problem (see eq. (6)) in two dimensions has been shown to be connected to the random matrix problem for the complex matrices [4].

# 3 Novel Correlations With an $N$-body Potential

In a recent paper, Murthy *et al.* [5] have proposed a model in two dimensions with two-body and three-body interactions. They were able to obtain the bosonic ground state and a class of excited states of the system exactly by adding an external (one body) harmonic oscillator potential. The model was constructed in such a way that the solutions have a novel correlation of the from

$$X_{ij} = x_i y_j - x_j y_i \tag{32}$$

built into them. Note that unlike the Jastrow-Laughlin form, $X_{ij}$ is a pseudo-scalar. However, unlike the Laughlin type of correlation, this correlation is not translationally invariant unless the radial degrees of freedom are frozen. I now show that the bosonic ground state and the radial excitations over it can also be obtained in case the oscillator potential is replaced by the $N$-body potential as given by eq. (1). Further, I also obtain the complete solution of the two-body problem. Notice that the two-body problem is quite nontrivial since the center of mass motion cannot be separated.

Following Murthy *et al.* [5], we start with the $N$-particle Hamiltonian (in the scaled variables) as given by

$$2H = -\sum_{i=1}^{N} \nabla_i^2 + g_1 \sum_{i \neq j}^{N} \frac{\mathbf{r}_j^2}{X_{ij}^2} + g_2 \sum_{i \neq j \neq k}^{N} \frac{\mathbf{r}_j \cdot \mathbf{r}_k}{X_{ij} X_{ik}} - \frac{2}{\sqrt{\sum \mathbf{r}_i^2}} \tag{33}$$

where $X_{ij}$ is as given by eq. (32) while $g_1$ and $g_2$ are dimension-less coupling constants of the two-body and the three-body interactions respectively.

It is easily shown that

$$\psi_0(x_i, y_i) = \prod_{i<j}^{N} \mid X_{ij} \mid^g exp(-\sqrt{2 \mid E_0 \mid \sum_i \mathbf{r}_i^2}) \tag{34}$$

is the exact ground state of the system with the corresponding ground state energy being (in units of $me^4/\hbar^2$)

$$E_0 = -\frac{1}{2[gN(N-1) + N - \frac{1}{2}]^2} \tag{35}$$

provided $g_1$ and $g_2$ are related to $g$ by

$$g_1 = g(g-1), \;\; g_2 = g^2 \,. \tag{36}$$

It may be noted that this $\psi_0$ is regular for $g \geq 0$ which implies that $g_1 \geq -1/4, g_2 \geq 0$.

There is a neat way of proving that we have indeed obtained the ground state by using the method of operators [14]. In particular, let us define the operators

$$
\begin{aligned}
Q_{x_i} &= p_{x_i} + ig\sum_{j\neq i}^{N}\frac{y_j}{X_{ij}} - \frac{i\alpha x_i}{\sqrt{\sum_j \mathbf{r}_j^2}} \\
Q_{y_i} &= p_{y_i} - ig\sum_{j\neq i}^{N}\frac{x_j}{X_{ij}} - \frac{i\alpha y_i}{\sqrt{\sum_j \mathbf{r}_j^2}}
\end{aligned} \tag{37}
$$

and their Hermitian conjugates $Q_{x_i}^{+}$ and $Q_{y_i}^{+}$. It is easy to see that the Q's annihilate the ground state (34) i.e. $Q_{x_i}\psi_0 = Q_{y_i}\psi_0 = 0$. Further, the Hamiltonian (33) can now be recasted in terms of these operators as

$$
H = \frac{1}{2}\sum_i \Big[Q_{x_i}^{+}Q_{x_i} + Q_{y_i}^{+}Q_{y_i}\Big] + E_0 \tag{38}
$$

where $E_0(= -\alpha^2/2)$ is as given by eq. (35). Since the operator on the right hand side is positive definite and annihilates the ground state wave function (34) hence $E_0$ must be the (bosonic) ground state energy of the system. Note that as in the other two- ( and multi-) dimensional problems, we are unable to obtain the fermionic ground state of the system analytically. In fact, we do not know of any $N$-body problem in two or more space dimensions whose fermionic ground state has been analytically obtained.

## 3.1 A class of Excited States

We now show that *a la* the oscillator case, even in our case, a class of excited states can be obtained analytically. To that end, we consider the ansatz

$$
\psi(x_i, y_i) = \prod_{i<j}^{N} \mid X_{ij} \mid^g exp(-\alpha\sqrt{\sum_i \mathbf{r}_i^2})\phi(x_i, y_i) \tag{39}
$$

On using eqs. (33) and (39) it is easily shown that $\alpha^2 = -2E$ while $\phi$ satisfies the eigenvalue equation

$$
\Big[-\frac{1}{2}\sum_i \nabla_i^2 + \alpha\sum_{i=1}\frac{\mathbf{r}_i\cdot\nabla_i}{\sqrt{\sum_j \mathbf{r}_j^2}} + \frac{A}{\sqrt{\sum_j \mathbf{r}_j^2}} + g\sum_{i\neq j}(x_j\frac{\partial}{\partial y_i} - y_j\frac{\partial}{\partial x_i})\Big]\phi = 0 \tag{40}
$$

where

$$A = \alpha[gN(N-1) + N - \frac{1}{2}] - 1 \tag{41}$$

The exact solutions of this differential equation are best studied in terms of the complex coordinates $z = x + iy, z^* = x - iy$, and their partial derivatives $\partial \equiv \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\partial^* \equiv \frac{\partial}{\partial z^*} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. In terms of these coordinates the differential equation for $\phi$ take the form

$$\left[-2\sum_i \partial_i\partial_i^* + \alpha\sum_i \frac{z_i\partial_i + z_i^*\partial_i^*}{\sqrt{\sum_j z_j z_j^*}} + \frac{A}{\sqrt{\sum_j z_j z_j^*}} + 2g\sum_{i\neq j}\frac{z_i\partial_i - z_i^*\partial_i^*}{z_i z_j^* - z_j z_i^*}\right]\phi = 0\,. \tag{42}$$

It is worth remarking that $\phi$ is also an eigenstate of the total angular momentum operator i.e. $L\Phi = l\Phi$. We now classify some exact solutions according to their angular momentum.

(a) $l = 0$ **Solutions**

Let us define an auxiliary parameter t by $t = 2\alpha\sqrt{\sum_i z_i z_i^*}$ and let $\phi = \phi(t)$. On using this in eq. (42), the differential equation for $\phi$ reduces to

$$t\phi''(t) + (b - t)\phi'(t) - a = 0 \tag{43}$$

where $b = 2N(N-1)g + 2N - 1$ , and $a = A/\alpha$. The allowed solutions are [15] confluent hypergeometric functions, $\phi(t) = M(a, b, t)$. Thus the polynomial solutions are obtained when $a = -n_r$ with $n_r$ being positive integer. Here, the subscript $r$ denotes radial excitations. The corresponding eigenvalues are

$$E_{n_r} = -\frac{1}{2[n_r + gN(N-1) + N - \frac{1}{2}]^2}\,. \tag{44}$$

Note that all these states have zero angular momentum.

(b) $l > 0$ **Solutions**

Let $t_z = \sum_i z_i^2$ and let $\phi = \phi(t_z)$. All the mixed derivative terms in eq. (42) drop out and we obtain

$$2\alpha t_z\phi'(t_z) + A\phi = 0 \tag{45}$$

whose solution is $\phi(t_z) = t_z^m$ where $m = -A/2\alpha$. Note that $\phi(t_z)$ is also an eigenfunction of the angular momentum operator with the eigenvalue $l = 2m$ . Hence the energy eigenvalues in this case are given by

$$E_l = -\frac{1}{2[gN(N-1) + N + l - \frac{1}{2}]^2}\,. \tag{46}$$

(c) $l < 0$ **Solutions**

Similarly let $t_z = \sum_i (z_i^*)^2$ and $\phi = \phi(t_{z^*})$. In this case $\phi$ satisfies the differential equation

$$2\alpha t_{z^*}\phi'(t_{z^*}) + A\phi = 0 \tag{47}$$

whose solution is $\phi(t_{z^*}) = t_{z^*}^m$ where $m = -A/2\alpha$. In this case $\phi(t_{z^*})$ is an eigenfunction of the angular momentum operator with the eigenvalue $l = -2m < 0$. Thus the energy eigenvalues in this case are given by

$$E_l = -\frac{1}{2[gN(N-1) + N - l - \frac{1}{2}]^2} \,. \tag{48}$$

(d) **Tower of Excited States**

We can now combine solutions in cases (b) and (c) (with nonzero $l$) with the solutions in (a) and obtain an even more general class of excited states. For example, let us define

$$\phi(z_i, z_i^*) = \phi_1(t)\phi_2(t_z) \tag{49}$$

where $\phi_1$ is the solution with $l = 0$, while $\phi_2$ is the solution with $l > 0$. The differential equation for $\phi_1$ is again a confluent hypergeometric equation and the energy eigenvalues are given by

$$E_{n_r,l} = -\frac{1}{2[n_r + gN(N-1) + N + l - \frac{1}{2}]^2} \tag{50}$$

One may repeat the same procedure to obtain exact solutions for a tower of excited states with $l < 0$. Combining all these states, it is then clear that the exact energy eigenvalues may be written in the form

$$E_{n_r,m} = -\frac{1}{2[n_r + gN(N-1) + N + 2 \mid m \mid - \frac{1}{2}]^2} \tag{51}$$

We thus see that for both $l > 0$ and $l < 0$, one has a tower of excited states. Actually, the existence of a tower is a general result applicable to all excited states of which the exact solutions shown above form a subset. In particular, following the arguments given in Bhaduri *et al.* [6], let us separate the coordinates $(x_i, y_i)$ into one radial coordinate $t = 2\alpha\sqrt{\sum_i \mathbf{r}_i^2}$ as above and

$2N-1$ angular coordinates collectively denoted by $\Omega_i$ say. Then eq. (40) can be expressed as

$$t\phi''(t) + [2N(N-1)g + 2N - 1 - t]\phi'(t) - [\frac{A}{\alpha} + \frac{2}{\alpha t}T]\phi = 0 \qquad (52)$$

where $T = D_2 + gD_1$. Here $D_n$ is an $n$'th order differential operator which only acts on functions of the angles $\Omega_i$. On further factorizing

$$\phi(x_i, y_i) = R(t)Y(\Omega_i) \qquad (53)$$

where $Y$ is a generalized spherical harmonics on the (2N-1)-dimensional sphere $S^{2N-1}$ it is easily seen that $Y$ is an eigenfunction of the operator $T$. Let us say that the corresponding eigenvalue is $\lambda$ (of course it must be admitted here that the hard part of the problem is to find $\lambda$). On further writing $R(t) = t^{\mu}\tilde{R}(t)$, it is easily shown that $\tilde{R}(t)$ satisfies a confluent hypergeometric equation

$$t\tilde{R}''(t) + (b - t)\tilde{R}'(t) - a\tilde{R}(t) = 0 \qquad (54)$$

provided

$$\mu = \left([N(N-1)g + N - 1]^2 + 4\lambda\right)^{1/2} - [N(N-1)g + N - 1]\,. \qquad (55)$$

Here $b = 2N(N-1)g + 2N - 1 + 2\mu$ while $a = \mu + A/\alpha$. We thus get normalizable polynomial solutions of degree $n_r$ where $a = -n_r$. Here $n_r = 0, 1, 2...$ denote the number of radial nodes. Hence the energy eigenvalues are given by

$$E_{n_r} = -\frac{1}{2[n_r + gN(N-1) + N - \frac{1}{2} + \mu]^2}\,. \qquad (56)$$

It may be noted that $\mu$ is in general a complicated function of the coupling constant $g$. We thus see that for a given value of $\mu$ (which in general is unknown) there is an infinite tower of energy eigenvalues for which $(-E)^{-1/2}$ is separated by a spacing of one unit. This tower structure is a characteristic of the $N$-body potential as given by eq. (1) or eq. (6) and will be present in any $N$-body problem in $D$-dimensions. Note that a similar tower structure also exists in the case of any $N$-body problem in $D$-dimensions if they are interacting via an oscillator potential except that in that case, for a given value of $\mu$, there is an infinite tower of energy eigenvalues separated by a spacing of two units. *A la* Bhaduri et al. [6], in our case too, the tower structure and the angular momentum are useful in organizing a numerical or analytical study of the energy spectrum. In particular, note that, the radial quantum number $n_r$, and the angular momentum $l$, are integers, and hence they cannot change as the parameter $g$ is varied continuously.

Before ending this discussion it is worth pointing out that the class of excited states can also be obtained if we add the $N$-body potential (5) to our $N$-body potential as given by eq. (1). In particular, on proceeding from eq. (39) and repeating the steps, instead of eq. (42) one now has

$$\Big[-2\sum_i \partial_i\partial_i^* \quad + \quad \alpha\sum_i \frac{z_i\partial_i + z_i^*\partial_i^*}{\sqrt{\sum_j z_j z_j^*}} + 2g\sum_{i\neq j}\frac{z_i\partial_i - z_i^*\partial_i^*}{z_i z_j^* - z_j z_i^*} + \quad \frac{A}{\sqrt{\sum_j z_j z_j^*}} + \frac{\delta^2}{2\sum_j z_j z_j^*}\Big]\phi = 0\,. \tag{57}$$

On following the steps as given above it is then easy to show that the exact energy eigenvalues are now given by

$$E_{n_r,m} = -\frac{1}{2[n_r + \gamma + \frac{1}{2}]^2} \tag{58}$$

where

$$\gamma = \Big([gN(N-1) + N + 2 \mid m \mid -1]^2 + \delta^2\Big)^{1/2}\,. \tag{59}$$

The corresponding energy eigenstates are also easily written down.

It is worth pointing out that if instead we add the $N$-body potential (5) to the oscillator potential, then on following the steps as given in [6], the exact energy eigenvalues can be shown to be

$$E_{n_r,m} = 2n_r + 1 + \gamma\,. \tag{60}$$

## 3.2 The Two-Body Problem: Complete Solution

As in the oscillator case [6], we now explicitly show that the two-body problem is integrable and exactly solvable when the two bodies are interacting via the $N$-body potential as given by eq. (1). As in [6] this is best done by going over to the hyperspherical formalism in two dimensions [16,17]. It may be noted that the two-body problem is quite non-trivial here since the center of mass cannot be separated.

The two body Hamiltonian is given by (see eq. (33))

$$H = -\frac{1}{2}(\nabla_1^2 + \nabla_2^2) - \frac{1}{\sqrt{\mathbf{r}_1^2 + \mathbf{r}_2^2}} + \frac{g_1}{2}\frac{(\mathbf{r}_1^2 + \mathbf{r}_2^2)}{X} \tag{61}$$

where $X = x_1 y_2 - x_2 y_1$. Note that the three-body term is obviously not there in the two-body problem ! The two body problem is best solved in the hyperspherical coordinates. To that end, let us parameterize the coordinates $\mathbf{r}_1, \mathbf{r}_2$ in terms of three angles and one length, $(R, \theta, \phi, \psi)$ as follows:

$$\begin{aligned} x_1 + iy_1 &= R(\cos\theta\cos\phi - i\sin\theta\sin\phi)exp(i\psi) \\ x_2 + iy_2 &= R(\cos\theta\sin\phi + i\sin\theta\cos\phi)exp(i\psi)\,. \end{aligned} \tag{62}$$

For a fixed $R$, these coordinates define a sphere in four dimensions with radius $R(R^2 = r_1^2 + r_2^2)$ and $\theta, \phi$ and $\psi$ are in the interval

$$-\pi/4 \le \theta \le \pi/4\,,\ -\pi/2 \le \phi \le \pi/2\,,\ -\pi \le \psi \le \pi\,. \tag{63}$$

The important point to note is that $X = x_1 y_2 - x_2 y_1 = \frac{R^2}{2}\sin(2\theta)$ depends only on $R$ and $\theta$ and is independent of $\phi$ and $\psi$. As a result the two integrals of motion are the angular momentum operator $L$

$$L = \sum_i (x_i p_{y_i} - y_i p_{x_i}) = -i\frac{\partial}{\partial\psi} \tag{64}$$

and the supersymmetry operator

$$Q = i[x_2\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial y_1} - x_1\frac{\partial}{\partial x_2} - y_1\frac{\partial}{\partial y_2}] = -i\frac{\partial}{\partial\phi}\,. \tag{65}$$

Thus with SUSY, the two body problem as given by eq. (61) is integrable with the four constants of motion being the Hamiltonian $H$, the angular part of $H, L$ and $Q$.

It is easy to check that the bosonic ground state has the quantum numbers $l$ and $q$ of the angular momentum and SUSY operators equal to zero. In this context it is worth noting that the eigenstates of the SUSY operator $Q$ are neither symmetric nor asymmetric, unless the eigenvalue $q = 0$. After finding a simultaneous eigenstate of $H, L$ and $Q$, we can separate it into symmetric (bosonic) and asymmetric (fermionic) parts. Note that these symmetric and anti-symmetric parts are eigenstates of $Q^2$ but not of $Q$.

The two-body Hamiltonian in terms of the hyperspherical coordinates is given by

$$H = -\frac{1}{2}\Big[\frac{\partial^2}{\partial R^2} + \frac{3}{R}\frac{\partial}{\partial R} - \frac{\Lambda^2}{R^2} + \frac{2}{R}\Big] + \frac{2g_1}{R^2 \sin^2(2\theta)} \tag{66}$$

where the operator $\Lambda^2$ is the Laplacian on the sphere $S^3$ and is given by

$$-\Lambda^2 = \frac{\partial}{\partial \theta^2} - \frac{2\sin(2\theta)}{\cos(2\theta)}\frac{\partial}{\partial \theta} + \frac{1}{\cos^2(2\theta)}\Big[\frac{\partial^2}{\partial \phi^2} + 2\sin(2\theta)\frac{\partial^2}{\partial\phi\partial\psi} + \frac{\partial^2}{\partial\psi^2}\Big]\,. \tag{67}$$

As in the oscillator case, the interaction term i.e. $2g_1/R^2\sin^2(2\theta)$ is independent of $\phi$ and $\psi$. Hence the operators $L$ and $Q$ commute with the Hamiltonian (66) (since they also commute with the noninteracting ($g_1 = 0$) Hamiltonian). Hence, for all $g_1$, we label the states with the eigenvalues of $L$ and $Q$. Each of these states is four-fold degenerate under parity $L \to -L$, $Q \to Q$ and the Hamiltonian is also invariant under parity. Hence the states labeled by the quantum numbers $(l, q)$ have the same energy as $(-l, q)$. The Hamiltonian is also invariant under the discrete transformation $\mathbf{r}_1 \to -\mathbf{r}_1$ and $\mathbf{r}_2 \to \mathbf{r}_2$. Note that under this transformation $L \to L$ and $Q \to -Q$. Thus the states labeled by the quantum numbers $(l, q)$ have the same energy as $(l, -q)$. In this way we obtain the four-fold degeneracy of the states. In fact we shall see below that the states with $(l, q\ )$ have the same energy as $(q, l)$ since interchanging $q$ and $l$ leaves the differential equation invariant. As a result, one has in fact an eight-fold degeneracy for the levels for which $\mid q \mid$ and $\mid l \mid$ are non-zero and different from each other. However, the degeneracy is only four-fold if $\mid l \mid = \mid q \mid \neq 0$. Similarly, there is a four-fold degeneracy between the states $(\pm l, 0)$ and $(0, \pm l)$ if $l \neq 0$. It must be noted here that the degeneracy that we are talking about is a subset of the degeneracy of the noninteracting system.

Let us now try to solve the eigenvalue equation $H\psi = E\psi$ with $H$ being as given by eq. (66). It is easily seen that if we write

$$\psi(R, \theta, \phi, \psi) = F(R)\Phi(\theta, \phi, \phi) \tag{68}$$

then the eigenvalue equation separates into angular and radial equations. In particular, the angular equation is given by

$$[\Lambda^2 + \frac{4g_1}{\sin^2 2\theta}]\Phi = \beta(\beta+2)\Phi \tag{69}$$

where $\beta \geq -1$, while the radial equation is given by

$$F''(R) + \frac{3}{R}F'(R) + (2E + \frac{2}{R} - \frac{\beta(\beta+2)}{R^2})F(R) = 0\,. \tag{70}$$

The radial equation is easily solved, yielding

$$F(R) = R^\beta exp(-\sqrt{2 \mid E \mid}R) M(a, b, 2\sqrt{2 \mid E \mid}R) \tag{71}$$

where $b = 2\beta + 3, a = 3/2 + \beta - 1/\sqrt{2 \mid E \mid}$ and $M(a, b, x)$ is the confluent hypergeometric function. Demanding $a = -n_r$, where $n_r$ is a positive integer, yields the bound state energy eigenvalues as

$$E = -\frac{1}{2(n_r + \beta + \frac{3}{2})^2}\,. \tag{72}$$

The tower structure of the eigenvalues built on the radial excitations of the ground state is obvious from eq. (72).

It must be emphasized here that $\beta$ is still unknown and has to be obtained by solving the angular equation (69). We now note that the angular equation is in fact identical in the oscillator and our case and it has been analyzed in great detail by Bhaduri *et al.* [6]. We can therefore borrow their results and draw conclusions about the value of $\beta$ and hence the spectrum as given by eq. (72). Some of the conclusions in our case are

1. On substituting

$$\Phi(\theta, \phi, \psi) = P(x) exp(iq\phi) exp(il\psi) \tag{73}$$

   where

$$P(x) = \mid x \mid^a (1-x)^b (1+x)^c \Theta^{a,b,c}(x) \tag{74}$$

   with $x = \sin\theta,\ b = \mid l + q \mid /4, c = \mid l - q \mid /4, a(a-1) = g_1 = g(g-1)$ and $l, q$ being integers, it is easily shown that $\Theta(x)$ satisfies the Heun's equation

$$\begin{aligned}(1 &- x^2)\Theta''(x) + 2[\frac{a}{x} - (b-c) - (a+b+c+1)x]\Theta'(x) \\ &+ [\frac{(\beta+1)^2}{4} - (a+b+c+\frac{1}{2})^2 + 2\frac{a(c-b)}{x}]\Theta(x) = 0 \quad (75)\end{aligned}$$

whose solutions are characterized by the so-called $P$-symbol [18]. Note that unlike the hypergeometric case, Heun's equation has four regular singular points.

2. In case $b = c$ (i.e. either $l = 0$ or/and $q = 0$), the Heun's eq. (75) is exactly solvable [6] with corresponding $\beta$ being

$$\beta = 2m + 2a + 4b\,, \quad m = 0, 1, 2, .... \quad (76)$$

Thus in this case $E^{-1/2}$ varies linearly with $a$ (as is the case for the exact solutions of the many-body problem). As pointed out in [5], this is an example of a conditionally exactly solvable (CES) problem [19].

3. There are quasi-exactly solvable (QES) [20] polynomial solutions of degree $p (p \geq 1)$ in case there is a specific relationship between $a, b$ and $c$ . As a result, $\beta$ and hence $E^{-1/2}$ is nonlinear in $a$. In particular, in this case $\beta$ is given by

$$\beta = 2a + 2b + 2c + 2p\,. \quad (77)$$

It is worth emphasizing that the equation is exactly solvable at an infinite number of isolated points in the space of parameters $(a, b, c)$. These are isolated points because if we vary $a$ slightly away from any one of them, the equation is not exactly solvable since $b, c$ can only take discrete values and hence cannot be varied continuously.

4. Bhaduri *et al.* [6] have done a detailed numerical, perturbative and large-$g$ analysis of $\beta$ as a function of $g$ both when $0 \leq g < 1$ and also for large $g$. Their analysis is also valid in our case. For example, their Figs. 1 and 2 are also valid in our case except that in our case, Fig.1 represents a plot of $\beta + 2n_r + 2$ as a function of $g$ while Fig.2 represents a plot of $\beta + 2 - 2g$ vs. $g$. Most of their discussion goes through in our case with this obvious modification.

5. The two body problem is also completely solvable in case we add the two-body potential as given by eq. (5) to the Hamiltonian as given by eq. (61). In this case, the entire discussion about the angular part as given above is still valid. The only modification is in the radial equation as given by (70). In particular, instead of (70) we now have

$$F''(R) + \frac{3}{R}F'(R) + (2E + \frac{2}{R} - \frac{\beta(\beta+2)+\delta^2}{R^2})F(R) = 0\,. \tag{78}$$

whose solution is

$$F(R) = R^{\gamma} exp(-\sqrt{2 \mid E \mid}R)M(a, b, 2\sqrt{2 \mid E \mid}R) \tag{79}$$

$$E = -\frac{1}{2(n_r + \gamma + \frac{3}{2})^2}\,. \tag{80}$$

where $b = 2\gamma+3, a = \gamma+3/2-1/\sqrt{2 \mid E \mid}$ with $\gamma = \sqrt{(1+\beta)^2+\delta^2}-1$, while $M(a, b, x)$ is the confluent hypergeometric function.

6. If instead if we add the two-body potential (5) to the corresponding oscillator problem, then it is easily seen that the angular part is again unchanged while the solution to the radial part is given by

$$E_{n_r} = 2n_r + \gamma + 1 \tag{81}$$

$$F(R) = R^{\gamma} exp(-R^2/2)M(a, b, R^2) \tag{82}$$

where $a = 1 + (\gamma - E)/2, b = \gamma + 2$.

# 4 Novel correlations in Arbitrary Dimensions

In the last section we have obtained the exact bosonic ground state with Novel correlation as well as radial excitations over it in case the $N$-particles in two dimensions also interact through the $N$-body potential. The purpose of this section is to generalize these arguments to arbitrary number ($D$) of dimensions. In this context it is worth recalling that in case the $N$ particles are interacting via the oscillator potential, such a generalization has recently been carried out by Ghosh [7]. Following him, let us consider the following many-body Hamiltonian in $D$-dimensions

$$
\begin{aligned}
H &= -\frac{1}{2}\sum_{i=1}^{N}\nabla_i^2 - \frac{1}{\sqrt{\sum_{i=1}\mathbf{r}_i^2}} + \frac{g_1}{2}\sum_R \frac{\mathbf{Q}^2_{i_2 i_3 \ldots i_D}}{P^2_{i_1 i_2 \ldots i_D}} \\
&+ \frac{g_2}{2}\sum_R \frac{\mathbf{Q}_{i_2 i_3 \ldots i_D} \times \mathbf{Q}_{j_2 j_3 \ldots j_D}}{P_{i_i i_2 \ldots i_D} P_{i_i j_2 \ldots j_D}}
\end{aligned} \tag{83}
$$

where $R$ denotes the sum over all the indices from 1 to $N$ with the restriction that no two indices can have the same value simultaneously. Here

$$
\begin{aligned}
\mathbf{Q}_{i_2 i_3 \ldots i_D} &= \mathbf{r}_{i_2} \times \mathbf{r}_{i_3} \times \ldots \times \mathbf{r}_{i_D} \\
P_{i_1 i_2 \ldots i_D} &= \mathbf{r}_{i_1} \cdot \mathbf{Q}_{i_2 i_3 \ldots i_D}\,.
\end{aligned} \tag{84}
$$

Thus the Hamiltonian has $D$-body and $(2D-1)$-body interactions only. Note that both $\mathbf{Q}_{i_2 i_3 \ldots i_D}$ and $P_{i_1 i_2 \ldots i_D}$ are antisymmetric under the exchange of particle coordinates. Further, $P_{i_1 i_2 \ldots i_D}$ vanishes when the relative angle between any two of the particles is zero or $\pi$.

For simplicity, let us first discuss the $D = 3$ case is some detail and then merely quote the results for the general case of $D$-dimensions. It is easily seen that the exact bosonic ground state in 3-dimensions is given by

$$
\Psi_0 = \prod_R \mid P_{ijk} \mid^g exp(-\alpha\sqrt{\sum_i \mathbf{r}_i^2}) \tag{85}
$$

where $E_0 = -\alpha^2/2$ provided

$$
g_1 = \frac{g}{2}(\frac{g}{2} - 1)\,,\ g_2 = \frac{g^2}{4}\,. \tag{86}
$$

The corresponding ground state energy $E_0$ is given by

$$
E_0 = -\frac{1}{2[\frac{3}{2}gN(N-1)(N-2) + \frac{3N}{2} - \frac{1}{2}]^2}\,. \tag{87}
$$

The fact that this is indeed the ground state is easily shown as follows. Consider the following annihilation operators in three dimensions (note that

$\mathbf{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k})$

$$
\begin{aligned}
A_{x_i} &= p_{x_i} + \frac{ig}{2}\sum_S \frac{(\mathbf{Q}_{jk})_{x_i}}{P_{ijk}} - \frac{i\alpha x_i}{\sqrt{\sum_j \mathbf{r}_j^2}} \\
A_{y_i} &= p_{y_i} - \frac{ig}{2}\sum_S \frac{(\mathbf{Q}_{jk})_{y_i}}{P_{ijk}} - \frac{i\alpha y_i}{\sqrt{\sum_j \mathbf{r}_j^2}} \\
A_{z_i} &= p_{z_i} + \frac{ig}{2}\sum_S \frac{(\mathbf{Q}_{jk})_{z_i}}{P_{ijk}} - \frac{i\alpha z_i}{\sqrt{\sum_j \mathbf{r}_j^2}}
\end{aligned}
\qquad (88)
$$

where $S$ denotes the sum over all the repeated indices from 1 to $N$ with the constraint that any two indices cannot have the same value simultaneously. It is easily shown that the Hamiltonian (83) in 3-dimensions can be written down as

$$
H = \frac{1}{2}\sum_{i=1}^{N}\left(A_{x_i}^+ A_{x_i} + A_{y_i}^+ A_{y_i} + A_{z_i}^+ A_{z_i}\right) + E_0 \qquad (89)
$$

Further, it is easily checked that the annihilation operators $A$'s annihilate the wave function $\psi_0$ as given by eq. (85) and hence $\psi_0$ is indeed the ground state with the ground state energy being $E_0$ as given by eq. (87).

The excited states can also be obtained in our case [7]. To that end we write

$$
\psi(x_i, y_i, z_i) = \psi_0(x_i, y_i, z_i)\phi(x_i, y_i, z_i) \qquad (90)
$$

where $\psi_0$ is as given by eq. (85) except that $\alpha$ is now given by $E = -\alpha^2/2$. On substituting this ansatz in the Schrödinger equation for the Hamiltonian (83), it is easily shown $\phi$ satisfies the equation

$$
\left[-\frac{1}{2}\sum_i \nabla_i^2 - \frac{g}{2}\sum \frac{\mathbf{Q}_{jk}\cdot\nabla_i}{P_{ijk}} + \alpha\sum\frac{\mathbf{r}_i\cdot\nabla_i}{\sqrt{\sum \mathbf{r}_i^2}} + \frac{A}{\sqrt{\sum \mathbf{r}_j^2}}\right]\phi = 0 \qquad (91)
$$

where

$$
A = \alpha\left[\frac{g}{2}N(N-1)(N-2) + \frac{3N}{2} - \frac{1}{2}\right] - 1\,. \qquad (92)
$$

On assuming that $\phi$ is a function of $t = 2\alpha\sqrt{\sum \mathbf{r}_i^2}$, it is easily seen that eq. (91) for $\phi(t)$ reduces to the confluent hypergeometric equation

$$
t\phi''(t) + (b-t)\phi'(t) - a\phi(t) = 0 \qquad (93)
$$

when $a = A/\alpha,\ b = 2(A+1)/\alpha$. Hence the admissible normalizable solutions of eq. (93) are $M(a = -n_r, b, t)$ with the corresponding energy eigenvalues being

$$E_{n_r} = -\frac{1}{2[n_r + \frac{g}{2}N(N-1)(N-2) + \frac{3N}{2} - \frac{1}{2}]^2}\,. \tag{94}$$

Note again the tower structure and the fact that $(-E_{n_r})^{1/2}$ is linear in $g$ for the exact solutions.

We can also obtain the exact solutions in case we add the $N$-body potential (5) to the Hamiltonian (83). In particular the energy eigenvalues are now given by

$$E_{n_r} = -\frac{1}{2[n_r + \gamma + \frac{1}{2}]^2} \tag{95}$$

where

$$\gamma = \left([\frac{3N + gN(N-1)(N-2)}{2} - 1]^2 + \delta^2\right)^{1/2}\,. \tag{96}$$

The corresponding eigenfunctions can be easily written down. if instead one adds the $N$-body potential (5) to the oscillator potential, then the energy eigenvalues are given by

$$E_{n_r} = 2n_r + 1 + \gamma\,. \tag{97}$$

The corresponding eigenfunctions can again be easily written down.

The generalization of the above results to $D$-dimensions is straight forward. In particular, it is easily shown that the exact bosonic ground state and the tower of excitations over it in the case of the $D$-dimensional many-body Hamiltonian (83) are given by

$$\psi_n = \prod_R \mid P_{i_1 i_2 \ldots i_D} \mid^g\, M(a = -n_r, b, t) exp(-\alpha\sqrt{\sum_i \mathbf{r_i^2}}) \tag{98}$$

where

$$E_{n_r} = -\frac{\alpha^2}{2} = -\frac{1}{2[n_r + \frac{DN}{2} + gD({}^N C_D) - \frac{1}{2}]^2} \tag{99}$$

provided

$$g_2 = (\frac{g}{(D-1)!})^2,\ g_1 = \frac{g}{(D-1)!}[\frac{g}{(D-1)!} - 1] \tag{100}$$

so that

$$g = \frac{(D-1)!}{2}[1 \pm \sqrt{1+4g_1}] \tag{101}$$

It is interesting to note that whereas the relation between $g_1$ and $g_2$ and also between $g$ and $g_1$ is $D$-dependent, the allowed ranges of $g_1$ are independent of $D$ i.e. in any dimensions, whereas the solutions in the upper branch are regular for $g_1 \geq -\frac{1}{4}$, in the lower branch the regular solutions are only allowed in the limited range $-\frac{1}{4} \leq g_1 \leq 0$. Here $a = A/\alpha, b = \frac{2(A+1)}{\alpha}$ where

$$A = \alpha\Big(\frac{D}{2}[N + 2g(^N C_D)] - \frac{1}{2}\Big) - 1\,. \tag{102}$$

The exact solutions are also obtained if we add the $N$-body potential (5) to the Hamiltonian (83) and the exact energy eigenvalues are given by

$$E_{n_r} = -\frac{1}{2[n_r + \gamma + \frac{1}{2}]^2} \tag{103}$$

where

$$\gamma = \Big([\frac{ND}{2} + gD(^N C_D) - 1]^2 + \delta^2\Big)^{1/2}\,. \tag{104}$$

Similarly, if we add the $N$-body potential (5) to the Hamiltonian (83) but with the oscillator potential then the exact solutions are given by

$$E_{n_r} = 2n_r + 1 + \gamma\,. \tag{105}$$

The corresponding eigenfunctions can be easily written down in both the cases.

# 5 Calogero-Sutherland Type Models in Higher Dimensions with $N$ body Interaction

In Sec.II, we have considered one possible generalization of the Calogero-Sutherland type models in higher dimensions. The key point there was to have a long-ranged three-body interaction term. This is over and above the long ranged two-body interaction term which is present in one dimension. Only then it was possible to obtain a class of exact solutions including the bosonic ground state. Another possible generalization of Calogero-Sutherland model to higher dimensions was considered recently by Ghosh [8] when he introduced two models with purely two-body long ranged interactions and in both the cases he was able to obtain the exact bosonic ground state and radial excitations over it in case the $N$-particle also interact via a (one-body) oscillator potential. In this section we shall show that the exact ground state and radial excitations over it can also be obtained in case the oscillator potential is replaced by the $N$-body potential as given by eq. (1).

Following Ghosh [8], let us consider the Hamiltonian

$$H = -\frac{1}{2}\sum_{k=1}^{N} \nabla_k^2 - \frac{1}{\sqrt{\sum_k \mathbf{r}_k^2}} + V_1(\beta) + V_2(\beta) + W_3(\beta) \tag{106}$$

where

$$\begin{aligned} V_1(\beta) &= \frac{\beta^2}{2} g(g-1) \sum_{k\neq j} \frac{\mid \mathbf{r}_k \mid^{2(\beta-1)}}{(\mid \mathbf{r}_k \mid^{\beta} - (\mathbf{r}_j \mid^{\beta})^2} \\ V_2(\beta) &= \frac{g\beta}{2}(D+\beta-2) \sum_{k\neq j} \frac{\mid \mathbf{r}_k \mid^{(\beta-2)}}{(\mid \mathbf{r}_k \mid^{\beta} - (\mathbf{r}_j \mid^{\beta})} \\ W_3(\beta) &= \frac{\beta^2}{2} g(g-1) \sum_{i\neq j\neq k} \frac{\mid \mathbf{r}_i \mid^{2(\beta-1)}}{(\mid \mathbf{r}_i \mid^{\beta} - \mid \mathbf{r}_j \mid^{\beta})(\mathbf{r}_i \mid^{\beta} - \mid \mathbf{r}_k \mid^{\beta})} \end{aligned} \tag{107}$$

where $g$ is a dimension-less constant. Note that for $\beta = 1$ and 2 the three body interaction term $W_3$ vanishes. Thus even though we give results for arbitrary positive values of $\beta$, it must be noted that one has long ranged two-body interaction alone only if $\beta = 1,2$.

We first note that the Hamiltonian (106) can be written as

$$H = \frac{1}{2}\sum_i \mathbf{A}_i^+ \cdot \mathbf{A}_i + E_0 \tag{108}$$

where the annihilation operators $\mathbf{A}_i$ are given by

$$\mathbf{A}_i = -i\nabla_i + i\beta g \sum_{i\neq j} \frac{\mid \mathbf{r}_i \mid^{\beta-2}}{(\mid \mathbf{r}_i \mid^{\beta} - \mid \mathbf{r}_j \mid^{\beta}} \mathbf{r}_i - \frac{i\alpha \mathbf{r}_i}{\sum_j \mathbf{r}_j^2} \tag{109}$$

while the ground state energy $E_0 (= -\alpha^2/2)$ is

$$E_0 = -\frac{1}{2[\frac{ND}{2} + \frac{g\beta}{2} N(N-1) - 1]^2}\,. \tag{110}$$

The fact that $E_0$ is the bosonic ground state energy is verified by noting that each of the operators $\mathbf{A}_i$ annihilate the ground state wave function

$$\psi_0 = \prod_{i<j}(\mid \mathbf{r}_i \mid^\beta - \mid \mathbf{r}_j \mid^\beta)^g exp(-\sqrt{2 \mid E_0 \mid \sum_i r_i^2})\,. \quad (111)$$

It is straight forward to obtain the radial excitation spectrum. In particular, it is easily shown that the corresponding exact eigenfunctions are

$$\psi = \prod_{i<j}(\mid \mathbf{r}_i \mid^\beta - \mid \mathbf{r}_j \mid^\beta)^g M(a=-n_r, b, t) exp(-\sqrt{2 \mid E \mid \sum_i r_i^2})\,. \quad (112)$$

with the corresponding eigenvalues being

$$E_{n_r} = -\frac{1}{2[n_r + \frac{ND}{2} + \frac{g\beta}{2}N(N-1) - 1]^2}\,. \quad (113)$$

Here $t = 2\alpha\sqrt{\sum_i \mathbf{r}_i^2}, a = A/\alpha, b = 2(A+1)/\alpha$ while $E = -\alpha^2/2$ and $A = \alpha[ND/2 + g\beta N(N-1)/2 - 1]$.

One can also obtain the spectrum in case one adds the $N$-body potential (5) to the Hamiltonian (106). In particular, it is easily shown that the corresponding energy eigenvalues are

$$E_{n_r} = -\frac{1}{2[n_r + \lambda + \frac{1}{2}]^2} \quad (114)$$

where

$$\lambda = \left([\frac{ND}{2} + \frac{g\beta N(N-1)}{2} - 1]^2 + \delta^2\right)^{1/2}\,. \quad (115)$$

The corresponding eigenfunctions can also be easily written down. Similarly, in the oscillator case too, the spectrum can be written down in the presence of the $N$-body potential (5).

# 6 Complete Bosonic spectrum of an $N$-body problem in $D$ dimensions

Over the years, the exact solution of an $N$-body problem in three (and even arbitrary) space dimensions has attracted considerable attention because of its obvious implications in several areas. So far, the only $N$-body problem which is completely solvable in two and higher dimensions is when $N$-bosons are interacting pair-wise via harmonic interaction. The purpose of this section is to show that there is one more $N$-body problem in two and higher dimensions for which the complete discrete spectrum can also be obtained. In particular I show that the full bound state spectrum can also be obtained in case $N$-bosons are interacting via the $N$-body potential as given by eq. (4).

In particular, consider the Hamiltonian

$$H = -\frac{1}{2}\sum_{i=1}^{N}\nabla_i^2 - \frac{1}{\sqrt{\sum_{i<j}(\mathbf{r}_i - \mathbf{r}_j)^2}}\,. \tag{116}$$

After the separation of the center of mass (which in this case moves as a free particle), the relative problem is usually discussed in terms of the Jacobi coordinates $\zeta_j (j = 1, 2, ..., N-1)$ defined by

$$\zeta_j = (\frac{j}{j+1})^{1/2}[\frac{1}{j}(\mathbf{r}_i + ... + \mathbf{r}_j) - \mathbf{r}_{j+1}]\,. \tag{117}$$

However, it is more convenient to consider the problem in hyperspherical coordinates $(\zeta, \omega)$ of the $(N-1)$-dimensional vector $\xi$. Here the hyper-radius $\zeta (0 \le \zeta \le \infty)$ is given by

$$\zeta^2 = \sum_{j=1}^{N-1} \zeta_j^2 \tag{118}$$

while $\omega$ is a set of $[(N-1)D\text{-}1]$ angular coordinates. With this choice, the Hamiltonian takes the form

$$H = -\frac{1}{2}\Big[\frac{\partial^2}{\partial\zeta^2} + \frac{(N-1)D-1}{\zeta}\frac{\partial}{\partial\zeta} - \frac{L^2(\omega)}{\zeta^2}\Big] - \frac{1}{\sqrt{N}\zeta} \tag{119}$$

where $L^2(\omega)$ is the "grand angular" operator whose eigen functions are the orthonormal hyper-spherical harmonics $Y_{[L]}(\omega)$ with eigenvalues $-k[k+(N-1)D-2]$. Here $k = 0,1,2,...$ is the grand angular quantum number. It is thus clear that if we write the eigenfunction $\psi$ in the form

$$\psi(\zeta, \omega) = \phi(\zeta) Y_{[L]}(\omega) \tag{120}$$

then in the $\zeta$ variable one has essentially a one dimensional Schrödinger equation

$$\Big[\frac{\partial^2}{\partial\zeta^2}+\frac{(N-1)D-1}{\zeta}\frac{\partial}{\partial\zeta}-\frac{k[k+(N-1)D-2]}{\zeta^2}+\frac{2}{\sqrt{N}\zeta}\Big]\phi(\zeta)=-2E\phi(\zeta) \tag{121}$$

for the Kepler problem in $(N-1)D$-dimensions. The corresponding discrete energy energy values and eigen functions are given by

$$E_{n_r,k}=-\frac{1}{2N[n_r+k+\frac{(N-1)D-1}{2}]^2} \tag{122}$$

$$\phi(\zeta)=y^k exp(-y/2)L^{2a}_{n_r}(y) \tag{123}$$

where $a=(N-1)D/2+k-1, y=2\sqrt{2\mid E\mid}\zeta$ and $n_r=0,1,2,...$ is the radial quantum number.

It is worth pointing out that in eq. (119), apart from the $N$-body potential (3), the only other potential for which the Schrödinger equation can be solved for all values of $k$ is the two-body harmonic oscillator potential. Besides, in both the cases one can also add the $N$-body potential (4) and the problem is still analytically solvable. In particular, instead of (122) the spectrum is now given by

$$E_{n_r,k}=-\frac{1}{2N[n_r+\frac{1}{2}+\lambda]^2} \tag{124}$$

where

$$\lambda=\sqrt{a^2+\delta^2}\,. \tag{125}$$

The corresponding eigenfunctions can also be easily written down. Note however that the degeneracy in the spectrum is now very much reduced.

Our $N$-body potential is in a way richer than the oscillator potential in that here one not only has an infinite number of negative energy bound states, but one also has positive energy scattering states. In particular, for $E>0$, the solution to the Schrödinger eq. (121) is given by

$$\phi_i(\zeta)=exp(ip\zeta)\zeta^k F\Big(k+\frac{(N-1)D-1}{2}-\frac{i}{p},(N-1)D+2K-1,-2ip\zeta\Big) \tag{126}$$

where $E = p^2/2$. Hence the phase shifts for this problem are

$$e^{2i\delta_p} = \frac{\Gamma(k + \frac{(N-1)D-1}{2} - \frac{i}{p})}{\Gamma(k + \frac{(N-1)D-1}{2} + \frac{i}{p})} \tag{127}$$

Unfortunately, we are unable to draw any simple conclusion about the $N$-particle scattering from this expression of the phase shift.

# 7 Summary and Open Problems

In this paper, we have discussed several $N$-body problems in two and higher dimensions and have provided support to the conjecture that whenever an $N$-body problem is (partially) solvable in case the $N$-bodies are interacting by the harmonic forces, then the same problem will also be (partially) solvable in case they are interacting by an $N$-body potential as given by eq. (1) (or (3)). Our conjecture is based on the following simple observation. In all the many-body problems with harmonic forces discussed in [2-8,13,17], after the short distance correlation etc. is taken out, the problem essentially reduces to that of harmonic oscillator in dimensions higher than one. Now the only two problems for which the bound state spectrum can be analytically obtained for all partial waves in two and higher dimensions are the oscillator and the Coulomb problems. This strongly suggests that those $N$-body problems for which the Schrödinger equation essentially reduces to Coulomb problem in two and higher dimensions, after the short distance correlation part etc. is taken out, must also be (partially) solvable. While we have no proof for this conjecture, we have not come across any counter example either. We have also studied other models in one and two dimensions with this $N$-body potential [9,10] which also provide support to this conjecture. It is clearly necessary to examine other solvable many-body problems with the harmonic forces and check if our conjecture is true in those cases or not.

Another well known fact in non-relativistic quantum mechanics is that both the Coulomb and the oscillator problems in two and higher dimensions are also exactly solvable for all partial waves if one adds a potential of the form $\hbar^2\delta^2/2mr^2$ to either one of them. Based on this observation, we have conjectured that for the $N$-body problems with the oscillator or our $N$-body potential as given by eq. (1) (or (3)), one can also add the $N$-body potential as given by eq. (5) (or by (4)) and the problem is still (partially) solvable but now the degeneracy in the spectrum is much reduced. While we have no proof for this conjecture, we as well as Gurappa *et al* [13] have provided support to this conjecture through several examples. Besides, we have not come across any counter example either.

Apart from the oscillator and the $N$-body potential (1), are there other potentials for which any of these $N$-body problems in higher dimensions can be (partially) solved? While we have no definite answer, it appears that all other potentials will only be quasi-exactly solvable [21] since we know of no other potentials in non-relativistic quantum mechanics in two and higher dimensions for which the bound state spectra can be analytically obtained for all partial waves.

There are several common features in all the $N$-body problems that we have studied in this paper and before [9,12] as well as those studied with the oscillator potential. Some of these are as follows.

1. By now several $N$-body problems exist in two and higher dimensions for which a class of eigenstates including the bosonic ground state have been exactly obtained in case they also interact either via the oscillator or the $N$-body potential (1). All these problems have long ranged two-body and in most cases also the long ranged three-body interactions. In the special case of one dimension, there are only long ranged two-body interactions and the complete bound state spectrum has been obtained in both the cases [2,10].

2. In all the $N$-body problems with the oscillator or the $N$-body potential studied so far, one always obtains the tower of states characteristic of the oscillator or the Coulomb problem as the case may be. We conjecture that this will be a general feature of any many-body problem with either of these forces. While we have no general proof, we have not come across any counter example either.

3. All the $N$-body problems in two and higher dimensions (that have been studied so far) with either the oscillator or the $N$-body potential are only partially solvable in the sense that while the bosonic ground state and radial excitations over it are known, the full bosonic spectrum is still unknown (except the one studied in the last section or its oscillator analogue). It is worth pointing out that the same is also true in the

case of $N$-anyons interacting either via the oscillator or via our $N$-body potential [17]. Further, for none of these problems, the ground state of $N$-fermions is analytically known. I conjecture that the same will be true for any other $N$-body problem in two and higher dimensions interacting via either of these potentials.

4. If anyon example is any guide, then it would seem that in the case of any $N$-body problem in two and higher dimensions with the oscillator potential, a part of the spectrum will be linear in the coupling constant (say $\Lambda_D$ in Sec.II) while a part of the spectrum will be nonlinear in the coupling constant. On the other hand, in the case of our $N$-body potential (1), a part of $(-E)^{-1/2}$ will be linear and a part will be nonlinear in the coupling constant. So far, in all the examples discussed in the literature, one has only been able to obtain the linear part of $E$ (or $(-E)^{1/2}$) in the case of the oscillator (or $N$-body) potential. While it is not clear if one has obtained the full linear spectrum in all the cases or not, it is certainly true that so far exact solution has not been obtained for *even one nonlinear state* in any of the $N$-body problems in two and higher dimensions with either of the potentials. I believe that if one can obtain exact solution for even one nonlinear state in any of these problems, it will be a major breakthrough.

It must be added here though that the oscillator potential is perhaps more useful than our $N$-body potential since whereas the clustering property holds in the case of the oscillator potential it does not hold in the case of the $N$-body potential. On the other hand, the $N$-body potential is in a way richer than the oscillator potential in that unlike it, one has both bound and continuous spectrum in this case. The most important and of course difficult problem is to find some physical application of the $N$-body potential. Hopefully, some application will be found in the near future.

## References


[1] See for example D.C. Mattis, *The Many Body Problems: An Encyclopedia of Exactly Solvable Models in One Dimension* (World Scientific, Singapore, 1995).

[2] F. Calogero, J. Math. Phys. **10** (1969) 2191, 2197, ibid **12** (1971) 419; For an exhaustive review of various exactly solvable N-body problems see, M.A. Olshanetsky and A.M. Perelomov, Phys. Rev. **71** (1981) 314; ibid **94** (1983) 6.

[3] F. Calogero and C. Marchioro, J. Math. Phys. **14** (1973) 182.

[4] A. Khare and K. Ray, Phys. Lett. **A230** (1997) 139.

[5] M.V.N. Murthy, R.K. Bhaduri and D. Sen, Phys. Rev, Lett. **76** (1996) 4103.

[6] R.K. Bhaduri, A. Khare, J. Law, M.V.N. Murthy and D.Sen, J. Phys. **A30** (1997) 2557.

[7] P.K. Ghosh, MRI-PHY/96-20, cond-mat/9607009.

[8] P.K. Ghosh, Phys. Lett. **A229** (1997) 365.

[9] A. Khare, Phys. Lett. **A221** (1996) 365.

[10] A. Khare, J. Phys. **A29** (1996) L45.

[11] F. Calogero, J. Phys. **A29** (1996) 6455.

[12] A. Khare, J. Phys. **A29** (1996) 6459.

[13] N. Gurappa, C.Nagaraja Kumar and P.K. Panigrahi, Mod. Phys. Lett. **A11** (1996) 1737.

[14] A.P. Polychronakos, Phys. Rev.Lett. **69** (1992) 703.

[15] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, (Reading, MA, U.S.A. Addison-Wesley), appendix D, p 496 (1958).

[16] J.E. Kilpatrick and S.Y. Larsen, Few Body Syst. **3** (1987) 75.

[17] A. Khare and J. McCabe, Phys. Lett. **B269** (1991) 330.

[18] A. Erdelyi, *Higher Transcendental Functions, Bateman manuscript Project, Vol. III*, McGraw Hill, New York, U.S.A., p 57 (1955).

[19] A. de Dutra, Phys. Rev. **A47** (1993) R2435; R. Dutt, A. Khare and Y.P. Varshni, J. Phys. **A28** (1995) L107.

[20] A. Ushveridze, *Quasi-exactly Solvable Models in Quantum Mechanics*, Inst. of Phys. Publishing, Bristol, U.K. (1994); F. Cooper, A. Khare and U.P. Sukhatme, Phys. Rep. **251** (1995) 267.

[21] A. Khare and B.P. Mondal, To Be Published.