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Invariant Measures of Horospherical Flows on Noncompact Homogeneous Spaces

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Introduction

In [18] Veech proves that the "horospherical flow" on G/Γ where G is a semisimple analytic group with no non-trivial compact factor and Γ is a discrete co-compact subgroup, is uniquely ergodic; that is, there is a unique Borel probability measure on G/Γ which is invariant under the flow.

On the other hand there exists a wide class of discrete subgroups Γ in semisimple groups G such that G/Γ admits a G-invariant probability measure, but is noncompact. The discrete subgroup $SL(2,\mathbb{Z})$ of integral unimodular matrices in $SL(2,\mathbb{R})$ the group of all unimodular matrices, constitutes the simplest example of the above phenomenon. It turns out however, that the horospherical flow on a non-compact homogeneous space cannot be uniquely ergodic. This is because there exist proper closed subgroups H containing horospherical subgroups, having closed orbits which admit a finite H-invariant measure. The objective of the present paper is to assert that for a certain class of horospherical flows including the example cited above all the ergodic invariant measures arise in the above manner. We recall that the set of ergodic measures also determines completely the set of all finite invariant measures.

Actually rather than restricting to the class of semisimple groups with no non-trivial compact factor we consider any reductive analytic group T. Apart from being more general this is also convenient in certain arguments. A subgroup U in T is said to be *horospherical* if there exists $t\in T$ such that

$$U=\{u\in T\mid t^j u t^{-j}\to e \text{ as } j\to\infty\}$$

e being the identity in T. Any horospherical subgroup is analytic; i.e. a connected Lie subgroup (cf. §4, [5]). Further the adjoint action of any element in the Lie subalgebra of U on the Lie algebra of T is nilpotent. A horospherical flow in the sense of [18] is the action of a *maximal* horospherical subgroup.

A discrete subgroup Γ of a locally compact group T is said to be a *lattice* if T/Γ admits a finite T-invariant regular Borel measure.

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A major part of the study of invariant measures of horospherical flows consists of obtaining a characterization of the Haar measure of a semisimple group G without compact factors (cf. Theorem 2.4) in terms of invariance under the right action of a given lattice Γ and the left action of a certain maximal horospherical subgroup N of G, suitably related to Γ (cf. Theorem 2.4). Our approach in obtaining the above characterization is based on the philosophy introduced by Fürstenberg [8], which was also employed in [18].

In Part III we extend the study to reductive Lie groups. Sections 8 and 9 deal with general technical details concerning the extension. Finally in Section 10 we specialize to reductive Lie groups T such that all the non-compact simple factors of the adjoint group of T are of $\mathbb{R}$-rank 1 and obtain a complete classification of (ergodic) invariant measures of the "horospherical flow". We prove the following.

Theorem A. *Let T be a reductive Lie group such that all noncompact simple factors of T are of $\mathbb{R}$-rank 1. Let U be a maximal horospherical subgroup and let Γ be a lattice in T. Let π be a finite U-invariant ergodic measure on T/Γ. Then there exists a closed subgroup L and $t \in T$ such that*

i) *$Lt\Gamma$ is closed and $\pi(T/\Gamma - Lt\Gamma/\Gamma)=0$ and*

ii) *L contains U, $t\Gamma t^{-1} \cap L$ is a lattice in L and π is the finite L-invariant measure on $Lt\Gamma/\Gamma \simeq L/t\Gamma t^{-1} \cap L$.*

Notice that assertions i) and ii) completely determine π (up to a scalar which is determined by the total measure).

We observe that since every compact minimal subset of any action of an ameanable group is the support of a finite ergodic invariant measure, in particular Theorem A characterizes all compact minimal subsets of the horospherical flow.

Actually our proof incorporates the case of uniform lattices considered in [18]. However for that case there now exist simpler proofs (cf. [3] and [6]). We may note that the proofs in [3] and [6] indeed hold for all reductive groups.

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§ 1. Measures on Homogeneous Spaces

(1.1) For any locally compact, second countable space X, we denote by $\mathcal{M}(X)$ the set of all positive regular Borel measures on X. Let β be a Borel measurable map, or briefly a transformation of a space X onto a space Y. Let $\pi \in \mathcal{M}(X)$. If either β is continuous and proper or π is finite then we have a measure $\beta\pi \in \mathcal{M}(Y)$ defined by $\beta\pi(E)=\pi(\beta^{-1}E)$ for any Borel subset E of Y. If H is a set of transformations of X, $\pi \in \mathcal{M}(X)$ is said to be H-invariant if $h\pi = \pi$ for all $h \in H$.

Let H be a locally compact, second countable group and let $\Phi: H \times X \to X$ be an action of H on a space X. The action is said to be measurable if the map Φ is measurable when $H \times X$ is endowed with product Borel structure. Let $\pi \in \mathcal{M}(X)$

be H-invariant; i.e. invariant under all $\Phi(h, \cdot)$, $h \in H$. Then π is said to be ergodic with respect to the H-action (or that π is H-ergodic or even that the H-action on X is ergodic with respect to π) if for any Borel subset E of X, the assertion $\pi(E \Delta hE)=0$ for all $h \in H$ implies that either $\pi(E)=0$ or $\pi(X-E)=0$. Here and in the sequel hE denotes the set $\Phi(h, \cdot)E$, provided that the action in question is clear from the context.

(1.2) For $\pi \in \mathcal{M}(X)$, $\mathcal{L}^2(X, \pi)$ denotes the Hilbert space of all measurable complex valued functions which are square integrable with respect to π. Let H be a measurable group of transformations of X and let $\pi \in \mathcal{M}(X)$ be H-invariant. Then $h \mapsto V_h$, where V_h is the operator on $\mathcal{L}^2(X,\pi)$ defined by $(V_h f)(x) = f(h^{-1}x)$, is a (strongly continuous) unitary representation of H on $\mathcal{L}^2(X,\pi)$. Following is a simple and useful criterion for ergodicity: a finite H-invariant measure π is ergodic with respect to the H-action if and only if for $f \in \mathcal{L}^2(X, \pi)$, $V_h f = f$ for all $h \in H$ implies that f is a constant function π-a.e.

(1.3) The following proposition asserts that the set of all ergodic H-invariant measures determines the cone of all finite H-invariant measures. The result may be considered folklore. A proof for the case of cyclic and one-parameter group may be found in [17]. The same proof readily generalizes to any locally compact separable group.

(1.4) **Proposition.** *Let (X,π) be a Lebesgue probability space and let Φ: $H \times X \to X$ be a measurable, measure-preserving action of a locally compact, separable group H. Then there exists a measurable partition ξ of X and a family $\{\pi_C\}_{C \in \xi}$ where π_C is a probability measure on X with support contained in C such that if $\bar{\pi}$ denotes the quotient measure of π on X/ξ then we have the following*

i) *For almost all $C \in X/\xi$ (with respect to $\bar{\pi}$), π_C is a H-invariant ergodic measure.*

ii) *Let E be a measurable subset of X. Then for almost all $C \in X/\xi$, $E \cap C$ is measurable, $\pi_C(E \cap C)$ is a measurable function on X/ξ and*

$$\pi(E) = \int_{X/\xi} \pi_C(E \cap C)\, d\bar{\pi}(C).$$

(1.5) We recall some results from [8] regarding measures on homogeneous spaces.

(1.6) **Proposition.** *Let H be a closed unimodular subgroup of a Lie group L. Let dh be a fixed Haar measure on H. Then there is a one-to-one correspondence between $\mathcal{M}(L/H)$ and the subset of $\mathcal{M}(L)$ of measures which are invariant under the right action of H on L. The correspondence is given by $\pi \leftrightarrow \omega$ where*

$$(1.7) \quad \int_L f(x)\, d\omega(x) = \int_{L/H} d\pi(xH) \int_H f(xh)\, dh$$

for all $f \in C_c(L)$, the space of all continuous functions with compact support.

(1.8) **Proposition.** *If H and H' be two closed unimodular subgroups of a Lie group L, then (1.7) also sets up a correspondence between measures in $\mathcal{M}(L/H)$ which are invariant under the left action of H' and the subset of $\mathcal{M}(L)$ consisting* *of measures which are invariant under the right action of H and the left action of H'.*

(1.9) **Corollary.** *Let ι denote the inversion map of L, i.e. $x \in L \mapsto x^{-1}$. Then the map $\sigma \to \iota\sigma$ of $\mathcal{M}(L)$ induces a one-to-one correspondence between the set of H'-invariant measures on L/H and the set of H-invariant measures on L/H'. Under the correspondence H'-ergodic measures correspond to H-ergodic measures; the L-invariant (resp. absolutely continuous with respect to the L-invariant) measure on L/H corresponds to L-invariant (resp. absolutely continuous with respect to the L-invariant) measure on L/H'.*

Definition. Let G be a Lie group, Γ a lattice in G and let N be a horospherical subgroup of G. A Γ-invariant measure π on G/N is said to be *Γ-finite* if the N-invariant measure ω on G/Γ corresponding to π under the above one-one correspondence is finite.

(1.10) Absolute continuity of invariant measures: Let X be a locally compact, second countable space and let G and H be locally compact groups of homeomorphisms of X. Assume that the actions of G and H commute with each other; i.e. $ghx = hgx$ for all $g \in G$, $h \in H$ and $x \in X$. Assume also that the cone of G-invariant measures on X is one dimensional. Let dh be a fixed Haar measure on H and let $\varphi \in C_c^+(H)$, the cone of positive continuous functions with compact support. For any $\pi \in \mathcal{M}(X)$ and any Borel subset of X define

(1.11) $$\pi_\varphi(E) = \int_H \pi(h^{-1}E)\,\varphi(h)\,dh.$$

Then $\pi_\varphi \in \mathcal{M}(X)$. Then we have the following (cf. Proposition 2.5, [18]).

(1.12) **Proposition.** *Let G, H and X be as above and let μ be a G-invariant measure on X. Let Γ be a subgroup of G such that the action of Γ on X is ergodic with respect to μ. If π is a Γ-invariant measure on X such that $\pi_\varphi \prec \mu$ for all $\varphi \in C_c^+(H)$, then π is a multiple of μ.*

(Here and in the sequel $\prec$ stands for absolute continuity with respect to the latter measure.)

Later in the application of the above proposition we need the following lemma (cf. Theorem 3 and Proposition 6, [15]).

(1.13) **Lemma.** *Let G be a semisimple Lie group with no non-trivial compact factor. Let N be a maximal horospherical subgroup in G and let Γ be a lattice in G. Then the action of Γ on G/N is ergodic with respect to the G-invariant measure on G/N.*

We will also often need the following.

(1.14) **Lemma.** *Let L be a locally compact second countable group and let Γ be a lattice in L. Let H be a (closed) subgroup of L such that H is normalized by Γ and $H\Gamma$ is a closed subgroup. Then $H \cap \Gamma$ is a lattice in H.*

Proof. Observe that since L/Γ admits a finite L-invariant measure $L/H\Gamma$ also admits a finite L-invariant measure. It follows from the criterion for existence of invariant measures on homogeneous spaces (cf. Ch.. II, §9, [20]) that the restrictions of the modular homomorphism of L to Γ and $H\Gamma$ are the modular homomorphisms of the respective subgroups. Hence in particular we get that $H\Gamma/\Gamma$ admits a $H\Gamma$ invariant measure. But in view of the Fubini-Weil formula (cf. Ch. II, §9, [20]) the $H\Gamma$ invariant measure on $H\Gamma/\Gamma$ must be finite. On the other hand, $H\Gamma/\Gamma$ is canonically isomorphic to $H/H\cap\Gamma$ – under the isomorphism the H-action on $H\Gamma/\Gamma$ corresponds to the H-action on $H/H\cap\Gamma$. Hence we conclude that $H/H\cap\Gamma$ admits a finite H-invariant measure; i.e. $H\cap\Gamma$ is a lattice in H.

Part II. A Characterisation of Haar Measure

Throughout this part G will denote a connected semisimple Lie group with no non-trivial compact factor and with trivial center. Then G is isomorphic to its adjoint group and hence can be considered as the connected component of the identity in the $\mathbb{R}$-algebraic group of all Lie automorphisms of its Lie algebra. The class of maximal horospherical subgroups is then exactly the class of maximal unipotent subgroups of G.

The object of Part II, as the title indicates, is to prove a characterisation of the G-invariant measure on G/N, N being a maximal horospherical subgroup, which will be formulated in §2. The proof follows in several steps through §7.

§2. Lattices and Fundamental Domains

Let G be a Lie group as described above and Γ be a lattice in G. We need a certain special fundamental domain for Γ which we shall describe in the present section.

Definition. A lattice Λ in a semisimple Lie group L is said to be *irreducible* if the only positive dimensional normal subgroup F for which $F\Lambda$ is closed is L itself.

Any lattice Γ in a semisimple Lie group without compact factors can be "decomposed" into irreducible lattices in normal subgroups: More precisely we have the following

(2.1) **Proposition** (cf. [4] *Appendix*). *Let G be a semisimple Lie group without compact factors and with trivial center. Let Γ be a lattice in G. Then there exist normal (semisimple) Lie subgroups G_i, $i\in I$ (a suitable indexing set) such that*

i) $G=\prod_{i\in I} G_i$ *(direct product).*

ii) *For each $i\in I$, $\Gamma_i=\Gamma\cap G_i$ is an irreducible lattice in G_i and*

iii) $\Gamma'=\prod_{i\in I}\Gamma_i$ *is a (normal) subgroup of finite index in Γ.*

The decomposition is unique upto reindexing.

Observe that as a consequence of the above, G/Γ is finitely covered by G/Γ' and the latter is a direct product of homogeneous spaces G_i/Γ_i, $i\in I$. We now classify the components G_i/Γ_i as follows. Put

$I_0=\{i\in I \mid G_i/\Gamma_i \text{ is compact}\}$,

$I_1=\{i\in I-I_0 \mid \mathbb{R}\text{-rank } G_i>1\}$

and

$$I_2 = I - I_0 - I_1.$$

For $i \in I_1$, G_i is a semisimple Lie group without compact factor with trivial center and Γ_i is a non-uniform lattice in G_i. Therefore by Margulis's arithmeticity theorem (cf. [13]) Γ_i is an arithmetic subgroup of G_i. More precisely, G_i admits a structure of the connected component of the identity in an $\mathbb{R}$-algebraic group defined over $\mathbb{Q}$, such that Γ_i is commensurable with group $(G_i)_{\mathbb{Z}}$ of integral elements in G_i (with respect to the $\mathbb{Q}$-structure). We shall henceforth assume to have fixed the $\mathbb{Q}$-structure on each G_i, $i \in I_1$ satisfying the above.

Next let $i \in I_2$. Then G_i is of $\mathbb{R}$-rank 1 and Γ_i is a non-uniform lattice in G_i. For these lattices we plan to use the results of Garland and Raghunathan (cf. [10]). For the present we recall that G_i admits a maximal horospherical subgroup N_i such that $N_i \cap \Gamma_i$ is a lattice in N_i. Let P_i be the normalizer of N_i in G_i. Then P_i is a minimal $\mathbb{R}$-parabolic subgroup of G_i. For future use we note that by conjugating N_i by a suitable element of Γ_i we can find a maximal horospherical subgroup N_i' which is not contained in P_i. If P_i' denotes the normalizer of N_i' then $P_i \neq P_i'$. Since $\mathbb{R}$-rank of G_i is 1, $P_i \cap P_i'$ is a (reductive) Levi subgroup in both P_i and P_i'.

With this information we proceed to introduce some terminology with respect to Γ, generalizing the corresponding notions for arithmetic groups. We must emphasize that this is done only for simplicity in writing the fundamental domain and that the material is essentially well-known.

Definition. A horospherical subgroup U in G is said to be *Γ-rational* if $U \cap \Gamma$ is lattice in U.

A Lie subgroup U is a maximal Γ-rational horospherical subgroup of G if and only if $U = \prod_{i \in I} U_i$ where for $i \in I_0$, $U_i = (e)$, for $i \in I_1$, U_i is a maximal unipotent $\mathbb{R}$-subgroup of G_i defined over $\mathbb{Q}$ and for $i \in I_2$, U_i is a maximal horospherical subgroup such that $U_i \cap \Gamma_i$ is a lattice in U_i. Let U be a maximal Γ-rational horospherical subgroup and let P be the normalizer of U in G. Then again we observe that $P = \prod_{i \in I} P_i$ where for $i \in I_0$, $P_i = G_i$ for $i \in I_1$, P_i is a minimal $\mathbb{R}$-parabolic subgroup defined over $\mathbb{Q}$ and for $i \in I_2$, P_i is a minimal $\mathbb{R}$-parabolic subgroup. The results which we now state can be easily proved by considering separately the components corresponding to indices $i \in I$ and using known results about respective classes of groups and above observations. Hence we shall often omit the details.

Definition. Let U_1 and U_2 be horospherical subgroups in G and let P_1 and P_2 be normalizers of U_1 and U_2 respectively. Then U_1 and U_2 are said to be *opposite* (to each other) if $P_1 \cap P_2$ is a (reductive) Levi subgroup in both P_1 and P_2.

Let U and U^- be two Γ-rational horospherical subgroups opposite to each other, and let P and P^- be the respective normalizers. Then there exists a unique maximal vector subgroup S contained in the center of (the reductive group) $P \cap P^-$ such that $\{\mathrm{Ad}\, s \mid s \in S\}$ is simultaneously diagonalizable over $\mathbb{R}$. Actually $S = \prod_{i \in I} S_i$ where $S_i = (e)$ for $i \in I_0$, S_i is the connected component of a maximal $\mathbb{Q}$-split $\mathbb{R}$-torus (algebraic) for $i \in I_1$ and a suitable one parameter group for $i \in I_2$.

Now let $\mathfrak{G}$ be the Lie algebra of G and $\mathscr{S}$ the Lie subalgebra corresponding to S. Then we get a decomposition of $\mathfrak{G}$ as

$$\mathfrak{G} = \mathfrak{Z}(\mathscr{S}) + \sum_{\lambda \in \Delta} \mathfrak{G}^\lambda$$

where $\Delta \subset \mathscr{S}^*$ (the dual of $\mathscr{S}$) is a root system for the pair $(\mathfrak{G}, \mathscr{S})$, $\mathfrak{Z}(\mathscr{S})$ the centralizer of $\mathscr{S}$ in $\mathfrak{G}$, and for each $\lambda \in \Delta$

$$\mathfrak{G}^\lambda = \{\xi \in \mathfrak{G} \mid [\alpha, \xi] = \lambda(\alpha)\xi \text{ for all } \alpha \in \mathscr{S}\}$$

the latter being of positive dimension for all $\lambda \in \Delta$. On Δ there exists an ordering such that if Δ^+ is the corresponding set of positive roots then $\sum_{\lambda \in \Delta^+} \mathfrak{G}^\lambda$ is the Lie subalgebra corresponding to U and $\sum_{\lambda \in \Delta^+} \mathfrak{G}^{-\lambda}$ is the Lie subalgebra corresponding to U^-. The last assertion may be proved by using Lemma 4.5.4, in [13] for the field $\mathbb{Q}$ in the case of arithmetic components (i.e. $i \in I_1$) and for $\mathbb{R}$ in the case of non-uniform lattices in $\mathbb{R}$-rank 1 groups (i.e. $i \in I_2$).

Now let (U, U^-) be a pair of opposite maximal Γ-rational horospherical subgroups and S and Δ^+ be as chosen above. For $t > 0$ we define

(2.2) $\quad S_t = \{s \in S \mid \lambda(\log s) \leqq t \text{ for all } \lambda \in \Delta^+\}$

where $\log: S \to \mathscr{S}$ is the inverse of the exponential map of $\mathscr{S}$ onto S.

We now state the main proposition of §2.

(2.3) **Proposition.** *Let (U, U^-) be a pair of opposite maximal Γ-rational horospherical subgroups of G and let S and S_t, $t > 0$ be as introduced above, corresponding to (U, U^-). Further let H be the connected component of the identity in the normalizer P^- of U^- and let K be a maximal compact subgroup of G. Then there exist a compact subset W of H, a finite subset J of G and $\delta > 0$ such that the following properties hold.*

a) $G = \Gamma J W S_\delta K$.

b) For all $j \in J$, $j^{-1} \Gamma j \cap U^-$ is a lattice in U^-.

c) Let $G = \prod_{i \in I} G_i$ be the decomposition of G as in Proposition 2.1 and let $i \in I$ be such that $\mathbb{R}$-rank $G_i > 1$ and $\Gamma_i = \Gamma \cap G_i$ is a non-uniform lattice in G_i. Then for all $j \in J$, $j^- \Gamma_i j$ is commensurable with Γ_i.

Proof. Let $G = \prod_{i \in I} G_i$ be the decomposition of G as in Proposition 2.1, with respect to Γ. It is obvious that assertions a), b) and c) follow if the corresponding assertions are proved for the lattice Γ_i in G_i for each $i \in I$. In other words in proving the Proposition we may further assume Γ to be an irreducible lattice in G. We now consider three cases separately.

Case i. Γ is a uniform lattice in G.

In this case both U and U^- are the identity subgroup, as there cannot be any Γ-rational horospherical subgroup of positive dimension. In particular $P^- = G$.

But certainly there exists a compact subset W of G such that $G=\Gamma W$. Putting $J=\{e\}$, a) is satisfied. b) is satisfied trivially.

Case ii. Assume that Γ is non-uniform and $\mathbb{R}$-rank $G>1$.

As recalled earlier, in this case Γ is an arithmetic subgroup of G_j (with respect to a $\mathbb{Q}$ structure considered fixed). U and U^- are maximal unipotent algebraic $\mathbb{R}$-subgroups defined over $\mathbb{Q}$, and S is the connected component of the identity in a maximal $\mathbb{Q}$-split (algebraic) $\mathbb{R}$-torus. Applying Borel's construction of fundamental domains for arithmetic groups (cf. Theorem 13.1, [1]) it follows that there exists a compact subset W' of H (actually of a smaller subgroup) a finite subset J' of $G_{\mathbb{Q}}$ and $\delta>0$ such that with

$$S^\delta=\{s\in S\,|\,\lambda(\log s)\leqq\delta \text{ whenever } -\lambda\in\Delta^+\}$$

(Δ^+ being the set of roots such that the corresponding Lie subalgebra of $\mathfrak{G}$ belongs to U) we have $G=KS^\delta W'J'\Gamma$. Thus $G=G^{-1}=\Gamma JW(S^\delta)^{-1}K$ where $J=(J')^{-1}$ and $W=(W')^{-1}$. Observe that

$$\begin{aligned}(S^\delta)^{-1}&=\{s^{-1}\,|\,s\in S,\ \lambda(\log s)\leqq\delta \text{ whenever } -\lambda\in\Delta^+\}\\ &=\{s\in S\,|\,\lambda(\log s)\leqq\delta \text{ for all } \lambda\in\Delta^+\}\\ &=S_\delta.\end{aligned}$$

Thus we have proved assertion a) in the case at hand. Since $J\subset G_{\mathbb{Q}}$ for each $j\in J$, $j^{-1}\Gamma j$ is commensurable with Γ. This prove c). To prove b) it is enough observe that for any $y\in G_{\mathbb{Q}}$, yUy^{-1} is also a unipotent $\mathbb{R}$-subgroup defined over $\mathbb{Q}$ and hence intersects an arithmetic group in a lattice.

Case iii. Γ is a non-uniform lattice in G and $\mathbb{R}$-rank of G is 1.

In this case we use the construction of fundamental domains by Garland and Raghunathan (cf. Theorem 0.6, [10]). Notice that in this case $G=KSU^-$ (notation as in the statement of the theorem) is an Iwasawa decomposition of G. Then by the theorem of Garland and Raghunathan there exists a compact subset W' of U^-, a finite subset J' of G and $\delta>0$ such that $G=KS^\delta W'J'\Gamma$ where

$$S^\delta=\{s\in S\,|\,\lambda(\log s)\leqq\delta \text{ whenever } -\lambda\in\Delta^+\}$$

(Δ^+ being the set of roots corresponding to U) and J' is such that for each $j\in J'$, $\Gamma\cap j^{-1}U^-j$ is a lattice in $j^{-1}U^-j$. Then $G=G^{-1}=\Gamma JW(S^\delta)^{-1}K$ where $J=(J')^{-1}$ and $W=(W')^{-1}$. As before we see that $(S^\delta)^{-1}=S_\delta$. Hence $G=\Gamma JWS_\delta K$. Also clearly for each $j\in J$, $j^{-1}\Gamma j\cap U^-$ is a lattice in U^-.

We now formulate a characterisation of the Haar measure whose proof is the subject of Part II. The notation U, U^-, S, K introduced earlier as also the subsets W, J and $\delta>0$ satisfying Proposition 2.3 shall be considered fixed. Also recall that P and P^- denote the normalizers of U and U^- respectively and that H denotes the connected component of the identity in P^-.

Now let A be a maximal vector subgroup containing S, contained in $P\cap P^-$ and such that $\{\mathrm{Ad}\,a\,|\,a\in A\}$ is diagonalizable over $\mathbb{R}$ and let N be a maximal horospherical subgroup (not necessarily Γ-rational) in G normalized by A and such that $U \subset N \subset P$. We denote $X = G/N$ and $X_0 = \bigcup_{j \in J} jP^- N/N$. Then $X - X_0$ is a union of lower dimensional manifolds. (This may be proved by using Bruhat decomposition on each component G_i $i \in I$.) Hence if μ is a G-invariant measure on X then $\mu(X - X_0) = 0$. We prove:

(2.4) **Theorem.** *Let π be a Γ-invariant, Γ-finite, ergodic (cf. §1 for definitions) measure on $X = G/N$ such that $\pi(X - X_0) = 0$. Then π is G-invariant.*

We close this section by recalling certain simple properties of the fundamental domains constructed in Proposition 2.3. Put $\Omega = JWS_\delta K/K \subset G/K$. Let μ_K denote a G-invariant measure on G/K. Then it is well-known that $\mu_K(\Omega) < \infty$ (cf. Lemma 12.5, [1]). We also need the following

(2.5) **Lemma.** *Let W_1 be any relatively compact subset of H. Then for any $t > 0$, $\bigcup_{h \in S_t} h^{-1} W_1 h$ is relatively compact.*

Proof. The subgroup H can be expressed as $Z \cdot U^-$ (semidirect product) where Z is the centralizer of S in G. Considering the components it is enough to prove the lemma for W_1 contained U^-. Any element of U^- can be expressed as $(\exp t_1 \xi_1)\,(\exp t_2 \xi_2) \ldots (\exp t_n \xi_n)$ where $\xi_1, \xi_2, \ldots, \xi_n \in \mathscr{U}^-$, the Lie subalgebra of U^-, are simultaneous eigenvectors of $\{\mathrm{Ad}\, s \mid s \in S\}$. For any $h \in S$, $t \in \mathbb{R}$ and $i \leqq n$ we have $h^{-1}(\exp t\xi_i) h = \exp \nu_i(h^{-1}) t \xi_i$ where ν_i is the character on S such that $(\mathrm{Ad}\, s)\,\xi_i = \nu_i(s)\,\xi_i$ for all $s \in S$. Since $\xi_i \in U^-$ clearly $-\log \nu_i \in \Delta^+$. Thus if $s \in S_t$, $\nu_i(h^{-1}) = \exp(-\log \nu_i(h)) \leqq e^t$. If W_1 is contained in

$$\{(\exp t_1 \xi_1)(\exp t_2 \xi_2) \ldots (\exp t_n \xi_n) \mid t_1, t_2 \ldots t_n \in [-t', t']\}$$

then $\bigcup h^{-1} W_1 h$ is contained in

$$\{(\exp t_1 \xi_1)(\exp t_2 \xi_2) \ldots (\exp t_n \xi_n) \mid t_1, t_2 \ldots t_n \in [-e^t t', e^t t']\}.$$

Hence it is relatively compact.

(2.6) *Remark.* $S = JWS_\delta K$ is contained in $JW_0 S^- K$ where $S^- = \{s \mid \lambda(\log s) \leqq 0$ for all $\lambda \in \Delta^+\}$, and W_0 is a compact subset of SH.

(2.7) *Note.* In the sequel the notations as in Proposition 2.3 and later shall be considered fixed. Also we shall denote by M the group of elements of K which centralize A. Then MAN is a minimal (real) parabolic subgroup of G.

§3. A Measure on the Fürstenberg Boundary

The homogeneous space G/MAN which is canonically isomorphic to K/M is the maximal Fürstenberg boundary of G. (cf. [7, 14].) We denote this space by B. We have a canonical projection of β: $X = G/N \to G/MAN = B$. We now fix a compact subset Q of X such that $\pi(Q) > 0$. Let π_Q be the measure on X defined by $\pi_Q(E) = \pi(E \cap Q)$ for any Borel subset E of X. Since Q is compact, π_Q is a finite measure. Thus we get a (regular, Borel) measure $\beta\pi_Q$ on B defined by

$\beta\pi_Q(Y)=\pi_Q(\beta^{-1}Y)$ for any Borel subset Y of B. Let m denote the K-invariant probability measure on B. The aim of this section is to prove the following:

(3.1) **Proposition.** *If $\beta\pi_Q \prec m$ then π is G-invariant.*

Proof. Observe that since the subgroup $Z=MA$ normalizes N there exists an action $z\mapsto\psi_z$ of Z on $X=G/N$ defined by $\psi_z(gN)=gzN$, which commutes with the left action of G on X. Therefore in view of Proposition 1.12 and Proposition 1.13 in order to prove the proposition it is enough to prove that for any $\varphi\in C_c^+(Z)$ the measure π_φ defined by

$$(3.2)\quad \pi_\varphi(E)=\int_Z \pi(\psi_z^{-1}E)\,\varphi(z)\,dz$$

for any Borel subset E of X, is absolutely continuous with respect to a G-invariant measure say μ. So let $\varphi\in C_c^+(Z)$ be fixed. Let $Q'=\{\psi_z x \mid z\in\text{support of }\varphi,\ x\in Q\}$ and let $\pi_{\varphi,Q}$ be the measure on X defined by

$$(3.3)\quad \pi_{\varphi,Q}(E)=\pi_\varphi(E\cap Q').$$

(3.4) **Lemma.** *If $\pi_{\varphi,Q}\prec\mu$ then $\pi_\varphi\prec\mu$.*

Proof. Since π is Γ-ergodic and $\pi(Q)>0$ we have $\pi(X-\Gamma Q)=0$. A direct computation now yields $\pi_\varphi(X-\Gamma Q')=0$. Therefore for any Borel subset E of X we have

$$\begin{aligned}(3.5)\quad \pi_\varphi(E)&=\pi_\varphi(E\cap\Gamma Q')\leq\sum_{\gamma\in\Gamma}\pi_\varphi(E\cap\gamma Q')\\ &=\sum_{\gamma\in\Gamma}\pi_\varphi(\gamma^{-1}E\cap Q')\\ &=\sum_{\gamma\in\Gamma}\pi_{\varphi,Q}(\gamma^{-1}E).\end{aligned}$$

Now if $\mu(E)=0$ then for any $\gamma\in\Gamma$, $\mu(\gamma^{-1}E)=0$. Therefore if $\pi_{\varphi,Q}\prec\mu$ by (3.5), $\mu(E)=0$ implies $\pi_\varphi(E)=0$.

Proof of Proposition (3.1). It is easy to verify that there exists a character χ: $Z\to\mathbb{R}^+$ such that for any Borel set $E\subset G/N=X$

$$\mu(E)=\int_B dm(xP)\int_Z \chi_E(\psi_z x)\,\chi(z)\,dz$$

(here and in the sequel χ_E denotes the characteristic function of E in the appropriate space). Assume that $\mu(E)=0$ and put

$$B_1=\{b=xP\in B \mid \int_Z \chi_E(\psi_z x)\,\chi(z)\,dz=0\}$$

Then $m(B_1)=1$. Put $E_1=E\cap\beta^{-1}B_1$ and $E_2=E-E_1$. Then

$$(3.6)\quad \pi_\varphi(E_1)=\int_X d\pi(x)\int_Z \chi_{E_1}(\psi_z x)\,\varphi(z)\,dz=0.$$

Put $B_2=B-B_1$. Then $E\subset\beta^{-1}B_2$ and $m(B_2)=0$. As in Lemma 3.4, we have

$$(3.7)\quad \pi(\beta^{-1}B_2)\leqq\sum_{\gamma\in\Gamma}\pi_Q(\gamma^{-1}\beta^{-1}B_2) =\sum_{\gamma\in\Gamma}\pi_Q(\beta^{-1}\gamma^{-1}B_2) =\sum_{\gamma\in\Gamma}\beta\pi_Q(\gamma^{-1}B_2).$$

It is easy to see that since $m(B_2)=0$, for all $\gamma\in\Gamma$, $m(\gamma^{-1}B_2)=0$. Therefore if $\beta\pi_Q\prec m$ by (3.7) we have $\pi(\beta^{-1}B_2)=0$. Therefore

$$\pi_{\varphi,Q}(E_2)\leqq\pi_{\varphi,Q}(\beta^{-1}B_2)=\int_X d\pi(x)\int_Z\chi_{\beta^{-1}B_2}(\psi_z x)\,\chi_{Q'}(\psi_z x)\,\varphi(z)\,dz =\int_X\chi_{\beta^{-1}B_2}(x)\,d\pi(x)\int_Z\chi_{Q'}(\psi_z x)\,\varphi(z)\,dz =0.$$

Combining with (3.6) we conclude that $\pi_{\varphi,Q}(E)=0$. Since E was an arbitrary Borel set with $\mu(E)=0$ we have thus proved that $\pi_{\varphi,Q}\prec\mu$. Hence by Lemma 3.4, $\pi_\varphi\prec\mu$. Again since $\varphi\in C_c^+(Z)$ was arbitrary as remarked in the beginning we get that π is G-invariant.

§ 4. Harmonic Functions on G/K

Recall that we have fixed an Iwasawa decomposition $G=KAN$. For $g\in G$ let $k(g)\in K$ and $a(g)\in A$, be the uniquely determined elements such that $g=k(g)\cdot a(g)\cdot n$ for some $n\in N$. Also for $g\in G$ let $H(g)=\log a(g)\in\mathfrak{A}$. The following useful relation is a simple consequence of the definition. If $g_1, g_2\in G$ then

$$(4.1)\quad H(g_1 g_2)=H(g_1 k(g_2))+H(g_2).$$

Let $\rho\in\mathfrak{A}^*$ be the element defined by $2\rho=\sum_{\lambda\in\Delta^+}(\dim\mathfrak{G}^\lambda)\,\lambda$. The Poisson kernel $\mathbb{P}$ on $G/K\times K/M$ is defined by

$$(4.2)\quad \mathbb{P}(gK,kM)=\exp-2\rho H(g^{-1}k).$$

$\mathbb{P}(\cdot,\cdot)$ is clearly well-defined. Using this kernel to each Borel measure σ on $B=K/M$ one associates a non-negative function h_σ on G/K defined by

$$(4.3)\quad h_\sigma(gK)=\int_B\mathbb{P}(gK,kM)\,d\sigma(kM).$$

Though we will not need it explicitly it may be recalled that h_σ defined as above is a harmonic function with respect to the Laplace-Beltrami operator (indeed with respect to any second order, elliptic G-invariant differential operator vanishing on constant functions) on G/K. Conversely any non-negative harmonic function h on G/K admits a representation as in (4.3) for some measure $\sigma=\sigma_h$ on B. Further the measure σ_h can be 'constructed' from h as follows. Let A^+ be the positive Weyl chamber in A with respect to Δ^+; viz.

$$A^+ = \{a \in A \mid \lambda(\log a) > 0 \text{ for all } \lambda \in \Delta^+\}.$$

For $a \in A^+$ let $\sigma(h, a)$ be the measure on B defined by

$$\sigma(h,a)(E) = \int_E h(kaK)\, dm(kM)$$

for any Borel subset E of B. Since $h(kmaK) = h(kaK)$ the integral is well-defined. Now define '$a \to \infty$' to mean that $\lambda(\log a) \to \infty$ for all $\lambda \in \Delta^+$. Then σ_h is determined by the following result of Knapp and Williamson [12].

(4.4) **Proposition.** *With notations as above*

$$\sigma_h = \lim_{a \to \infty} \sigma(h, a)$$

in the space of all bounded measures which is assumed to be endowed with the weak-topology.*

(4.5) **Corollary** (cf. [18]). *Let σ be a finite Borel measure on B and h_σ the harmonic function defined by (4.3). If there exists a sequence $\{a_j\}$ in A^+, $a_j \to \infty$ in the above sense, such that*

$$\limsup_{j \to \infty} \int_K h_\sigma(ka_jK)^2\, dk = l^2 < \infty$$

then $\sigma \prec m$.

Proof. By Proposition 4.4 and Schwartz inequality for any continuous function ψ on B, we get

$$\begin{aligned} \left|\int_B \psi(kM)\, d\sigma(kM)\right| &= \lim_{j \to \infty} \left|\int_B \psi(kM)\, h_\sigma(kaK)\, dm(kM)\right| \\ &\leq \limsup_{j \to \infty} \left|\int_B \psi(kM)^2\, dm\right|^{1/2} \left|\int_B h_\sigma(kaK)\, dm\right|^{1/2} \\ &= l \left|\int_B \psi(kM)^2\, dm\right|^{1/2} \end{aligned}$$

which shows that $\sigma \prec m$.

On G/K there exists a G-invariant metric defined by

(4.6) $$D(g_1K, g_2K) = \max_{k \in K} |\rho H(g_1^{-1} g_2 k)|$$

(cf. [18], §5). Further it is also proved that the distance is given by

$$D(g_1K, g_2K) = \rho(\log a)$$

where $a \in \operatorname{cl} A^+$ is the unique element such that $g_1^{-1} g_2 = k_1 a k_2$ (Cartan decomposition) where k_1, $k_2 \in K$.

The *disc of radius* τ in G/K with center K is the set

(4.7) $U_\tau = \{gK \mid D(gK, K) \leqq \tau\}$.

Let μ_K be a G-invariant measure on G/K. Proofs of the following lemmas may be found in [18].

(4.8) **Lemma** (cf. Lemma 5.10, [18]). *For* $\tau_0 > 0$ *there exists a constant* $c(\tau_0)$ *such that for all* $\tau > 0$, $\mu_K(U_{\tau+\tau_0}) \leqq c(\tau_0)\, \mu_K(U_\tau)$.

(4.9) **Lemma** (cf. Lemma 5.11, [18]). *Let* h *be a locally integrable, non-negative function on* G/K. *If there exists a constant* c *such that for all* $\tau > 0$

$$\int_{U_\tau} h(gK)^2 \, d\mu_K \leqq c\, \mu_K(U_\tau)$$

then there exists a sequence $\{a_j\}_1^\infty$ in A^+, $a_j \to \infty$ such that

$$\limsup_{j\to\infty} \int_K h(k a_j K)^2 \, dkM < \infty.$$

(4.10) *Remark.* Let f be the harmonic function on G/K corresponding to the meaure $\beta \pi_Q$ on B; viz.

$$f(gK) = \int_B \mathbb{P}(gK, kM)\, d\beta\pi_Q(kM).$$

In view of Proposition 3.1, Corollary 4.5 and Lemma 4.9, if there exists a constant c such that for all $\tau > 0$

$$\int_{U_\tau} f(gK)^2 \, d\mu_K \leqq c\, \mu(U_\tau)$$

then π is G-invariant.

§5. Estimation of $\mathscr{L}^2$-Norms on Discs

Recall that we have $G/K = \Gamma\Omega$. Now let $\gamma \in \Gamma$, $b \in \Omega$ and $g \in G$ be such that $gK = \gamma b$. We have for $f(gK)$ as in the last section

$$\begin{aligned} f(gK) &= \int_B \exp -2\rho H(g^{-1}k)\, d\beta\pi_Q(kM) \\ &= \int_X \exp -2\rho H(g^{-1}k(x))\, d\pi_Q(x). \end{aligned}$$

(Notice that the functions k and a defined on G are invariant under the right action of N and hence define functions on X which also we denote by the same letters.) Since Q is a compact subset of X there exists a constant c_1 such that for all $x \in Q$

(5.1) $c_1^{-1} \leqq \exp -2\rho H(a(x)) \leqq c_1$.

Since $H(g^{-1}k(x))=H(g^{-1}x)-H(a(x))$ using (5.1) we conclude that

$$f(gK)\leqq c_1 \int_X \exp -2\rho H(g^{-1}x)\,d\pi_Q(x)$$
$$=c_1 \int_X \chi_Q(x) \exp -2\rho H(b^{-1}\gamma^{-1}x)\,d\pi(x).$$

Here by $H(b^{-1}\gamma^{-1}x)$ we mean $H(b_0^{-1}\gamma^{-1}x)$ for some $b_0\in G$ such that $b=b_0K$, the function being obviously independent of the representative. Now since π is Γ-invariant we get

$$f(gK)\leqq c_1 \int_X \chi_Q(\gamma x) \exp -2\rho H(b^{-1}x)\,d\pi(x).$$

For the sake of brevity henceforth let the function $\exp -2\rho H(\cdot)$ on G or X be denoted by $F(\cdot)$ (on G or X respectively). Thus

$$(5.2)\quad f(gK)\leqq c_1 \int_X \chi_Q(\gamma x)\, F(b^{-1}x)\,d\pi(x).$$

Now let $\tau>0$ and U_τ be the disc of radius τ and center K. Put $\Gamma_\tau=\{\gamma\in\Gamma\,|\,\gamma\Omega\cap U_\tau$ is nonempty$\}$ and for $\gamma\in\Gamma_\tau$ let $D_\gamma=\Omega\cap\gamma^{-1}U_\tau$. Then $U_\tau=\bigcup_{\gamma\in\Gamma_\tau}\gamma D_\gamma$. Hence

$$\int_{U_\tau} f(gK)^2\,d\mu_K \leqq \sum_{\gamma\in\Gamma_\tau}\int_\Omega f(\gamma bK)^2\,\chi_{D_\gamma}(b)\,d\mu_K(b)$$
$$\leqq c_1 \sum_{\gamma\in\Gamma_\tau}\int_\Omega \chi_{D_\gamma}(b)\,d\mu_K(b) \int_{X\times X} \chi_Q(\gamma x)\,\chi_Q(\gamma y)\,F(b^{-1}x)$$
$$\cdot F(b^{-1}y)\,d\pi(x)\,d\pi(y)$$

in view of (5.2). Using Fubini theorem, which henceforth we apply repeatedly without mention, the last expression may be written as

$$(5.3)\quad c_1 \int_\Omega d\mu_K(b) \int_{X\times X} F(b^{-1}x)\,F(b^{-1}y)\,E(x,y,b)\,d\pi(x)\,d\pi(y)$$

where for all $x,y\in X$ and $b\in\Omega$

$$(5.4)\quad E(x,y,b)=\sum_{\gamma\in\Gamma_\tau}\chi_Q(\gamma x)\,\chi_Q(\gamma y)\,\chi_{D_\gamma}(b)$$
$$=\sum_{\gamma\in\Gamma}\chi_Q(\gamma x)\,\chi_Q(\gamma y)\,\chi_{U_\tau}(\gamma b).$$

We first obtain an integral which majorizes $E(x,y,b)$. Let V be a compact, symmetric neighborhood of the identity e in G with $V^2\cap\Gamma=\{e\}$. Then there exists $\tau_0>0$ such that $VU_\tau\subset U_{\tau+\tau_0}$. Also put $Q_1=KVQ$. With this notation it is straightforward to verify that there exists a constant c_2 such that for any $x,y\in X$ and $b\in\Omega$

$$(5.5)\quad \chi_Q(x)\,\chi_Q(y)\,\chi_{U_\tau}(b)\leqq c_2 \int_V \chi_{Q_1}(gx)\,\chi_{Q_1}(gy)\,\chi_{U_{\tau+\tau_0}}(gb)\,dg$$

where dg is a fixed Haar measure on G. Combining (5.4) and (5.5), we deduce that

$$E(x,y,b) \leqq c_2 \sum_{\gamma\in\Gamma} \int_V \chi_{Q_1}(g\gamma x)\,\chi_{Q_1}(g\gamma y)\,\chi_{U_{\tau+\tau_0}}(g\gamma b)\,dg$$
$$= c_2 \sum_{\gamma\in\Gamma} \int_{V\gamma} \chi_{Q_1}(gx)\,\chi_{Q_1}(gy)\,\chi_{U_{\tau+\tau_0}}(gb)\,dg.$$

Since V is symmetric and $V^2 \cap \Gamma = \{e\}$, $\{V\gamma\}_{\gamma\in\Gamma}$ is a family of pairwise disjoint subsets of G. Hence

$$E(x,y,b) \leqq c_2 \int_G \chi_{Q_1}(gx)\,\chi_{Q_1}(gy)\,\chi_{U_{\tau+\tau_0}}(gb)\,dg.$$

Now for any $\tau'>0$ let $G_{\tau'} = \{g\in G \mid gK \in U_{\tau'}\}$. We also fix a Borel cross-section $O: \Omega \to G$ of the canonical projection of G onto G/K, such that $O(\Omega)$ is contained in JWS_t. For any $b\in\Omega$ let b_0 denote $O(b)$. Then for $x, y\in X$ and $b\in\Omega$

$$\text{(5.6)}\quad E(x,y,b) \leqq c_2 \int_G \chi_{Q_1}(gx)\,\chi_{Q_1}(gy)\,\chi_{U_{\tau+\tau_0}}(gb_0K)\,dg$$
$$= c_2 \int_{G_{\tau+\tau_0}} \chi_{Q_1}(gb_0^{-1}x)\,\chi_{Q_1}(gb_0^{-1}y)\,dg.$$

Now consider the Cartan decomposition $G = K(\operatorname{cl} A^+)K$ where A^+ is the positive Weyl chamber in A. Let $g = k_1\alpha k$ be the expression for the generic element in G. Correspondingly the Haar measure dg has an expression $\nu(\alpha)\,dk_1\,d\alpha\,dk$ where $\nu(\alpha)$ is a function on $\operatorname{cl} A^+$, $d\alpha$ is a Haar measure on A and dk_1 and dk are normalized Haar measures on K.

Since $KQ_1 = Q_1$ the integral in (5.6) is invariant under the left action of K. Also $G_{\tau+\tau_0}$ is invariant under the left action. Therefore using the above decomposition of dg we get

$$\text{(5.7)}\quad E(x,y,b) \leqq \int_{A^+_{\tau+\tau_0}} \nu(\alpha)\,d\alpha \int_K \chi_{Q_1}(\alpha k b_0^{-1}x)\,\chi_{Q_1}(\alpha k b_0^{-1}y)\,dk$$

where $A^+_{\tau+\tau_0} = \{\alpha\in A^+ \mid \alpha K \in U_{\tau+\tau_0}\}$. Here we have also used the fact that the complement of A^+ in $\operatorname{cl} A^+$ has zero measure. Now for $\alpha\in A^+$, put $Q_\alpha = K\alpha^{-1}Q_1$. Then the integral over K appearing in (5.7) is nonzero only if $x, y\in b_0 Q_\alpha$. Further, using Schwartz lemma we may conclude that

$$\text{(5.8)}\quad \int_K \chi_{Q_1}(\alpha k b_0^{-1}x)\,\chi_{Q_1}(\alpha k b_0^{-1}y)\,dk$$
$$\leqq \chi_{Q_\alpha}(b_0^{-1}x)\,\chi_{Q_\alpha}(b_0^{-1}y)\Big\{\int_K \chi_{Q_1}(\alpha k b_0^{-1}x)\,dk\Big\}^{1/2}\Big\{\int_K \chi_{Q_1}(\alpha k b_0^{-1}y)\,dk\Big\}^{1/2}.$$

Fortunately for us an estimate of the integrals appearing in (5.8) which is sufficient for our purposes has indeed been obtained in [18].

(5.9) **Proposition** (Lemma 6.7 in [18]). *For any compact subset Q_1 of X there exist constants $\varepsilon>0$ and c_3 such that for all $\alpha\in A^+$ and $x\in X$*

$$\int_K \chi_{Q_1}(\alpha k x)\,dk \leqq c_3\,F(x)^\varepsilon\,F(\alpha)^\varepsilon.$$

Therefore the right hand side of (5.8) is majorized by

(5.10) $c_3^2\,\chi_{Q_\alpha}(b_0^{-1}x)\,\chi_{Q_\alpha}(b_0^{-1}y)\,F(b^{-1}x)^{\varepsilon/2}\,F(b^{-1}y)^{\varepsilon/2}\,F(\alpha)^\varepsilon.$

Putting together (5.3), (5.6) and (5.10) we now conclude that there exists a constant c_4 such that

$$(5.11)\quad \int_{U_\tau} f(gK)^2\,d\mu_K \leqq c_4 \int_\Omega d\mu_K(b) \int_{X\times X} F(b^{-1}x)^{1+\frac{\varepsilon}{2}}\,F(b^{-1}y)^{1+\frac{\varepsilon}{2}}\,d\pi(x)\,d\pi(y)$$
$$\int_{A_\tau^+ + \tau_0} \chi_{Q_\alpha}(b_0^{-1}x)\,\chi_{Q_\alpha}(b_0^{-1}y)\,F(\alpha)^\varepsilon\,\nu(\alpha)\,d\alpha$$
$$= c_4 \int_\Omega d\mu_K(b) \int_{A_\tau^+ + \tau_0} F(\alpha)^\varepsilon\,\nu(\alpha)\,d\alpha$$
$$\cdot\Big\{\int_X \chi_{Q_\alpha}(b_0^{-1}x)\,F(b^{-1}x)^{1+\frac{\varepsilon}{2}}\,d\pi(x)\Big\}^2.$$

Now consider the integral

$$I(b,\alpha) = \int_X \chi_{Q_\alpha}(b_0^{-1}x)\,F(b_0^{-1}x)^{1+\frac{\varepsilon}{2}}\,d\pi(x).$$

Recall that $b_0 \in JWS_t \subset JS^- W_0$ where W_0 is a compact subset of SH (cf. Corollary 2.6). In view of (4.1) there exists a constant c_5 such that for all $x' \in X$ and $w \in W_0$

$$F(w^{-1}x') \leqq c_5\,F(x')$$

Hence if $b_0 = jhw$ where $j \in J$, $h \in S^-$ and $w \in W_0$ then

$$(5.12)\quad I(b,\alpha) = \int_X \chi_{Q_\alpha}(w^{-1}h^{-1}j^{-1}x)\,F(w^{-1}h^{-1}j^{-1}x)^{1+\frac{\varepsilon}{2}}\,d\pi(x)$$
$$\leqq c_5^{1+\frac{\varepsilon}{2}} \int_X \chi_{W_0 Q_\alpha}(h^{-1}j^{-1}x)\,F(h^{-1}j^{-1}x)^{1+\frac{\varepsilon}{2}}\,d\pi(x)$$
$$= c_5^{1+\frac{\varepsilon}{2}} \int_X \chi_{W_0 Q_\alpha}(h^{-1}x)\,F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\,d\pi_j(x)$$

where π_j is the measure on X defined by

(5.13) $\pi_j(E) = \pi(jE)$

for any Borel subset E of X. Then π_j is $j^{-1}\Gamma j$-invariant and $\pi_j(X - P^-N/N) = 0$.

(5.14) **Proposition.** *There exists a compact subset E of H and a constant c_6 such that for all $j \in J$, $h \in S^-$ and any measurable subset Q of X*

$$\int_Q F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\,d\pi_j(x) \leqq c_6 \int_{EQ} F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\,d\mu(x).$$

Proof of this proposition is postponed until the next section. Using the Proposition we now complete the proof of Theorem 2.4. Firstly from (5.12) we have

$$I(b,\alpha)\leqq c_5^{1+\frac{\varepsilon}{2}} \int_{hW_0Q_\alpha} F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\, d\pi_j(x)$$

$$\leqq c_7 \int_{EhW_0Q_\alpha} F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\, d\mu(x) \quad (\text{where } c_7=c_6\, c_5^{1+\frac{\varepsilon}{2}})$$

$$=c_7 \int_{h(h^{-1}Eh)W_0Q_\alpha} F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\, d\mu(x)$$

$$\leqq c_7 \int_{hE_0W_0Q_\alpha} F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\, d\mu(x)$$

where $E_0=\bigcup_{h\in A^-} h^{-1}Eh$. Since E is a compact subset of H by Lemma 2.5, E_0 is also a relatively compact subset of H. How

$$\text{(5.15)}\quad I(b,\alpha)\leqq c_7 \int \chi_{E_0W_0Q_\alpha}(h^{-1}x)\, F(h^{-1}x)^{1+\frac{\varepsilon}{2}}\, d\mu(x)$$

$$=c_7 \int \chi_{E_0W_0Q_\alpha}(x)\, F(x)^{1+\frac{\varepsilon}{2}}\, d\mu(x).$$

Using the Iwasawa decomposition $G=KAN$, X may be identified with $K\times A$ (topologically). It is well-known that under this identification $d\mu(x)$ has the expression (up to a scalar multiple) $\exp 2\rho H(a)\, dk\, da$. Since $F(x)=\exp -2\rho H(a(x))$ it follows that

$$\text{(5.16)}\quad \int_X \chi_{E_0W_0Q_\alpha}(x)\, F(x)^{1+\frac{\varepsilon}{2}}\, d\mu(x)=\int_{E_0W_0Q_\alpha} F(x)^{\varepsilon/2}\, dk\, da$$

$$=\int_{E_0W_0K\alpha^{-1}Q_1} F(x)^{\varepsilon/2}\, dk\, da.$$

(5.17) **Proposition** (Lemma 6.21 in [18]). *There exists a constant c_8 depending only on $E'=E_0W_0$, Q_1 and ε such that for all $\alpha\in A^+$*

$$\int_{E'K\alpha^{-1}Q_1} F(x)^{\varepsilon/2}\, dk\, da\leqq c_8\, F(\alpha)^{-\varepsilon/2}.$$

Combining (5.11), (5.15), (5.16) and Proposition 5.17, we now have

$$\int_{U_\tau} f(gK)^2\, d\mu_K\leqq c_9 \int_\Omega d\mu_K(b) \int_{A^+_{\tau+\tau_0}} v(\alpha)\, d\alpha$$

where c_9 is a suitable constant. Now recall that $\mu_K(\Omega)<\infty$ and that $\mu(U_{\tau+\tau_0})\leqq c(\tau_0)\,\mu_K(U_\tau)$. Thus finally we may conclude that there exists a constant c such that for all $\tau>0$

$$\int_{U_\tau} f(gK)^2\, d\mu_K\leqq c\,\mu(U_\tau).$$

By Remark 4.10 this implies that π is G-invariant, thus proving Theorem 2.4.

§ 6. A Convolution Technique

In this section we develop some techniques to compare certain integrals with respect to a measure which is invariant under a lattice with the corresponding integrals with respect to a G-invariant measure. The ideas have their origin in [8] and [18].

In the sequel we let Γ, U, U^-, P, P^-, S, S^-, N, J etc. be the same as before. Recall that their choice depended on Γ. However we now let Λ denote an arbitrary lattice in G. The results in the sequel will be applicable when we set $\Lambda = j^{-1}\Gamma j, j \in J$. In what follows μ' will denote a G-invariant measure on $X' = G/U$.

(6.1) **Lemma.** *Let σ be a Λ-invariant, Λ-finite (cf. § 1 for definition) measure on X'. Let V be a compact neighborhood of the identity e in G. Then there exists a constant d_1 such that for any non-negative measurable function φ on X'*

$$\int_{X'}\int_{G} \varphi(gx)\,\chi_V(g)\,dg\,d\sigma(x) \leqq d_1 \int_{X'} \varphi(x)\,d\mu'(x).$$

χ_V being the characteristic function of V and dg being a Haar measure on G.

Proof. For any $\varphi \in C_c(X')$ put

$$\nu(\varphi) = \int_{X'}\int_{G} \varphi(gx)\,\chi_V(g)\,dg\,d\sigma(x).$$

Then ν defines a (locally finite, Borel) measure on X'. It is easy to see, as in § 1, that ν is absolutely continuous with respect to μ'. To prove the Lemma we only need to show that the Radon-Nikodym derivative is bounded (a.e.).

Let σ' and ν' be the measures on G corresponding to σ and ν respectively under the correspondence introduced in Proposition 1.6. Clearly it is enough to prove that there exists a bounded measurable function ζ which is invariant under the right action of N and is a Radon-Nikodym derivative for ν' with respect to the Haar measure dg on G.

It is straightforward to verify from the definitions of σ' and ν' that for any $\psi \in C_c(G)$

$$(6.2)\quad \int_{G} \psi(y)\,d\nu'(y) = \int_{G}\int_{G} \psi(gy)\,\chi_V(g)\,dg\,d\sigma'(y).$$

Using right invariance of dg we get

$$(6.3)\quad \begin{aligned} \int_{G} \psi(y)\,d\nu'(y) &= \int_{G}\int_{G} \psi(g)\,\chi_V(gy^{-1})\,dg\,d\sigma'(y) \\ &= \int_{G}\int_{G} \psi(g)\,\chi_{V^{-1}g}(y)\,dg\,d\sigma'(y) \\ &= \int_{G} \psi(g)\,\sigma'(V^{-1}g)\,dg. \end{aligned}$$

In other words $\sigma(V^{-1}g)$ is a representative in the a.e. class $\frac{d\nu'}{dg}$. Also $\sigma(V^{-1}g)$ is invariant under the right action of N. We now show that $\zeta(g) = \sigma'(V^{-1}g)$ is a bounded function on G.

Let D be a compact, symmetric neighbourhood of e in G such that for any $\gamma \in \Lambda - \{e\}$ the sets D and $D\gamma$ are disjoint. Since V is compact there exist $g_1, g_2, \ldots, g_k$ in G such that $V \subset \bigcup_1^k g_i D$. Hence $V^{-1} g \subset \bigcup_1^k D g_i^{-1} g$. Hence it is enough to prove that $\sigma'(Dg)$ is a bounded function on G. Observe that for any $\gamma \in \Lambda - \{e\}$ and $g \in G$ the sets Dg and γDg are disjoint. Hence if σ'' is the measure on $\Lambda \backslash G$ corresponding to σ' then $\sigma'(Dg) \leqq \sigma''(\Lambda \backslash G)$. Since σ is Λ-finite $\sigma''(\Lambda \backslash G) < \infty$. Hence $\sigma'(Dg)$ is bounded.

(6.4) **Corollary.** *Let σ be as in Lemma 6.1. and let V be any compact neighbourhood of e in G. Then there exists a constant d_2 such that for any measurable subset E of X'*

$$\sigma(E) \leqq d_2 \mu'(VE).$$

Proof. Let E be any measurable subset of X'. Then for all $g \in V$ we have

$$\chi_E(x) \leqq \chi_{VE}(gx)$$

for all $x \in X$. Thus for every $g \in V$ we have

$$\sigma(E) = \int_{X'} \chi_E(x)\, d\sigma(x) \leqq \int_{X'} \chi_{VE}(gx)\, d\sigma(x).$$

Integrating both sides of the inequality over V we get

$$\sigma(E) \leqq d' \int_V \int_{X'} \chi_{VE}(gx)\, d\sigma(x)\, dg$$

where d' is the inverse of the Haar measure of V. By Lemma 6.1 (and Fubini theorem) it follows that

$$\sigma(E) \leqq d'\, d_1 \mu'(VE).$$

We now specialize to Λ-invariant measures on $X' = G/U$ for which the measure of $X' - P^- U/U$ is zero. Denote $Y = P^- U/U$. Y is the unique open orbit of P^- on X' and its complement is a union of lower dimensional manifolds. The restriction of the G-invariant measure μ' on G/U to Y, being P^--invariant, is a constant multiple of the quotient of $d_l h$ on Y. Here $d_l h$ denotes a left Haar measure on P^-. By choosing $d_l h$ appropriately we may assume, as we do, that for any measurable function φ on Y

$$\text{(6.5)} \quad \int_{P^-} \varphi(hU)\, d_l h = \int_Y \varphi(y)\, d\mu'(y)$$

(6.6) **Lemma.** *Let Λ be any lattice in G such that $\Lambda \cap U^-$ is a lattice in U^- and let σ be a Λ-invariant measure on X' such that $\sigma(X' - Y) = 0$. Then there exists a neighbourhood Ω of the identity e in P^- and a constant d_3 such that for any non-negative measurable function φ on Y*

$$\int_Y \int_{P^-} \varphi(hy)\, \chi_\Omega(h)\, d_l h\, d\sigma(y) \leqq d_3 \int_Y \varphi(y)\, d\mu'(y).$$

Proof. Let H be the connected component of e in P^-. Then $H=U^-\cdot Z$ (semidirect product) where Z is the connected component of e in the centralizer of S in G. Let $D\subset U^-$, $F\subset Z$ and $V\subset G$ be compact neighbourhoods of e in the respective subgroups such that DF is contained in V and $V^{-1}V\cap\Lambda=\{e\}$. Put $\Omega=(DF)^{-1}$. Let ν be the measure on Y defined by

$$(6.7)\quad \int_Y \varphi(y)\,d\nu(y)=\int_Y\int_{P^-}\varphi(hy)\,\chi_\Omega(h)\,d_lh\,d\sigma(x).$$

We claim that $\nu\prec\mu'/Y$, the restiction of μ to Y. For if $E\subset Y$ is a Borel set such that $\mu'(E)=0$, then for any $y\in Y$ say $y=h_0U$

$$\begin{aligned}\int_{P^-}\chi_E(hy)\,d_lh&=\int_{P^-}\chi_E(hh_0U)\,d_lh\\&=\delta(h_0)\int_{P^-}\chi_E(hU)\,d_lh\\&=\delta(h_0)\,\mu'(E)\qquad\text{(by (6.5))}\\&=0.\end{aligned}$$

Here and in the sequel δ denotes the modular homomorphism of P^-; i.e. for any integrable function θ on P^- and $h_0\in P^-$

$$(6.8)\quad \int_{P^-}\theta(hh_0)\,d_lh=\delta(h_0)\int_{P^-}\theta(h)\,d_lh.$$

To prove the Lemma it is enough to prove that the Radon-Nikodym derivative $d\nu/d\mu'$ (on Y) is bounded. Observe that the map $\tau\colon P^-\to Y$ defined by $h\to hy_0$ where $y_0\in Y$ is fixed (arbitrarily) is bijective. Therefore using τ we now identify Y with P^-. It is easy to see that under this identification the Radon-Nikodym derivative may be written as

$$(6.9)\quad \frac{d\nu}{d\mu'}(h)=\int_Y\chi_\Omega(hy^{-1})\,\delta(y)\,d\sigma(x).$$

Since $\Omega\subset H$ is compact and δ is a continuous homomorphism, there exists a constant d_4 such that whenever $hy^{-1}\in\Omega$, $\delta(y)\leqq d_4\delta(h)$. Therefore

$$\begin{aligned}(6.10)\quad \frac{d\nu}{d\mu'}(h)&\leqq d_4\,\delta(h)\int_Y\chi_\Omega(hy^{-1})\,d\sigma(y)\\&=d_4\delta(h)\,\sigma(\Omega^{-1}h).\end{aligned}$$

Recall that $\Omega^{-1}=DF$ and consider the function $\sigma(\Omega^{-1}h)=\sigma(DFh)$. Let $p_1,p_2,\dots,p_k$ be a set of representatives for P^-/H. Then any element h of P^- can be uniquely represented as $h=u^-p_jz$ where $u^-\in U^-$, $z\in Z$ and $1\leqq j\leqq k$.

Since $\Lambda\cap U$ is a lattice in U^- there exists a relatively compact Borel subset C of U^- such that i) for $\gamma\in U^-\cap\Lambda$, $\gamma\neq e$ the sets C and γC are disjoint and ii) $U^-=\bigcup_{\gamma\in U^-\cap\Lambda}\gamma C$, that is C is a fundamental domain for $U^-\cap\Lambda$ in U^-. Recall also that any lattice in a nilpotent Lie group is uniform. Hence a relatively compact fundamental domain exists. Now let $h\in P^-$ and $h=u^-p_jz$ be the representation as above. For each $\zeta \in F$ and $\gamma \in U^- \cap \Lambda$ denote

$$C(\gamma, \zeta) = \{\gamma^{-1} D \zeta u^- \zeta^{-1}\} \cap C$$

and for each $\gamma \in U^- \cap \Lambda$ let

$$C_\gamma = \bigcup_{\zeta \in F} C(\gamma, \zeta) \zeta p_j z.$$

It is easy to see that for each $\gamma \in U^- \cap \Lambda$, C_γ is a Borel subset of P^-. Clearly $\{\gamma C_\gamma\}_{\gamma \in U^- \cap \Lambda}$ is a family of pairwise disjoint sets whose union is $DFu^- p_j z$. Therefore using Λ-invariance of σ we now get

$$(6.11) \quad \sigma(\Omega^{-1} h) = \sigma(DFu^- p_j z) = \sum_{\gamma \in U^- \cap \Lambda} \sigma(\gamma C_\gamma) = \sum_{\gamma \in U^- \cap \Lambda} \sigma(C_\gamma).$$

We next claim that C_γ, $\gamma \in U^- \cap \Lambda$ are pairwise disjoint. For otherwise let $\gamma \neq \gamma'$ be such that $h_0 = u_0^- p_l z_0 \in C\gamma \cap C\gamma'$ where $u_0^- \in U^-$, $z_0 \in Z$ and $1 \leqq l \leqq k$. In other words there exist $\zeta, \zeta' \in F$, $u_1^- \in C(\gamma, \zeta)$ and $u_2^- \in C(\gamma', \zeta')$ such that

$$(6.12) \quad h_0 = u_0^- p_l z_0 = u_1^- \zeta p_j z = u_2^- \zeta' p_j z.$$

We may write $\zeta p_j = n_1^- p_j z_1$ and $\zeta' p_j = n_2^- p_j z_2$ where $n_1^-, n_2^- \in U^-$ and $z_1, z_2 \in Z$. Substituting in (6.12) and using uniqueness of the representation we get $u_1^- n_1^- = u_2^- n_2^-$ and $z_1 = z_2$. But then $\zeta' \zeta^{-1} = (\zeta' p_j)(\zeta' p_j)^{-1} = n_2^- (n_1^-)^{-1} = u_2^- (u_1^-)^{-1}$. Since $H = U^- \cdot Z$ is a semidirect product it follows that $\zeta = \zeta'$ and $u_1^- = u_2^-$. However this means that $\gamma^{-1} D \cap \gamma'^{-1} D$ is non-empty as it contains $u_1^- (\zeta u^- \zeta^{-1})^{-1}$. But since $D \subset V$ and $\{\gamma V, \gamma \in \Lambda\}$ is a family of pairwise disjoint sets we arrive at a contradiction. Therefore C_γ, $\gamma \in U^- \cap \Lambda$ are pairwise disjoint. Also each C_γ is contained in $CFp_j z$. Therefore by (6.11)

$$\sigma(\Omega^{-1} h) \leqq \sigma(CF_0 z)$$

where $F_0 = \bigcup_1^k F p_j$. Recalling the identifications and using Corollary 6.4 we conclude that

$$(6.13) \quad \sigma(\Omega^{-1} h U) = \sigma(CF_0 z U) \leqq d_2 \mu'(VCF_0 z U).$$

Recall that since Z normalizes U there is an action $z \to \psi_z$ of Z on $X' = G/U$ given by $\psi_z(gU) = gzU$. Since this action commutes with the G-action on left there exists a character $\chi\colon Z \to \mathbb{R}^+$ such that for any Borel subset E of X', $\mu'(\psi_z(E)) = \chi(z) \mu'(E)$. We claim that $\chi(z) = \delta(z)^{-1}$. To see this let Q be a compact subset of H with non-empty interior and let $E = QU/U$. Then E is a compact subset Y with non-empty interior. Then in view of (6.5) we have

$$\chi(z)\,\mu'(E)=\mu'(\psi_z(E))=\mu'(QzU/U)=\int_H \chi_Q(hz^{-1})\,d_l h$$
$$=\delta(z^{-1})\int_H \chi_Q(h)\,d_l h=\delta(z^{-1})\,\mu'(E).$$

Since $0<\mu'(E)<\infty$ we must have $\chi(z)=\delta(z^{-1})$. Using (6.10) and (6.13) we now conclude that

$$(6.14)\quad \frac{d\nu}{d\mu'}(h)\leqq d_4 d_2\,\delta(h)\,\mu'(VCF_0 zU/U)$$
$$=d\,\delta(h)\,\delta(z^{-1})$$

where $d=d_4 d_2\mu'(VCF_0 U/U)$. Since VCF_0 is a relatively compact subset, d is finite. It is easy to see that the restriction of the modular homomorphism δ to U^- is identically 1. Hence $\delta(h)=\delta(u^- p_j z)=\delta(p_j)\,\delta(z)$. Therefore by (6.14) $d\nu/d\mu'\leqq d \sup_{1\leqq j\leqq k} \delta(p_j)$. This proves Lemma 6.6.

(6.15) **Corollary.** *Let σ be a Λ-invariant measure on $X=G/N$ such that $\sigma(X-P^- N/N)=0$. Then there exists a neighbourhood Ω of e in H and a constant $d_3'>0$ such that for any non-negative measurable function φ on X*

$$\int_X\int_H \varphi(hx)\,\chi_\Omega(h)\,d_l h\leqq d_3'\int_X \varphi(x)\,d\mu(x).$$

Proof. Let Ω be the neighbourhood of e in H as in Lemma 6.6. In view of the one-one correspondence of measures on X and X' with those on G introduced in Proposition 1.6 the corollary is an obvious consequence of Lemma 6.6.

§ 7. Proof of Proposition (5.14)

Recall that the function F on X involved in the integrals in Proposition 5.14 is defined by the (right) N-invariant function on G, (also denoted by F) defined by

$$F(g)=\exp -2\rho H(g)$$

where ρ and $H(g)$ are as introduced in the beginning of § 4. We now need the following alternative formulation of F.

Let $\mathfrak{G}$ be the Lie algebra of G and let $\mathfrak{G}_{\mathbb{C}}$ be its complexification. Let $\mathscr{C}$ be a Cartan subalgebra of $\mathfrak{G}_{\mathbb{C}}$ containing the Lie subalgebra $\mathfrak{A}$ of A. Let $\Delta_{\mathbb{C}}^+$ be a system of positive roots for the pair $(\mathfrak{G}_{\mathbb{C}},\mathscr{C})$ such that the set of restrictions of elements of $\Delta_{\mathbb{C}}^+$ to $\mathfrak{A}$ contains all roots of $\mathfrak{A}$ on $\mathscr{N}$, the Lie subalgebra of N. Let $\Lambda\in\mathscr{C}^*$, the dual of $\mathscr{C}$ be such that 2Λ is the sum of all roots in $\Delta_{\mathbb{C}}^+$. Then there exists a unique finite dimensional irreducible representation $d\theta$ of $\mathfrak{G}_{\mathbb{C}}$ with highest weight Λ, say on a vector space $\mathscr{L}$. Let θ be the corresponding representation of G on $\mathscr{L}$. Then $\mathscr{L}$ is endowed with a norm $\|\cdot\|$ such that for all $g\in G$

(7.1) $\quad \|\theta(g)\,v_0\| = \exp\rho H(g)$

where v_0 is an element in the weight space corresponding to the highest weight.

Now let as before H be the connected component of e in P^- and let $\mathfrak{H}$ be the Lie subalgebra corresponding to H. Using (7.1) in this section we first prove the following result.

(7.2) **Lemma.** *Let σ be any measure on $X=G/N$. Let $\eta\in\mathfrak{H}$ be a simultaneous eigenvector of $\{\operatorname{Ad} a\,|\,a\in A\}$ such that $\operatorname{ad}\eta$ is a nilpotent transformation of $\mathfrak{G}$. Let q and $t_0>0$ be given. Let $E_\eta=\{\exp t\eta\,|\,|t|\leqq t_0\}$. Then there exists a constant d_5 such that for any Borel subset Q of X and $\xi\in S^-$ (cf. § 2 for notation)*

$$\int_Q F(\xi^{-1}x)^q\,d\sigma(x) \leqq d_5 \int_{E_\eta Q} F(\xi^{-1}x)^q\,d\sigma_\eta(x)$$

where σ_η is the measure on X such that for all $\varphi\in C_c(X)$

$$\int_X \varphi(x)\,d\sigma_\eta(x) = \int_X \int_{\mathbb{R}} \varphi(\exp t\eta\cdot x)\,\chi_{E_\eta}(\exp t\eta)\,dt\,d\sigma(x).$$

Proof. Since η is a nilpotent element of $\mathfrak{G}$, $\tau=d\theta(\eta)$ is a nilpotent linear transformation of $\mathscr{L}$. Consequently there exists $p\in\mathbb{Z}^+$ such that

(7.3) $\quad \theta(\exp t\eta) = \sum_{i=0}^{p-1} a_i t^i \tau^i$

where a_i, $0\leqq i<p$ are the first p coefficients in the exponential series. For $v\in\mathscr{L}$ and $t'\in\mathbb{R}$, put

$$\begin{aligned}\mathscr{P}(v,t') &= \frac{d}{dt}\,\|\theta(\exp t\eta)\cdot\theta(\exp t'\eta)\,v\|\,\big|_{t=0}\\ &= \frac{d}{dt}\,\|\theta(\exp t\eta)\,v\|\,\big|_{t=t'}.\end{aligned}$$

We claim that for any (fixed) $v\in\mathscr{L}$ either $\mathscr{P}(v,t')=0$ for all $t'\in\mathbb{R}$ or the number of solutions of $\mathscr{P}(v,t')=0$ is bounded by $2p$. We first observe that for $v\neq 0$, $\mathscr{P}(v,t')=0$ if and only if

$$\frac{d}{dt}\,\|\theta(\exp t\eta)\,v\|^2\,\big|_{t=t'} = 0.$$

Now let $\{v_j\}_1^n$ be coordinates of v with respect to an orthonormal basis of $\mathscr{L}$. Then in view of (7.1) the left hand side is a polynomial in t' whose degree is at most $2p$. Hence if $\mathscr{P}(v,t')$ is not identically zero it can have at most $2p$ solutions.

Now for any $t_1>0$ define

$$\begin{aligned}\mathscr{L}_0(t_1) &= \{v\in\mathscr{L}\,|\,\mathscr{P}(v,t')=0 \text{ for some } t'\in[-t_1,t_1]\},\\ \mathscr{L}^+(t_1) &= \{v\in\mathscr{L}\,|\,\mathscr{P}(v,0)>0\} - \mathscr{L}_0(t_1)\end{aligned}$$

and

$$\mathscr{L}^-(t_1) = \{v \in \mathscr{L} \mid \mathscr{P}(v,0) < 0\} - \mathscr{L}_0(t_1).$$

Then obviously for any $t_1 > 0$ the above three sets constitute a partition of $\mathscr{L}$.

(7.5) **Sub-Lemma.** *In above notation if either $v \in \mathscr{L}^+(t_0)$ and $t \in [-t_0, 0]$ or $v \in \mathscr{L}^-(t_0)$ and $t \in [0, t_0]$ then*

$$\|\theta(\exp t\eta)\, v\| \leqq \|v\|.$$

Proof. We consider the former case; the latter can be dealt with similarly. Since $v \in \mathscr{L}^+(t_0)$ either the desired inequality holds for all $t \leqq 0$ or there exists $t' > 0$ such that

(7.6) $\quad \|\theta(\exp t\eta)\, v\| < \|v\| \qquad \text{for } t \in (-t', 0)$

and

$$\|\theta(\exp t'\eta)\, v\| = \|v\|.$$

By Rolle's theorem there exists $t'' \in (-t', 0)$ such that

$$\mathscr{P}(v, t'') = \frac{d}{dt} \|\theta(\exp t\eta)\, v\|\,|_{t=t''} = 0.$$

Since $v \notin \mathscr{L}_0(t_0)$ we must have $-t'' > t_0$. Hence $t' > -t'' > t_0$ in view of (7.6) the claim is proved.

(7.7) **Corollary.** *Let $\xi \in S^-$. Then if either $v \in \theta(\xi)\, \mathscr{L}^-(t_0)$ and $t \in [0, t_0]$ or $v \in \theta(\xi)\, \mathscr{L}^+(t_0)$ and $t \in [-t_0, 0]$ then*

$$\|\theta(\xi^{-1} \exp t\eta)\, v\| \leqq \|\theta(\xi^{-1})\, v\|.$$

Proof. Let λ be the character on A such that $(\operatorname{Ad} a)\,\eta = \lambda(a)\,\eta$ for all $a \in A$. Since $\eta \in \mathfrak{H}$ it follows that for any $\xi \in S^-$, $0 < \lambda(\xi^{-1}) \leqq 1$. Thus if $t \in [-t_0, 0]$ then $\xi^{-1}(\exp t\eta)\,\xi = \exp t\lambda(\xi^{-1})\,\eta = \exp t'\eta$ for some $t' \in [-t_0, 0]$. Hence by Sub-Lemma 7.5, in either of the cases we get

$$\begin{aligned} \|\theta(\xi^{-1} \exp t\eta)\, v\| &= \|\theta(\xi^{-1} (\exp tn)\, \xi)\, \theta(\xi^{-1})\, v\| \\ &\leqq \|\theta(\xi^{-1})\, v\|. \end{aligned}$$

(7.8) **Sub-Lemma.** *Let $t_0 > 0$ be given. Then there exists a function ζ defined on $\mathscr{L}_0(2t_0)$ and a constant $d_6 > 0$ such that* i) *for $v \neq 0$, $\zeta(v) > 0$*

ii) *for all $v \in \mathscr{L}_0(t_0)$ and $t \in [-t_0, t_0]$, $\zeta(\theta(\exp t\eta)\, v) = \zeta(v)$ and*

iii) *$v \in \theta(\xi)\, \mathscr{L}_0(t_0)$ where $\xi \in S^-$ and $t \in [-t_0, t_0]$,*

$$d_6^{-1}\, \zeta(\theta(\xi^{-1})\, v) \leqq \|\theta(\xi^{-1} \exp t\eta)\, v\| \leqq d_6\, \zeta(\theta(\xi^{-1})\, v).$$

Proof. On $\mathscr{L}_0(2t_0)$ consider the partition D whose elements are connected components of $\{\theta(\exp t\eta)\, v \mid t \in \mathbb{R}\} \cap \mathscr{L}_0(2t_0)$, $v \in \mathscr{L}_0(2t_0)$. Let $\sim$ denote the corresponding equivalence relation. Now for $v \in \mathscr{L}_0(2t_0)$ define

(7.9) $\zeta(v) = \inf\{\|v'\| \mid v' \in \mathscr{L}_0(2t_0) \text{ and } v' \sim v\}$.

We first observe that since η is nilpotent either $\|\theta(\exp t\eta) v\| \to \infty$ as $t \to \pm\infty$ or v is a fixed point of $\theta(\exp t\eta)$, $t \in \mathbb{R}$. Hence for any $v \neq 0$, $\zeta(v) > 0$.

Now if $v \in \mathscr{L}_0(t_0)$ and $t \in [-t_0, t_0]$ then $\theta(\exp t\eta) v \in \mathscr{L}_0(2t_0)$ and $\theta(\exp t\eta) v \sim v$. Therefore ζ satisfies ii). Next let $v \in \mathscr{L}_0(t_0)$ and consider the function $\mathscr{P}(v,t)$. First suppose $\mathscr{P}(v,t) = 0$ for all $t \in \mathbb{R}$. Then $\|\theta(\exp t\eta) v\|$ is the constant function $\|v\|$. On the other hand if $\mathscr{P}(v,t)$ is not identically zero then, as observed earlier, the set of solutions of $\mathscr{P}(v,t) = 0$ has at most $2p$ elements. Consequently the "length" of the equivalence class $D(v)$ of v, defined by

$$l(v) = \operatorname{Sup}\left\{ |t_2 - t_1| \;\middle|\; \begin{array}{l} \theta(\exp t_1\eta) v, \theta(\exp t_2\eta) v \in \mathscr{L}_0(2t_0) \\ \text{and } \theta(\exp t_1\eta) v \sim \theta(\exp t_2\eta) v \end{array} \right\}$$

satisfies $l(v) \leqq 8pt_0$. Therefore if we put

$$d_6 = \sup\{\|\theta(\exp t\eta) v'\| / \|v'\| \mid v' \in \mathscr{L}, |t| \leqq 8pt_0\}$$

then for any $v_1, v_2 \in D(v)$ we have

$$d_6^{-1} \|v_1\| \leqq \|v_2\| \leqq d_6 \|v_1\|.$$

Also by compactness of $D(v)$, $\zeta(v) = \|v'\|$ for some $v' \in D(v)$. Thus we have proved iii) in Lemma 7.8 for the case when $\xi = e$. For arbitrary $\xi \in S^-$, iii) can be proved by applying the above to $\theta(\xi^{-1}) v$ and observing as in Corollary 7.7 that $\{\xi^{-1}(\exp t\eta)\xi \mid |t| \leqq t_0\}$ is contained in $\{(\exp t\eta) \mid |t| \leqq t_0\}$.

We now complete the proof of Lemma 7.2. Let Q be a Borel subset of $X = G/N$. Let $v_0 \in \mathscr{L}$ be as in (7.1). Now for $\xi \in S^-$ define

$$Q_0(\xi) = \{gN \in Q \mid \theta(g) v_0 \in \theta(\xi) \mathscr{L}_0(t_0)\},$$
$$Q^+(\xi) = \{gN \in Q \mid \theta(g) v_0 \in \theta(\xi) \mathscr{L}^+(t_0)\},$$
$$Q^-(\xi) = \{gN \in Q \mid \theta(g) v_0 \in \theta(\xi) \mathscr{L}^-(t_0)\}.$$

Then for each ξ the above three sets form a partition of Q. We intend to prove the inequality in Lemma 7.2 by proving it separately on each of these sets.

Firstly consider $Q^+(\xi)$. Since $F(\xi^{-1}x) = \|\theta(\xi^{-1}g) v_0\|^{-2}$ in view of Corollary 7.7 for $x \in Q^+(\xi)$ and $t \in [-t_0, 0]$ we get

$$\begin{aligned} F(\xi^{-1}x) &\leqq \|\theta(\xi^{-1}(\exp t\eta) g) v_0\|^{-2} \\ &= F(\xi^{-1}(\exp t\eta) x). \end{aligned}$$

Let $E_\eta^- = \{\exp t\eta \mid t \in [-t_0, 0]\}$. Then for given $q > 0$ and $t \in [-t_0, 0]$, we have

$$\chi_{Q^+(\xi)}(x) F(\xi^{-1}x)^q \leqq \chi_{E_\eta^- Q}((\exp t\eta) x) F(\xi^{-1}(\exp t\eta) x)^q.$$

for all $x \in X$. Therefore

(7.10) $$\int_{Q^+(\xi)} F(\xi^{-1}x)^q\,d\sigma(x)=\frac{1}{t_0}\int_{E_\eta^-} dt\int_X \chi_{Q^+(\xi)}(x)\,F(\xi^{-1}x)^q\,d\sigma(x)$$
$$\leqq\frac{1}{t_0}\int_{E_\eta^-} dt\int_X \chi_{E_\eta^- Q}((\exp t\eta)\,x)\,F(\xi^{-1}(\exp t\eta)\,x)^q\,d\sigma(x)$$
$$\leqq\frac{1}{t_0}\int_{E_\eta Q} F(\xi^{-1}x)^q\,d\sigma_\eta(x)$$

where σ_η and E_η are as in the statement of the lemma. Similarly we can prove

(7.11) $$\int_{Q^-(\xi)} F(\xi^{-1}x)^q\,d\sigma(x)\leqq\frac{1}{t_0}\int_{E_\eta Q} F(\xi^{-1}x)^q\,d\sigma_\eta(x).$$

Now consider $\int_{Q_0(\xi)} F(\xi^{-1}x)^q\,d\sigma(x)$. For $x\in Q_0(\xi)$, in view of iii) in Sub-Lemma 7.8 and (7.1), for $t\in[-t_0,t_0]$ we have

$$d_6^{-2}\zeta^{-2}(\xi^{-1}x)\leqq F(\xi^{-1}(\exp t\eta)x)\leqq d_6^2\zeta^{-2}(\xi^{-1}x).$$

Hence in particular for any $t\in[-t_0,t_0]$, $x\in Q_0(\xi)$

$$F(\xi^{-1}x)^q\leqq d_6^{2q}\zeta^{-2}(\xi^{-1}x)^q\leqq d'F(\xi^{-1}(\exp t\eta)\,x)^q$$

where $d'=d_6^{4q}$. Hence for any $t\in[-t_0,t_0]$ and $x\in X$

$$\chi_{Q_0(\xi)}(x)\,F(\xi^{-1}x)^q\leqq d'\chi_{E_\eta Q}((\exp t\eta)\,x)\,F(\xi^{-1}(\exp t\eta)\,x)^q.$$

Integrating both sides with respect to x and $t\in[-t_0,t_0]$ we conclude that

(7.12) $$\int_{Q_0(\xi)} F(\xi^{-1}x)^q\,d\sigma(x)\leqq\frac{d'}{2t_0}\int_{E_\eta Q} F(\xi^{-1}x)^q\,d\sigma_\eta(x).$$

Combining (7.10), (7.11) and (7.12) we conclude that there exists a constant d_5 (independent of ξ) such that

$$\int_Q F(\xi^{-1}x)^q\,d\sigma(x)\leqq d_5\int_{E_\eta Q} F(\xi^{-1}x)^q\,\sigma_\eta(x).$$

(7.13) **Lemma.** *Let $G=KAN$ be the Iwasawa decomposition as fixed in §2. Let M be the subgroup of K consisting of elements which centralize A. Let Σ be a compact neighbourhood of e in MA. Let σ be any measure on $X=G/N$ and let $q>0$ be given. Then there exists a constant d_7 such that for any Borel subset Q of X*

$$\int_Q F(\xi^{-1}x)^q\,d\sigma(x)\leqq d_7\int_{\Sigma Q} F(\xi^{-1}x)^q\,d\sigma_\Sigma(x)$$

where σ_Σ is the measure on X defined by putting for $\varphi\in C_c(X)$

$$\int_X \varphi(x)\,d\sigma_\Sigma(x)=\int_X\int_{MA}\varphi(zx)\,\chi_\Sigma(z)\,dz\,d\sigma(x)$$

dz being a fixed Haar measure on MA.

Proof. In the notation used earlier in the section define

$$d_8 = \mathrm{Sup}\{\|\theta(z)v\|/\|v\| \mid v\in\mathcal{L}-(0), z\in\Sigma\}.$$

Since Σ is compact d_8 is finite. Then using (7.1) for any $z\in Z$, $\xi\in S^-$ and $x=gN\in X$ we have

$$\begin{aligned} F(\xi^{-1}x) &= \|\theta(\xi^{-1}g)v_0\|^{-2} \\ &\leqq d_8^2\,\|\theta(z\xi^{-1}g)v_0\|^{-2} \\ &= d_8^2\,F(z\xi^{-1}x) \\ &= d_8^2\,F(\xi^{-1}zx). \end{aligned}$$

For the last step recall that $S\subset A$. Now as before for the Borel set Q and $z\in\Sigma$ we can write

$$\chi_Q(x)\,F(\xi^{-1}x)^q \leqq d_8^{2q}\,\chi_{\Sigma Q}(zx)\,F(\xi^{-1}zx)^q \ldots \qquad \text{for all } x\in X.$$

Let d_9 be the inverse of the Haar measure of Σ in MA. Then

$$\begin{aligned} \int_Q F(\xi^{-1}x)^q\,d\sigma(x) &= d_9\int_\Sigma dz\int_X \chi_Q(x)\,F(\xi^{-1}x)^q\,d\sigma(x) \\ &\leqq d_9 d_8^{2q}\int_X\int_\Sigma \chi_{\Sigma Q}(zx)\,F(\xi^{-1}zx)^q\,dz\,d\sigma(x) \\ &= d_7\int_{\Sigma Q} F(\xi^{-1}x)^q\,d\sigma_\Sigma(x) \end{aligned}$$

where $d_7 = d_9 d_8^{2q}$.

(7.14) **Proposition.** *Let Λ be a lattice in G such that $\Lambda\cap U^-$ is a lattice in U^-. Let σ be a Λ-invariant measure on $X=G/N$ such that $\sigma(X-Y)=0$ where $Y=P^-N/N$. Let $q>0$ be given. Then there exists a compact neighbourhood E of e in $H\subset P^-$ and a constant d such that for any $\xi\in S^-$ and any Borel subset Q of X*

$$\int_Q F(\xi^{-1}x)^q\,d\sigma(x) \leqq d\int_{EQ} F(\xi^{-1}x)^q\,d\mu(x).$$

Proof. Let Ω be a compact neighbourhood of e in H, (the connected component of P^-) for which Corollary 6.15 is satisfied. In the Lie subalgebra $\mathfrak{H}$ of H there exists elements $\eta_1, \eta_2, \ldots, \eta_r$, which are nilpotent elements of $\mathfrak{G}$ and such that the map $\mathcal{M}$: $\mathbb{R}^r\times MA\to H$ defined by $(t_1, t_2, \ldots, t_r, z)\mapsto(\exp t_1\eta_1)\cdot(\exp t_2\eta_2)\ldots(\exp t_r\eta_r)\cdot z$ is regular at $(0,0,\ldots,0,e)$. Hence there exists a neighbourhood Σ of e in MA and t_0 such that if $R=\{(t_1,t_2,\ldots,t_r,z)\,\|\,|t_i|\leqq t_0$ for $1\leqq i\leqq r$ and $z\in\Sigma\}$ and $E_i=E_{\eta_i}=\{\exp t\eta_i \mid |t_i|\leqq t_0\}$ then $\mathcal{M}/R$ is a diffeomorphism onto $E=E_1E_2\cdot\ldots\cdot E_r\Sigma$ and the latter is a neighbourhood of e in H. By choosing smaller Σ and t_0 we may also assume that $E\subset\Omega$.

Applying Lemma 7.13 and then Lemma 7.2 repeatedly we conclude the following: For any $q>0$ there exists a constant d' such that for any $\xi\in S^-$ and any Borel subset Q of X

$$\int_Q F(\xi^{-1}x)^q\,d\sigma(x) \leqq d'\int_{EQ} F(\xi^{-1}x)^q\,d((\ldots(\sigma_\Sigma)_{\eta_r})_{\eta_{r-1}})_{\eta_1}. \tag{7.15}$$

Since $\mathcal{M}/R$ is a diffeomorphism onto E there exists a differentiable function ψ on E such that for any $\varphi \in C_c(H)$

$$\int_{\Sigma} dz \int_{E_r} dt_r \int_{E_{r-1}} \dots \int_{E_1} dt_1 \{\varphi((\exp t_1\eta_1)(\exp t_2\eta_2)\dots(\exp t_r\eta_r)z\} = \int_{E} \varphi(h)\psi(h)\, d_l h.$$

Thus if φ is a non-negative function on X

$$\int_X \varphi(x)\, d((\dots(\sigma_\Sigma)_{\eta_r})\dots)_{\eta_1}(x) = \int_X \int_H \varphi(hx)\chi_E(h)\psi(h)\, d_l h\, d\sigma(x)$$

$$\leqq d'' \int_X \int_H \varphi(hx)\chi_\Omega(h)\, d_l h\, d\sigma(x)$$

where $d'' = \sup_{h\in E} \psi(h) < \infty$. By Lemma 6.6 we further have

$$\int_X \int_H \varphi(hx)\chi_\Omega(h)\, d_l h\, d\sigma(x) \leqq d_3 \int_X \varphi(x)\, d\mu(x).$$

In particular for $\varphi(x) = F(\xi^{-1}x)^q \chi_{EQ}(x)$ we get

$$\text{(7.16)} \quad \int_X F(\xi^{-1}x)^q \chi_{EQ}(x)\, d((\dots(\sigma_\Sigma)_{\eta_r}))\dots)_{\eta_1} \leqq d'' d_3 \int_X F(\xi^{-1}x)^q \chi_{EQ}(x)\, d\mu.$$

Combining (7.15) and (7.16) we conclude that for any $\xi \in S^-$

$$\int_Q F(\xi^{-1}x)^q\, d\sigma(x) \leqq d \int_{EQ} F(\xi^{-1}x)^q\, d\mu(x)$$

where $d = d'd''d_3$. This proves Proposition.

Proof of Proposition (5.14). Recall that for each $j \in J$, $j^{-1}\Gamma j \cap U^-$ is a lattice in U^-. Further the measure π_j defined by (5.13) is $j^{-1}\Gamma j$-invariant and $\pi_j(X-Y) = 0$. Thus for each $j\in J$ there exists a compact neighbourhood E_j of e in H, and a constant $d_j > 0$ such that for any $\xi \in S^-$ and any Borel subset Q of X

$$\int_Q F(\xi^{-1}x)^{1+\frac{\varepsilon}{2}}\, d\pi_j(x) \leqq d_j \int_{E_jQ} F(\xi^{-1}x)^{1+\frac{\varepsilon}{2}}\, d\mu(x).$$

Since J is finite, putting $E = \bigcup_{j\in J} E_j$ and $c_6 = \max_{j\in J} d_j$ we get the Proposition.

Part III. Classification of Invariant Measures

We now consider the general case. In the sequel T denotes a (connected) reductive Lie group. This means that the adjoint representation is completely reducible. The adjoint group T^* admits a direct product decomposition $T^* = C \cdot G$ wher C and G are normal (and hence semisimple) subgroups of T^*, C is compact and G is a product of noncompact simple Lie groups. We denote by ρ: $T \to G$ the canonical projection homomorphism of T onto G. There exists in T a unique maximum normal semisimple analytic (connected) subgroup G_0

without compact factors. The restriction of ρ to G_0 is a covering of G. Any horospherical subgroup U of T (see Introduction for definition) is contained in G_0.

Let U be a maximal horospherical subgroup of T, and let Γ be a lattice in T. In § 8 we first show that any ergodic U-invariant measure on T/Γ is concentrated on a translate of a closed orbit of a reductive subgroup L of T containing G_0 such that the L-invariant measure on the orbit is U-ergodic. This reduces the task of proving Theorem A only in the special case when the action of U on T/Γ is ergodic with respect to the Haar measure. Then using Proposition 1.12 and the duality principle (Corollary 1.9) in § 9 we reduce the problem to the special case of semisimple groups with trivial center and without compact factors. In § 10 we show that if the measure (or more precisely the dual of it) fails to satisfy the condition of Theorem 2.4 then it is concentrated on a closed orbit of a proper subgroup. We are then able to produce an inductive argument to complete the proof of Theorem A.

§ 8. Ergodicity of Haar Measure

(8.1) **Lemma.** *Let H be any Lie group (not necessarily reducitve) and Λ be a lattice in H. Let V be a normal analytic subgroup of H. Then there exists an analytic subgroup L of H such that*

(i) *L is normalized by Λ, $L\Lambda$ is a closed subgroup and $L\cap\Lambda$ is a lattice in L.*

ii) *L contains V and the action of V on $L/L\cap\Lambda$ is ergodic with respect to the L-invariant measure on $L/L\cap\Lambda$.*

Proof. We construct inductively, a decreasing sequence $\{L_i\}_0^\infty$ of analytic subgroups of H such that for each i, L_i contains V, L_i is normalized by Λ and $L_i\Lambda$ is closed. Put $L_0=H$ and suppose $L_0, L_1, \dots, L_{s-1}$ are constructed satisfying the above conditions. Let L_s be the connected component of the identity in $cl(V\Lambda_s)$ where $\Lambda_s=L_{s-1}\cap\Lambda$. Then L_s contains V and is normalized by Λ. We show that $L_s\Lambda$ is closed. Let $x_\alpha\gamma_\alpha$ be a net in $L_s\Lambda$ converging to $y\in H$. Since $L_s\subset L_{s-1}$ and $L_{s-1}\Lambda$ is closed (by induction hypothesis) y has the form $x\gamma$ where $x\in L_{s-1}$ and $\gamma\in\Lambda$. Hence $x_\alpha(\gamma_\alpha\gamma^{-1})\to x\in L_{s-1}$. But $L_{s-1}\Lambda$ being closed, is a discrete union of left cosets of certain elements of Λ. Therefore $\gamma_\alpha\gamma^{-1}$ must eventually belong to L_{s-1}. Choosing a subnet we may assume that for all α, $\gamma_\alpha\gamma^{-1}\in\Lambda\cap L_{s-1}=\Lambda_s$. But clearly $L_s\Lambda_s$ is closed. Hence $x_\alpha(\gamma_\alpha\gamma^{-1})$ converges in $L_s\Lambda_s$. Thus $x_\alpha\gamma_\alpha$ converges in $L_s\Lambda$. Hence $L_s\Lambda$ is closed.

Now put $L=\bigcap_{i=0}^{\infty} L_i$. Since $\{L_i\}_0^\infty$ is a decreasing sequence of analytic subgroups there exists $j\geqq 0$ such that $L=L_j$. It follows that L is normalized by Λ and that $L\Lambda$ is closed. By Lemma 1.14 $L\cap\Lambda$ is a lattice in L.

Clearly L contains V. Further since $L=L_j=L_{j+1}$ we conclude that $V(L\cap\Lambda)$ is dense in L. The ergodicity of the action of V on $L/L\cap\Lambda$ is now a consequence of the following Lemma (due to Mostow) which we separate out because of its usefulness.

(8.2) **Lemma** (Mostow). *Let H be a locally compact second countable group and let Λ be a closed subgroup such that H/Λ admits a finite H-invariant measure, say σ. Let V be a normal subgroup H such that $V\Lambda$ is dense in H. Then the action of V on H/Λ is ergodic with respect to σ.*

Proof. Let $f\in\mathscr{L}^2(H/\Lambda,\sigma)$ such that $f(gx\Lambda)=f(x\Lambda)$ a.e. σ for all $g\in V$. Let $\tilde{f}$ be the function on H defined by $\tilde{f}(x)=f(x\Lambda)$. For any $g\in V$ and $\gamma\in\Lambda$ we have

$$\text{(8.3)}\quad \tilde{f}(xg\gamma)=\tilde{f}((xgx^{-1}x\gamma)=f((xgx^{-1}x\Lambda)=f(x\Lambda)=\tilde{f}(x)$$
a.e. on $V\times H$.

But observe that $\tilde{f}$ is a locally integrable function on H. Also the action of H (on right) on the space of locally integrable functions on H is continuous when the latter space is endowed with the topology induced by local seminorms. Since $V\Lambda$ is dense (8.3) now implies that $\tilde{f}$ is a constant function. Hence so is f. Therefore the action of V on $H/H\cap\Lambda$ is ergodic with respect to σ.

We now return to the earlier notation—as introduced in the introduction to Part III.

(8.3) **Proposition.** *There exists a reductive analytic subgroup L of T containing G_0 such that*

i) L is normalized by Γ, $L\Gamma$ is closed and $L\cap\Gamma$ is a lattice in L.

ii) The action of U on $L/L\cap\Gamma$ is ergodic with respect to the L-invariant measure.

iii) If π is any U-invariant ergodic measure on T/Γ then there exists $t\in T$ such that $\pi(T/\Gamma-tL\Gamma/\Gamma)=0$.

Proof. By Lemma 8.1 there exists an alaytic subgroup L containing G_0 for which i) is satisfied, and the action of G_0 on $L/L\cap\Gamma$ is ergodic. We also observe that any analytic subgroup containing G_0 is reductive.

The assertion ii) is a consequence of Calvin Moore's ergodicity theorem. Observe that if p_i is a canonical projection of G_0 onto any simple factor of the adjoint group then $\mathrm{cl}(p_i(U))$ is noncompact. The ergodicity theorem asserts that any subgroup satisfying the above has the following property (cf. Theorem 1, [15] and also [4]). Let τ be a unitary representation of G_0 on a separable Hilbert space $\mathscr{H}$. If $\psi\in\mathscr{H}$ is such that $\tau(u)\psi=\psi$ for all $u\in U$ then $\tau(g)\psi=\psi$ for all $g\in G_0$. Applying the result to the unitary representation corresponding to the action of G_0 on $L/L\cap\Gamma$, we get that if $f\in\mathscr{L}^2(L/L\cap\Gamma)$ such that $f(ux)=f(x)$ a.e. for all $u\in U$ then $f(gx)=f(x)$ a.e. for all $g\in G_0$. Since the action of G_0 is ergodic such an f must be a constant function; i.e. the action of U on $L/L\cap\Gamma$ is ergodic.

We now prove iii). Since $L\Gamma$ is a closed subgroup of T the partition of T/Γ into $\{xL\Gamma/\Gamma \mid x\in T\}$ is countably separated; i.e. there exists a countable family $\{E_j\}$ of Borel subsets such that each E_j is a union of certain elements of the partition and given any two distinct elements of the partition there exists j such that E_j contains one of them but not the other. Since L contains G_0 which is a normal subgroup containing U any element of the above partition is U-invariant. Therefore if π is a U-invariant ergodic measure on T/Γ there exists $t\in T$ such that $\pi(T/\Gamma-tL\Gamma/\Gamma)=0$.

(8.4) *Remark.* In view of Proposition 8.3, Theorem A needs to be proved only under the additional hypothesis that the action of U on T/Γ is ergodic with respect to the T-invariant measure. For suppose this is done. Let π be a U-invariant ergodic measure on T/Γ. Let L be an analytic subgroup and $t \in T$ be as obtained in Proposition 8.3. Let π' be the measure on $L/L \cap \Gamma \simeq L\Gamma/\Gamma$ defined by $\pi'(E) = \pi(tE)$ for any Borel subset E of $L\Gamma/\Gamma$. Then π' is a $t^{-1}Ut$-invariant measure on $L/L \cap \Gamma$. Thus we get a subgroup H of L and $x \in L$ such that $H(L \cap \Gamma)$ is closed and π' is the xHx^{-1}-invariant measure supported on $xH(L \cap \Gamma)/L \cap \Gamma \simeq xH\Gamma/\Gamma$. It follows that π is the $txHx^{-1}t^{-1}$ invariant measure supported on $txH\Gamma/\Gamma$.

§ 9. Reduction to Semisimple Groups

In this section we reduce proving Theorem A to the special case when T is a semisimple group without compact factors. Recall that G is the product of all noncompact simple normal subgroups of the adjoint group of T and $\rho: T \to G$ is the canonical projection of T onto G. Let $\bar{\rho}: T/\Gamma \to G/\rho(\Gamma)$ be the map defined by $\bar{\rho}(t\Gamma) = \rho(t)\,\rho(\Gamma)$ for all $t \in T$.

(9.1) **Lemma.** *$\rho(\Gamma)$ is a lattice in G and $\bar{\rho}$ is proper.*

Proof. Let L denote the connected component of the identity in $\ker \rho$. Also let R be the radical of T. Then R is also the radical of L and L/R is compact. Now by a theorem of L. Auslander (cf. Theorem 8.24, [16]) the connected component A of the identity in $\operatorname{cl} R\Gamma$ is solvable. Since L/R is compact AL is a closed subgroup. Further AL is normalized by Γ and $AL\Gamma$ is a closed subgroup. Since Γ is a lattice in T it follows easily that the subgroup $A\Gamma L/L$ has the Selberg property (A subgroup H of T is said to have Selberg property if for any neighborhood Ω of the identity in T and any $g \in T$ there exists $\gamma \in \Gamma$ and $n \in \mathbb{Z}^+$ such that $g^n \in \Omega\gamma\Omega^{-1}$). Therefore by Borel's density theorem (cf. Theorem 5.5, [16], see also [9]) $A\Gamma L/L$ is a Zariski dense subgroup of T/L. In particular it follows that the connected component AL/L is normal in T/L. But since A is solvable and T/L is semisimple this means that $AL = L$ or that A is contained in L. Consequently $L\Gamma/L$ is a discrete subgroup. Further by Lemma 1.14 it follows that $L\Gamma/L$ is a lattice in T/L. Next notice that $\ker \rho/L$ is a central subgroup of T/L. Therefore again because of Borel's density theorem $(\ker \rho)\,\Gamma/L$ is discrete (cf. Corollary 5.17, [16]). Thus $\rho(\Gamma)$ is discrete and indeed a lattice.

To prove that $\bar{\rho}$ is proper we proceed as follow: Since A is a closed subgroup (notation as above) normalized by Γ and $A\Gamma$ is closed it follows that $A \cap \Gamma$ is a lattice in A. Since A is solvable it follows from a result of Mostow (cf. Corollary 3.5 and Theorem 2.1, [16]) that $A/A \cap \Gamma$ is compact. Since L/A is compact it follows that $L/L \cap \Gamma$ is compact. Now the same argument as above shows that $(\ker \rho) \cap \Gamma$ is a lattice in $\ker \rho$. Since each orbit of L on $\ker \rho/(\ker \rho) \cap \Gamma$ is open it follows that there are only finitely many orbits. Also each orbit, being topologically isomorphic to $L/L \cap \Gamma$ is compact. Hence $\ker \rho/(\ker \rho) \cap \Gamma$ is compact. Hence $\bar{\rho}$ is proper.

(9.2) **Lemma.** *Let Q be an analytic group. Let U' be a normal analytic subgroup of Q such that Q/U' is reductive. Let U be an analytic subgroup containing U' such that U/U' is a maximal horospherical subgroup of Q/U'. Let V be the smallest normal analytic subgroup of Q containing U. Then the restriction of any ergodic measure-preserving action of V to U is ergodic.*

Proof. Since U' is a normal subgroup contained in U it is enough to show that any ergodic action of V/U' restricts to an ergodic action of U/U'. But V/U' is clearly the maximum normal semisimple subgroup without compact factors in Q/U'. Also U/U' is a maximal horospherical subgroup in V/U'. The desired result now follows from C.C. Moore's ergodicity in the same way as argued in Proposition 8.3, ii).

(9.3) **Proposition.** *Let π be a U-invariant ergodic measure on T/Γ. Suppose that there exists an analytic subgroup H of G and $g_0 \in G$ such that*

a) H contains $N=\rho(U)$. There exists an analytic subgroup N' of N which is normal in H and H/N' is reductive.

b) $H \cap g_0 \rho(\Gamma) g_0^{-1}$ is a lattice in H, $H g_0 \rho(\Gamma)$ is closed and $\operatorname{supp} \bar{\rho}\pi = H g_0 \rho(\Gamma)/\rho(\Gamma)$.

c) $\bar{\rho}\pi$ is H-invariant.

Then there exists an analytic subgroup L of $\rho^{-1}(H)$ and $t_0 \in T$ such that

i) $L \cap t_0 \Gamma t_0^{-1}$ is a lattice in L, $L t_0 \Gamma$ is closed and $\operatorname{supp} \pi = L t_0 \Gamma/\Gamma$ *and*

ii) L contains U and π is L-invariant.

Proof. Let $t_1 \in \rho^{-1}(g_0)$. Put $Q=\rho^{-1}(H)$ and $\Lambda = Q \cap t_1 \Gamma t_1^{-1}$. Since $g_0 \rho(\Gamma) g_0^{-1} \cap H$ is a lattice in H and $\bar{\rho}$ is proper it follows that Λ is a lattice in Q. Also Λ is the isotropy subgroup for the action of Q at the point $t_1 \Gamma$. Since $\bar{\rho}\pi$ is supported on $H g_0 \rho(\Gamma)/\rho(\Gamma)$ it follows that the support of π is contained in $Q t_0 \Gamma/\Gamma \simeq Q/\Lambda$. Therefore π may be visualized as a U-invariant, ergodic measure on Q/Λ.

Recall that the restriction of ρ to G_0 is a covering of G. Hence there exists a unique analytic subgroup U' of U such that the restriction of ρ to U' is an isomorphism of U' onto N'. Further it is clear from Lie algebra considerations that U' is normal in Q and that Q/U' is reductive. Also U/U' is a maximal horospherical subgroup in Q/U'.

Now let V be the smallest normal analytic subgroup of Q containing U. By Lemma 8.1 there exists an analytic subgroup L' of Q such that L' is normalized by Λ, $L'\Lambda$ is a closed subgroup, $L' \cap \Lambda$ is a lattice in L', L' contains V and the action of V on $L'/L' \cap \Lambda$ is ergodic. By Lemma 9.2 it follows that the action of U on $L'/L' \cap \Lambda$ is ergodic. Also since $\{tL'\Lambda \mid t \in Q\}$ is a countably separated partition as in the proof of Proposition 8.3 we conclude that there exists $t_2 \in Q$ such that the U-invariant ergodic measure is supported on $t_2 L'\Lambda/\Lambda$. Under the canonical isomorphism of Q/Λ with $Q t_1 \Gamma/\Gamma$ the supporting set corresponds to $t_2 L' t_1 \Gamma/\Gamma$. Now put $L = t_2 L' t_2^{-1}$ and $t_0 = t_2 t_1$. Then π is supported on $L t_0 \Gamma/\Gamma \simeq L/\Delta$ where $\Delta = L \cap t_0 \Gamma t_0^{-1}$ is a lattice in L. Observe that U is contained in L and the action of U on L/Δ is ergodic. This is because ergodicity of the action of a subgroup on a homogeneous space depends only on the conjugacy class of the subgroup.

We also observe that the restriction of ρ to L maps L onto H. For, since $\bar{\rho}$ is proper from considerations of supports of π and $\bar{\rho}\pi$ we can deduce that $\rho(Lt_0\Gamma) = Hg_0\rho(\Gamma)$. In other words $\rho(L)$ acts transitively on the H-orbit $Hg_0\rho(\Gamma)/\rho(\Gamma) \simeq H/H \cap g_0\rho(\Gamma)g_0^{-1}$. However since $\rho(\Gamma)$ is a discrete subgroup of G and $\rho(L)$ is analytic this is possible only if $\rho(L)=H$.

Let $\rho': L/\Delta \to H/\rho(\Delta)$ be the map induced by the restriction of ρ to L. Clearly ρ' is a proper map. Now we may visualize π as a U-invariant ergodic measure on L/Δ, such that $\rho'\pi$ is a H-invariant measure and we need to prove that π is L-invariant. Since ρ' is proper we can decompose π along the fibers of ρ': There exists a family $\{\pi_p\}_{p \in H/\rho(\Delta)}$ where π_p is a probability measure on $\rho'^{-1}(p)$ such that for any $\varphi \in C_c(L/\Delta)$

$$\int_{L/\Delta} \varphi(x)\, d\pi(x) = \int_{H/\rho(\Delta)} d\rho'\pi(p) \int_{\rho'^{-1}(p)} \varphi(x)\, d\pi_p(x).$$

Now let $R = (\ker \rho) \cap L$. Let dr be a Haar measure on R. For any $\theta \in C_c^+(R)$ let π_θ be the measure defined by

$$\int_{L/\Delta} \varphi(x)\, d\pi_\theta(x) = \int_{L/\Delta} \int_R \varphi(rx)\, \theta(r)\, dr\, d\pi(x)$$

for all $\varphi \in C_c(L/\Delta)$. Since the actions of the subgroups R and U commute it follows that each π_θ is U-invariant. We claim that for all $\theta \in C_c(R)$, π_θ is absolutely continuous with respect to the L-invariant measure on L/Δ. Note that R acts transitively on each $\rho'^{-1}(p)$. Since $\rho'\pi$ is the H-invariant measure, in view of Fubini-Weil formula for invariant measures on homogeneous spaces it is enough to assert that for each $p \in H/\rho(\Delta)$ and $\theta \in C_c^+(R)$ the measure $(\pi_p)_\theta$ defined by

$$\int_{\rho'^{-1}(p)} \varphi(x)\, d(\pi_p)_\theta = \int_{\rho'^{-1}(p)} \int_R \varphi(rx)\, \theta(r)\, dr\, d\pi_p(x)$$

where $\varphi \in C_c(\rho'^{-1}(p))$ is absolutely continuous with respect to the R-invariant measure on $\rho'^{-1}(p)$. But this obviously follows in the same way as in the proof of Lemma 6.6.

Now recall the one-to-one correspondence between the U-invariant measures on L/Δ and Δ-invariant measures on L/U established in Corollary 1.9. (Notice that since L contains a lattice it is necessarily unimodular.) Let ω and ω_θ, $\theta \in C_c^+(R)$ denote the Δ invariant measures on L/U corresponding to π and π_θ, $\theta \in C_c^+(R)$ respectively. It is straightforward to check that for any $\theta \in C_c^+(R)$ and any Borel subset E of L/U

$$\omega_\theta(E) = \int_R \varphi(\psi_r^{-1} E)\, \theta(r)\, dr$$

where $r \to \psi_r$ is the action of R on L/U on right defined by $\psi_r(xU) = xrU$ (it is well-defined since R normalizes U). Since the L-invariant measure on L/Δ is ergodic for the action of U it follows that the L-invariant measure on L/U is ergodic for the action of Δ. Since for each $\theta \in C_c^+(R)$, π_θ is absolutely continuous with respect to the L-invariant measure, each ω_θ is absolutely continuous with respect to the L-invariant measure on L/U. Therefore it now follows from Proposition 1.12 that ω is a L-invariant measure on L/U. Hence π is a L-invariant measure on L/Δ.

§ 10. Applications

Using the characterization of Haar measure obtained in earlier sections we now determine all ergodic invariant measures of the 'horospherical flow' on G/Γ where G is a semisimple analytic group such that $\mathbb{R}$-rank of each factor of G is 1. A generalization of the result — without the condition on rank of the factors — involves study of orbits of the horospherical flow and will be considered later.

(10.1) **Theorem.** *Let G be a semisimple analytic group with trivial center and such that $\mathbb{R}$-rank of each factor of G is 1. (In particular G has no compact factors.) Let Γ be a lattice in G. Let U be a maximal horospherical subgroup of G. Let π be an ergodic U-invariant finite measure on G/Γ. Then there exists an analytic subgroup L of G and $g_0 \in G$ such that*

i) *$Lg_0\Gamma$ is closed, $L \cap g_0 \Gamma g_0^{-1}$ is a lattice in L and* $\operatorname{supp} \pi = Lg_0\Gamma/\Gamma$, *and*

ii) *L contains U and π is L-invariant. Also*

iii) *There exists an analytic subgroup U' of U which is normal in L and such that L/U' is reductive.*

(10.2) *Remark.* If there exists a maximal horospherical subgroup of G for which the result holds then it holds for all maximal horospherical subgroups.

Proof of Remark. Recall that all maximal horospherical subgroups are conjugate. Assume the theorem to be true for U and let $U' = gUg^{-1}$ where $g \in G$. Let π' be a U'-invariant ergodic measure on G/Γ. Let π be the measure defined by $\pi(E) = \pi'(gE)$ for any Borel set E. Then π is a N-invariant ergodic measure on G/Γ. If L is an analytic subgroup and $g_0 \in G$ be such that i)–iii) are satisfied for N and π then $L' = gLg^{-1}$ and $g_0' = gg_0$ clearly have the desired properties with respect to N' and π'.

(10.3) **Lemma.** *Let G be as in Theorem* 10.1 *and let Λ be any lattice in G. Let $G = \prod_{i \in I} G_i$ be the decomposition such that $\Lambda_i = \Lambda \cap G_i$ is an irreducible lattice in G_i (cf. Proposition* 2.1*). Let $i \in I$ be such that G_i/Λ_i is non-compact. Then G_i is a simple Lie group and in particular $\mathbb{R}$-rank of G_i is* 1.

Proof. Let $G_i = \prod_{j=1}^{l} H_j$ be the decomposition of G_i into simple factors and if possible let $l > 1$. Let $U \neq (e)$ be any Λ_i-rational horospherical subgroup of G_i. Also U can be expressed as $\prod_{j=1}^{l} U_j$ where U_j, $1 \leq j \leq l$ is a horospherical subgroup H_j. Since $\mathbb{R}$-rank of each H_j is 1, U_j is either the identity subgroup (e) or a maximal horospherical subgroup in H_j. If there exists j such that $U_j = (e)$ then we get a normal subgroup F of G_i containing U and such that $(e) \underset{+}{\subset} F \underset{+}{\subset} G_i$. However since Λ_i is an irreducible lattice in G_i no proper subgroup contains a non-trivial element of Λ_i (cf. Lemma 1.5 [13]). Thus we conclude that any non-trivial Λ_i-rational horospherical subgroup U is a maximal horospherical subgroup of G_i.

However recall that by arithmeticity theorem (cf. [13]) of Margulis, G_i admits a $\mathbb{Q}$-structure such that Λ_i is an arithmetic lattice. For that $\mathbb{Q}$-structure the conclusion of last paragraph implies that $\mathbb{Q}$-rank G_i is same as the $\mathbb{R}$-rank of G_i. Since $\mathbb{Q}$-rank (being equal to the maximum length of an increasing chain of non-trivial Λ_i-rational *non-trivial* horospherical subgroups) is clearly 1 so is the $\mathbb{R}$-rank. Since each H_j is non-compact $\mathbb{R}$-rank of each H_j is positive. This gives a contradiction unless $l=1$.

(10.4) *Remark.* Let G and Γ be as in the statement of Theorem 10.1. Let G be decomposed as $G=G_0\cdot G_1$ where G_0 is the maximum normal Lie subgroup of G such that $G_0\cap\Gamma$ is a uniform lattice in G (product of all G_i, $i\in I_0$ (in the notation of § 2) and G_1 is such that $G_1\cap\Gamma$ is a (non-uniform) lattice in G_1. Let U, U^-, P, P^-, S, A, N etc. be as in § 2. Then in view of Lemma 10.3 we have the following:

i) $U=G_1\cap N$, and U is a maximal horospherical subgroup in G_1

ii) $P=G_0\cdot(P\cap G_1)$; $P^-=G_0\cdot(P^-\cap G_1)$. $P\cap G_1$ and $P^-\cap G_1$ are minimal $\mathbb{R}$-parabolic subgroups of G_1 and we have Langlands decompositions (cf. [19], pp. 75) as

$$P\cap G_1=M\cdot(A\cap G_1)\cdot U$$

and

$$P^-\cap G_1=M\cdot(A\cap G_1)\cdot U^-$$

where M is the centralizer of $A\cap G_1$ in a maximal compact subgroup G_1.

iii) $$P^-U=G_0\cdot(P^-\cap G_1)U=(G_0\cdot M\cdot(A\cap G_1)\cdot U^-)U =U^-(G_0\cdot M(A\cap G_1)\cdot U)=U^-P.$$

iv) Using the Bruhat decomposition of G_1 with respect to $A\cap G_1$ we have a Bruhat decomposition of G as

$$(10.4)\quad G=G_0\cdot G_1=G_0\bigcup_{\omega\in W}U\omega(P\cap G_1)\quad\text{(disjoint union)}$$
$$=\bigcup_{\omega\in W}U\omega P$$

where W is the Weyl group of G_1 with respect to $A\cap G_1$.

v) There exists a unique $\omega_0\in W$ such that $\omega_0^{-1}U\omega_0=U^-$. Also since $\mathbb{R}$-rank of each factor of G is 1 it follows that for $\omega\neq\omega_0$, $(\omega^{-1}U\omega)P$ is contained in a proper parabolic subgroup say $Q(\omega)$ of G. Thus by (10.4) we have

$$(10.5)\quad G=\bigcup_{\omega\in W}\omega(\omega^{-1}U\omega)P$$
$$=\omega_0U^-P\cup\bigcup_{\omega\neq\omega_0}\omega Q(\omega).$$

We may also note that each $Q(\omega)$, $\omega\neq\omega_0$ is of the form $L\cdot P'$ where L is a normal (semisimple) Lie subgroup of G and P' is a minimal parabolic subgroup in a semisimple normal Lie subgroup L' of G such that $G=L\cdot L'$. Indeed $P' = P\cap L'$.

Proof of Theorem (10.1). We proceed by induction on the dimension of G (satisfying the hypothesis of the theorem).

In view of Remark 10.2 it is also enough to prove the theorem for the maximal horospherical subgroup N as chosen in §2. Let π be a finite, N-invariant ergodic measure on G/Γ and σ be the Γ-invariant measure on G/N corresponding to π (cf. Corollary 1.3). Let $X=G/N$ and $X_0=\bigcup_{j\in J} jP^- U/N$ with notations as in §2. If $\sigma(X-X_0)=0$ then by Theorem 2.4, σ is G-invariant. Hence so is π. Then clearly G itself has the desired properties.

Now consider the complementary possibility viz. there exists $j\in J$ such that $\sigma(X-jP^- U/N)>0$. In view of (10.5) it follows that there exists $\omega\neq\omega_0$ such that $\sigma(j\omega_0^{-1}\omega Q(\omega)/N)>0$. Since σ is ergodic we conclude that

$$\sigma(X-\Gamma j\omega_0^{-1}\omega Q(\omega)/N)=0.$$

Put $p=j\omega_0^{-1}\omega$ and $\Gamma'=p^{-1}\Gamma p$. Then we have

$$(10.6)\quad \sigma(X-p\Gamma' Q(\omega)/N)=0.$$

Recall from Remark 10.4, v) that $Q(\omega)=L\cdot(P\cap L')$ where L and L' are normal subgroup of G such that $G=L\cdot L'$ (direct product). Let $P\cap L'=M'A'N'$ be a Langlands decomposition of $P\cap L'$ in L' where $A'=A\cap L'$ and $N'=N\cap L'$. Put $Q' =LM'N'$. We claim that a) $\Gamma'\cap Q(\omega)$ is contained and is a lattice in Q' and b) $\Gamma' Q'$ is closed.

Firstly consider $\Gamma'\cap N'$. It is clear from the definition of L' that ω centralizes L'. Hence $\Gamma'\cap N'=(j\omega_0^{-1}\omega)^{-1}\Gamma(j\omega_0^{-1}\omega)\cap N'=\omega_0 j^{-1}\Gamma j\omega_0^{-1}\cap N'$. By choice of J for each $j\in J$, $j^{-1}\Gamma j\cap U^-$ is a lattice in U^- (cf. Proposition 2.3). Hence $\omega_0 j^{-1}\Gamma j\omega_0^{-1}\cap U$ is a lattice in U. Since by Lemma 10.3 for any lattice Λ, $(\Lambda\cap L)\cdot(\Lambda\cap L')$ is a subgroup of finite index in Λ and N' is the L'-component of U, the above implies that $\omega_0 j^{-1}\Gamma j\omega_0^{-1}\cap N'$ is a lattice in N'. Since M' is compact we may further conclude that $\Gamma'\cap M'N'$ is a lattice in $M'N'$.

By Lemma 10.3 in particular $L\Gamma'$ is closed. Since $\Gamma'\cap M'N'$ is a (necessarily uniform) lattice in $M'N'$, straightforward verification now shows that $\Gamma' Q' =\Gamma' LM'N'$ is closed. Further clearly $\Gamma'\cap Q'$ is lattice in Q'. To complete the claim we only need to show that $\Gamma'\cap Q(\omega)=\Gamma'\cap Q'$. Suppose this is not true. Then again in view of Lemma 10.3 there exists a simple normal subgroup V of L' such that $V\cap\Gamma'$ is a lattice in V, and $V\cap Q(\omega)\cap\Gamma'\supsetneqq V\cap Q'\cap\Gamma'$. Repeating the earlier argument we see that $V\cap Q'\cap\Gamma'$ is a lattice $V\cap Q'$. Since $\mathbb{R}$-rank of each simple factor of G is 1, $V\cap Q'$ is of codimension one in $V\cap Q(\omega)$. Hence if $V\cap Q(\omega)\cap\Gamma'\supsetneqq V\cap Q'\cap\Gamma'$ it follows that $V\cap Q(\omega)\cap\Gamma'$ is a lattice in $V\cap Q(\omega)$. However this is a contradiction since $V\cap Q(\omega)$ is clearly not unimodular. Hence $\Gamma'\cap Q(\omega)=\Gamma'\cap Q'$.

We now return to the measure σ. Observe that $\Gamma p Q(\omega)=p\Gamma' Q' A'$. Since A' normalizes N we have

$$\Gamma p\, Q(\omega)/N=\{\Gamma p\,\Gamma' Q'/N\}\, A'.$$

Since $p\Gamma'Q'$ is closed it follows that $\{p\Gamma'Q'aN/N \mid a\in A'\}$ is a countably separated partition of $p\Gamma'Q(\omega)/N$, each element of which is invariant under the Γ-action (on left). Since σ is a Γ-invariant ergodic measure it now follows that there exists $a\in A'$ such that

$$\sigma(X-p\Gamma'Q'aN/N)=0$$

i.e. $\sigma(X-\Gamma q\, Q'/N)=0$ where $q=pa$. Under the one-to-one correspondence of Γ-invariant measures on G/N and N-invariant measures on G/Γ the last assertion corresponds to the following: $\pi(G/\Gamma-Q'q\Gamma/\Gamma)=0$.

Recall that $q^{-1}\Gamma q=a^{-1}\Gamma' a$ intersects Q' as well as N' in lattices in respective subgroups. Therefore $\Lambda=\{q^{-1}\Gamma q\cap Q'\}N'/N'$ is a lattice $Q'/N'=T$ (say). Let π' be the measure on T/Λ defined by $\pi'(E)=\pi(\bar{\eta}^{-1}E)$ where $\bar{\eta}$: $Q'q^{-1}\Gamma/\Gamma\simeq Q/q^{-1}\Gamma q\cap Q'\to T/\Lambda$ is induced by the projection homomorphism $\eta: Q'\to Q'/N'$. Since $N'\cap\Gamma'$ is a lattice in N', η is a proper map. Hence π is a regular measure on T/Λ. Clearly π' is invariant under the action of $U'=\eta(N)$ and is ergodic. It is easy to verify that U' is a maximal horospherical subgroup in T. Therefore by induction hypothesis and Proposition 9.3 there exists an analytic subgroup H of T and $t_0\in T$ such that assertions i)–iii) are satisfied (for appropriate subgroups). Put $L=\eta^{-1}(H)$ and $g_0=tq^{-1}$ where $t\in\eta^{-1}(t_0)$. Then it is evident that assertions i)–iii) of Theorem 10.1 are satisfied.

Combining Theorem 10.1 and Proposition 9.3 we conclude Theorem A.

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