Supersymmetry, Shape Invariance and Solvability of A_{N-1} and BC_N Calogero-Sutherland Model

Pijush K. Ghosh[a,1], Avinash Khare [b,2] and M. Sivakumar[c,3]

[a] *The Mehta Research Institute of Mathematics & Mathematical Physics,*

Chhatnag Road, Jhusi, Allahabad-221 506, INDIA.

[b] *Institute of Physics, Sachivalaya Marg, Bhubaneswar-751 005, INDIA.*

[c] *School of Physics, University Of Hyderabad, Hyderabad-500 046, INDIA.*

Abstract

Using the ideas of supersymmetry and shape invariance we re-derive the spectrum of the A_{N-1} and BC_N Calogero-Sutherland model. We briefly discuss as to how to obtain the corresponding eigenfunctions. We also discuss the difficulties involved in extending this approach to the trigonometric models.

PACS numbers: 03.65.Ge, 05.30.-d

Typeset using REVTEX

---

[1] Electronic address: pijush@mri.ernet.in

[2] Electronic address:khare@iop.ren.nic.in

[3] Electronic address:mssp@uohyd.ernet.in

I. INTRODUCTION

In recent years, supersymmetric quantum mechanics (SUSY QM) has provided a deeper understanding of the exact solvability of several well known potentials in 1-dimensional QM. In particular, using the ideas of shape invariance (SI), it provides a procedure for getting the spectrum, the eigenfunctions and the S-matrix (i.e. the reflection and transmission coefficients) algebraically [1]. There also exist interesting connections between SUSY QM and soliton solutions. Despite these (and more) interesting developments in SUSY QM for one particle system in one dimension, so far, not many of these results could be extended either for N-particle systems in one dimension or for one particle systems in more than one dimension.

In recent times there is a revival of interest in the N-body problems in one dimension with inverse square interaction which were introduced and studied by Calogero [2] and developed by Sutherland [3] and others [4]. These models have several interesting features, like exact solvability, classical and quantum integrability and also have interesting applications in several branches of physics [5,6]. Apart from the well known translational invariant inverse square interaction models, referred to as A_{N-1} Calogero-Sutherland Model (CSM), there also exist generalizations of this model, but without the translational invariance, referred to as BC_N, B_N, D_N models. These nomenclatures refer to the relationship of these models to the root system of the classical Lie group. It might be added here that these models also share with A_{N-1} CSM, features like exact solvability, and integrability and have also found application in certain physical systems.

The purpose of this note is to enquire if the ideas of one dimensional SUSY QM could be extended to the N-particle case. In particular, whether the spectrum of the celebrated Calogero and other models could be obtained algebraically by using the ideas

of SI and SUSY QM. The first step in that direction was taken recently by Efthimiou and Spector [7] who showed that the well known Calogero model (also termed as A_{N-1} CSM) exhibits SI. However, they were unable to obtain the spectrum algebraically. This is because using SUSY they were unable to relate the eigenspectra of the two SUSY partner potentials. In this paper we demonstrate that using SUSY QM, SI and exchange operator formalism [8], the spectrum of the rational A_{N-1} CSM, and also of all its generalizations like B_N, D_N and BC_N can be obtained algebraically. It is worth mentioning that the SI in our case is somewhat different from that of Efthimiou and Spector [7]. So far as we are aware off, this is the first instance when an N-particle quantum system has been solved using the techniques of SUSY QM and SI.

The plan of the paper is the following. We briefly review the ideas of one dimensional SUSY QM in Sec. II with the main emphasis on solvability using SI.. In Sec. II.A, we apply these ideas to the Calogero model, i.e., the rational A_{N-1} model. We show that the spectrum of such a model can be derived using the ideas of SUSY QM, SI and the exchange operator formalism[8]. We also briefly discuss as to how to obtain the corresponding eigen-functions. In Sec. II.B, we treat the rational BC_N model, a translationally non-invariant system, in the same spirit. The full spectrum is obtained and the method for obtaining the exact eigen-functions is explicitly spelled out. It is also shown in this section that the BC_N model posses SI even if the exchange operator formalism is not employed. This is a generalization of Efthimiou et al's work [7] on A_{N-1} model to the BC_N case. Finally, in Sec. III, we summarize our results and discuss the possible directions to be followed in order to have a viable formalism of many-body SUSY QM. We also point out the difficulties involved in extending these results to the trigonometric case. In Appendix we show that the BC_N trigonometric model is also shape invariant.

II. SUSY, SI AND SOLVABILITY

It may be worthwhile to first mention the key steps involved in obtaining the eigen-spectrum of a one body problem by using the concepts of SUSY QM and SI. One usually defines the SUSY partner potentials H_1 and H_2 by

$$H_1 = A^\dagger A \,, \ H_2 = AA^\dagger \ , \tag{1}$$

where $(\hbar = 2m = 1)$

$$A = \frac{d}{dx} + W(x) \,, \ A^\dagger = -\frac{d}{dx} + W(x) \ . \tag{2}$$

In the case of unbroken SUSY, the ground state wave function is given in terms of the superpotential $W(x)$ by,

$$\psi_0(x) \propto e^{-\int^x W(y)dy} \ , \tag{3}$$

while the energy eigenvalues and the wave functions of H_1 and H_2 are related by, $(n = 0, 1, 2, ...)$

$$E_n^{(2)} = E_{n+1}^{(1)} \ , \quad E_0^{(1)} = 0 \ , \tag{4}$$

$$\psi_n^{(2)} = [E_{n+1}^{(1)}]^{-1/2} A \psi_{n+1}^{(1)} \ , \quad \psi_{n+1}^{(1)} = [E_n^{(2)}]^{-1/2} A^\dagger \psi_n^{(2)} \ . \tag{5}$$

Let us now explain precisely what one means by SI. If the pair of SUSY partner Hamiltonians H_1, H_2 defined above are similar in shape and differ only in the parameters that appear in them, then they are said to be SI. More precisely, if the partner Hamiltonians $H_{1,2}(x; a_1)$ satisfy the condition,

$$H_2(x; a_1) = H_1(x; a_2) + R(a_1), \tag{6}$$

where a_1 is a set of parameters, a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of x, then $H_1(x; a_1)$ and $H_2(x; a_1)$ are said to be SI. The property of SI permits an immediate analytic determination of the energy eigenvalues, eigenfunctions and the scattering matrix [1]. In particular the eigenvalues and the eigenfunctions of H_1 are given by $(n = 1, 2, ...)$

$$E_n^{(1)}(a_1) = \sum_{k=1}^{n} R(a_k) , \quad E_0^{(1)}(a_1) = 0 , \tag{7}$$

$$\psi_n^{(1)}(x; a_1) \propto A^\dagger(x; a_1) A^\dagger(x; a_2) .... A^\dagger(x; a_n) \psi_0^{(1)}(x; a_{n+1}) , \tag{8}$$

$$\psi_0^{(1)}(x; a_1) \propto e^{-\int^x W(y; a_1) dy} . \tag{9}$$

A. Rational A_{N-1} Calogero Model

We now apply the exchange operator formulation to the rational A_{N-1} CSM. The Hamiltonian of the rational A_{N-1} CSM is given by

$$H_{CSM} = \sum_i \frac{1}{2} p_i^2 + l(l \mp 1) \sum_{i<j} (x_i - x_j)^{-2} + \omega \sum_i \frac{1}{2} x_i^2 . \tag{10}$$

The sign $\mp$ in (10) refers to the fact that H acts on completely anti-symmetric or symmetric functions, respectively. Let us now define an operator D_i

$$D_i = -i\partial_i + il \sum_j{}' (x_i - x_j)^{-1} M_{ij}, \tag{11}$$

known as the Dunkl operator in the literature. Hereafter $'$ means $i = j$ is excluded in the summation. The exchange operator M_{ij} have the following properties [8]

$$M_{ij}^2 = 1 , \quad M_{ij}^\dagger = M_{ij} , \quad M_{ij}\psi^\pm = \pm\psi^\pm ,$$

$$M_{ij}D_i = D_j M_{ij} , \quad M_{ij}D_k = D_k M_{ij} , \quad k \neq i, j ,$$

$$M_{ijk} = M_{ij}M_{jk} , \quad M_{ijk} = M_{kij} = M_{jki} , \tag{12}$$

5

where $\psi^{\pm}$ is a(an) symmetric(antisymmetric) function. Note that the Dunkl operator is hermitian by construction and $[D_i, D_j] = 0$. If we now define

$$a_i = D_i - i\omega x_i , \ \ a_i{}^{\dagger} = D_i + i\omega x_i , \tag{13}$$

then it is easy to see that

$$[a_i, a_j{}^{\dagger}] = 2w\delta_{ij}(1 + l{\sum_k}' M_{ik}) - 2(1 - \delta_{ij})lwM_{ij} . \tag{14}$$

Let us now consider the SUSY partner potentials H and $\tilde{H}$ defined by

$$H = \frac{1}{2}\sum_i a_i{}^{\dagger}a_i , \ \ \ \tilde{H} = \frac{1}{2}\sum_i a_i a_i{}^{\dagger} . \tag{15}$$

Using eqs. (11) and (13) it is easily shown that

$$H_{CSM} = H + E_0^{CSM} , \ \ \ E_0^{CSM} = [\frac{N}{2} \mp \frac{l}{2}N(N - 1)]\omega \ .. \tag{16}$$

Thus, by construction, the ground state energy of H is zero.

Using eqs. (11) and (13) it is easily shown that if ψ is the eigenstate of H with eigenvalue $E(> 0)$, then $A_1\psi$ is the eigenstate of $\tilde{H}$ with eigenvalue $E + \delta_1$ i.e.

$$\tilde{H}(A_1\psi) = [E + \delta_1](A_1\psi), \tag{17}$$

where,

$$A_1 = \sum_i a_i , \ \ \ \delta_1 = [(N - 1) \pm lN(N - 1)]\omega \ . \tag{18}$$

Similarly, if $\tilde{\psi}$ is the eigenfunction of $\tilde{H}$ with eigenvalue $\tilde{E}$, then $A_1^{\dagger}\psi$ is the eigenfunction of H with eigenvalue $\tilde{E} - \delta_1$ i.e.

$$H(A_1^{\dagger}\tilde{\psi}) = [\tilde{E} - \delta_1](A_1^{\dagger}\tilde{\psi}). \tag{19}$$

This proves one to one correspondence between the non-zero energy eigen values of H and $\tilde{H}$. Thus it follows from here that the energy eigenvalues and eigenfunctions of the two partner Hamiltonians H and $\tilde{H}$ are related by

$$\tilde{E}_n = E_{n+1} + \delta_1 \,, \quad E_0 = 0 \,, \quad n = 0, 1, 2, ... \tag{20}$$

$$\tilde{\psi}_n = \frac{A_1 \psi_{n+1}}{\sqrt{E_{n+1} + \delta_1}} \,, \quad \psi_{n+1} = \frac{A_1^\dagger \tilde{\psi}_n}{\sqrt{E_{n+1}}} \,. \tag{21}$$

Note that δ_1 vanishes for $N = 1$ and we recover the usual results of SUSY QM with one degree of freedom. It is worth noting that unlike the case of one dimensional QM, in this case the (positive) energy levels of H and $\tilde{H}$ are not degenerate.

Using eqs.. (11) and (13) it is also easily shown that H and $\tilde{H}$ satisfy the shape invariance condition

$$\tilde{H}(\{x_i\}, l) = H(\{x_i\}, l) + R(l), \tag{22}$$

where

$$R(l) = [N \pm lN(N-1)]\omega = \omega + \delta_1 \,. \tag{23}$$

As a result, using the formalism of SUSY QM, and the relation between E_{n+1} and $\tilde{E}_n$ as given by eq. (20), the spectrum of H is given by

$$E_n = \sum_i R(l_i) - n\delta_1 \,. \tag{24}$$

Note that in this particular case all l_i are identical so that using δ_1 and $R(l)$ as given by eqs. (18) and (23), the spectrum turns out to be

$$E_n = n(R - \delta_1) = n\omega \,. \tag{25}$$

Using eq. (16) we then get the correct spectrum of the Calogero A_{N-1} model.

Let us now discuss as to how to obtain the eigenfunctions of CSM using the formalism of SUSY QM. We have seen that A_1 and $A_1^\dagger$ relate the non-zero eigen states of the partner Hamiltonians H and $\tilde{H}$. Once a particular state of $H(\tilde{H})$ with non-zero eigen value is

known, the use of eq. (21) enables us to find the corresponding state of $\tilde{H}(H)$. In particular, using eq. (8) and the fact that in this case all the l_i are identical, it follows that all the eigen-functions can be obtained from the ground state wave functions ψ_0 as, $\psi_n = (A_1^\dagger)^n \psi_0$. Note that this is justified from the operator algebra also, since A_1 and $A_1^\dagger$ can be identified as the annhilation and the creation operator respectively. In particular, one can show using eqs. (17), (19), (22) and (23) that $[H, A_1] = -A_1$ and $[H, A_1^\dagger] = A_1$.

This procedure for obtaining the eigen-functions is similar to that of Isikov *et al.* [9]. To see this, define a set of operators,

$$A_n = \sum_{i=1}^{N} a_i^n, \qquad n \le N, \tag{26}$$

which are symmetric in the particle indices. These operators satisfy relations which are analogous to those given by eqs. (17) and (19) for any n (see the next paragraph). It is easily checked that $[H, A_n] = -nA_n$ and $[H, A_n^\dagger] = nA_n^\dagger$. Following [9], the k-th eigen-state is given by,

$$\psi_{\{n_i\}} = \prod_{i=1}^{N} \left(A_i^\dagger\right)^{n_i} \psi_0, \qquad a_i \psi_0 = 0, \qquad k = \sum_{i=1}^{N} n_i . \tag{27}$$

Note that $\psi_{\{n_i\}}$ incorporates all the degenerate states corresponding to a particular value of k and all the corresponding states of $\tilde{H}$ can be obtained by applying the same A_1 on $\psi_{\{n_i\}}$.

Let us now ask the question whether or not A_1 is the only operator which relates the states with nonzero eigen values of the partner Hamiltonians. The answer obviously is negative and in fact, any operator which is symmetric in the particle indices can be used to relate the non-zero eigenstates of the partner Hamiltonians. However, none of these operators are useful in deriving the full spectrum of the A_{N-1} CSM model. For example, if ψ is an eigen-function of H with non-zero energy eigen-value E, then,

$$\tilde{H}(A_n\psi) = [E + \delta_n] (A_n\psi) \quad , \qquad \delta_n = [(N - n) \pm lN(N - 1)]\omega \ . \tag{28}$$

Note that the above equation is valid only if ψ is at least the n-th excited state, since $A_n(A_n^\dagger)$ anhilates(creates) n states. Similarly, one can show that any state $\tilde{\psi}$ of $\tilde{H}$, which represents at least the $(n - 1)$-th excited state with energy eigen value $\tilde{E}$, is related to a state $A_n^\dagger\tilde{\psi}$ of H with the eigen value $\left(\tilde{E} - \delta_n\right)$. This again proves one to one correspondence between the n-th excited state of H and the $(n-1)$-th excited state of $\tilde{H}$. As a result, the use of SI gives only the spectrum beginning with the n'th excited state of H and not the full spectrum.

It is worth pointing out that for the B_N type models, however, the symmetry arguments force us to replace A_1 by A_2 in order to derive the full spectrum using SUSY QM. This is discussed below in detail.

B. Rational BC_N Calogero Model

The Hamiltonian for BC_N Calogero model is given by

$$H_{BC_N} = \frac{1}{2}[\sum_i p_i^2 + l(l \mp 1) \sum_{i,j}{}' \left[(x_i - x_j)^{-2} + (x_i + x_j)^{-2}\right]$$
$$+ (l_1 - 1)l_1 \sum_i x_i^{-2} + \frac{(l_2)(l_2 - 1)}{2} \sum_i x_i^{-2} + \frac{\omega}{2} \sum_i x_i^2] \ . \tag{29}$$

The sign $\mp$ in front of the second term implies that H is restricted to act on the space of anti-symmetric (symmetric) wave-functions only. This model reduces to CSM of B_N, C_N and D_N type in the limit $l_2 = 0$, $l_1 = 0$ and $l_2 = l_1 = 0$, respectively. Without loss of generality, in this section, we therefore only study the B_N type model, i.e. $l_2 = 0$. The other cases are easily obtained from here.

It is interesting to observe that this system also shares the property of SI as found in [7] in the case of the A_{N-1} model. Superpotential corresponding to this model is

$$W_i = \frac{\partial G(x_1...x_N)}{\partial x_i} = \frac{\partial (ln\psi_0)}{\partial x_i} , \qquad (30)$$

where ψ_0 is the ground state wave function of H_{B_N} and G is given by

$$G = +l_1 \sum_i \ln(x_i) + l \sum_{i>j} \ln(x_i - x_j)(x_i + x_j) - \frac{\omega}{2} \sum_i x_i^2 . \qquad (31)$$

Thus, the superpotential takes the form

$$W_i = +l \sum_j{}' \left[(x_i - x_j)^{-1} + (x_i + x_j)^{-1} \right] + l_1 x_i^{-1} - \omega x_i . \qquad (32)$$

Following [7], define $\mathcal{A}_i(\mathcal{A}_i^\dagger) = \pm \partial_i + W_i$, from which Hamiltonian (29) with $l_2 = 0$ can be expressed as

$$H^{B_N} = \sum_i \mathcal{A}_i^\dagger \mathcal{A}_i - \left[\frac{N}{2} - lN(N-1) - l_1 N \right] \omega , \qquad (33)$$

Shape invariance follows due to the identity,

$$\sum_i \mathcal{A}_i \mathcal{A}_i^\dagger(l, l_1) = \sum_i \mathcal{A}_i^\dagger \mathcal{A}_i(l+1, l_1+1) . \qquad (34)$$

Shape invariance as observed in [7] for A_{N-1} CSM, is present not only in the rational B_N, D_N, BC_N models, but also in their trigonometric counterparts. For trigonometric BC_N models this is shown in the Appendix .

As in the A_{N-1} case, the SI condition does not help us in obtaining the spectrum of the rational BC_N models unless we employ the exchange operator formalism. Further, the Hamiltonian in eq. (29) can also be cast in a diagonal form using exchange operator method. This however requires including a reflection operator (t_i) where t_i commutes with x_j and anti-commutes with x_i. The Dunkl derivative operator (analogous to the A_{N-1} case) is given by

$$\mathcal{D}_i = -i\partial_i + il \sum_j{}' \left[(x_i - x_j)^{-1} M_{ij} + (x_i + x_j)^{-1} \tilde{M}_{ij} \right] + il_1 x_i^{-1}, \quad \tilde{M}_{ij} = t_i t_j M_{ij} . \quad (35)$$

The reflection operator t_i satisfies the following relations

$$t_i^2 = 1 \ , \quad t_i \psi(x_1, \ldots, x_i, \ldots, x_N) = \psi(x_1, \ldots, -x_i, \ldots, x_N) \ ,$$

$$M_{ij} t_i = t_j M_{ij} \ , \quad \tilde{M}_{ij}^\dagger = \tilde{M}_{ij} \ , \quad t_i \mathcal{D}_i = -\mathcal{D}_i t_i \ , \quad t_i \mathcal{D}_j = \mathcal{D}_j t_i \ , \quad j \neq i,$$

$$\tilde{M}_{ij} \mathcal{D}_i = -\mathcal{D}_i \tilde{M}_{ij} \ . \tag{36}$$

It follows from eq. (35) that $[\mathcal{D}_i, \mathcal{D}_j] = 0$ and

$$[x_i, \mathcal{D}_j] = i\delta_{ij} \left(1 + l \sum_k{}' (M_{ik} + \tilde{M}_{ik}) + 2l_1 t_i \right) - i(1 - \delta_{ij}) l(M_{ij} - \tilde{M}_{ij}) \ . \tag{37}$$

Defining, $\hat{a}_i$ and $\hat{a}_i^\dagger$ with the same defintion as in the previous case (see eq. (13)) and using the above equations, one finds

$$[\hat{a}_i, \hat{a}_j^\dagger] = 2\omega \delta_{ij} \left(1 + l \sum_k{}' (M_{ik} + \tilde{M}_{ik}) + 2l_1 t_i \right) - 2(1 - \delta_{ij}) l\omega (M_{ij} - \tilde{M}_{ij}) \ . \tag{38}$$

As before, the SUSY partner Hamiltonians $\mathcal{H}$ and $\tilde{\mathcal{H}}$ for the B_N case are defined as

$$\mathcal{H} = \frac{1}{2} \sum_i \hat{a}_i^\dagger \hat{a}_i \ , \quad \tilde{\mathcal{H}} = \frac{1}{2} \sum_i \hat{a}_i \hat{a}_i^\dagger \ . \tag{39}$$

It can be seen that,

$$H_{B_N} = \mathcal{H} + E_0^{B_N} \ , \quad E_0^{B_N} = [\frac{N}{2} \mp \frac{l}{2} N(N-1) + l_1 N]\omega \ . \tag{40}$$

The operator which brings in a correspondence between the eigenstates ψ and $\tilde{\psi}$ are respectively

$$\hat{A}_2 = \sum_i \hat{a}_i^2 \ , \quad \hat{A}_2^\dagger = \sum_i (\hat{a}_i^\dagger)^2 \ . \tag{41}$$

One can show that if $\psi(\tilde{\psi})$ is the eigenfunction of $\mathcal{H}(\tilde{\mathcal{H}})$ with eigenvalue $\mathcal{E}(\tilde{\mathcal{E}})$ then

$$\mathcal{H}(\hat{A}_2^\dagger \psi) = (\tilde{\mathcal{E}} - \hat{\delta}_2)(\hat{A}_2^\dagger \tilde{\psi}) \ , \quad \tilde{\mathcal{H}}(\hat{A}_2 \psi) = (\mathcal{E} + \hat{\delta}_2)(\hat{A}_2 \psi) \ . \tag{42}$$

where,

$$\hat{\delta}_2 = [N - 2 \pm 2lN(N-1) + 2l_1 N]\omega \ . \tag{43}$$

Now the question is why we should take $\hat{A}_2$ instead of $\hat{A}_1$ (note that $\hat{A}_n = \sum_i \hat{a}_i^n$)? The point is, unlike the A_{N-1} case, the BC_N Hamiltonian has the reflection symmetry, $x_i \rightarrow -x_i$. Such a symmetry on the wave-functions is ensured only if one uses $\hat{A}_2$ and not $\hat{A}_1$.

Following the treatment in the A_{N-1} case, it is easy to show that H and $\tilde{H}$ of the B_N model also satisfy the SI condition i.e.

$$\tilde{H}(\{x_i\}, l, l_1) = H(\{x_i\}, l, l_1) + R_2(l, l_1) \tag{44}$$

where

$$R_2(l, l_1) = [N \pm 2lN(N-1) + 2l_1 N]\omega \ . \tag{45}$$

Since in this case also all the l_i are identical, hence it is easy to see that the spectrum is given by

$$E_n = n(R_2 - \hat{\delta}_2) = 2n\omega \ . \tag{46}$$

Note that now the spectrum is given by $2n\omega$, instead of $n\omega$ as in the case of $\hat{A}_{N-1}$. This spectrum was also obtained earlier in [10], but by different method.

Thus we have shown that for the N-body Calogero models, the spectrum can also be obtained by using the ideas of SQM, SI and exchange operator formalism.

III. SUMMARY & DISCUSSIONS

In this paper, we have generalized the ideas of SUSY QM with one degree of freedom to the rational-CSM, which is a many-body problem. In particular, we have shown that the exchange operator formalism is suitable for relating the non-zero eigen states of the

partner Hamiltonians of CSM. The shape invariance in this formalism becomes trivial compared to the case discussed in [7]. In fact, the potentials of the partner Hamiltonians differ by a constant and this is reminiscent of the usual harmonic oscillator case. As a result, the operator method employed in [9] for solving the rational-CSM algebraically and the SUSY method described here are not very different from each other.

One of the nontrivial check of the applicability of the SUSY QM and the SI ideas to the many-body problems lies in solving the trigonometric CSM, since unlike the oscillator case, in this case the energy spectrum is not linear in the radial quantum number. Unfortunately, the generalized momentum operator D_i for all types of models, rational as well as trigonometric CSM associated with the root structure of A_n, B_n, D_n and BC_n, are hermitian by construction. So, we can not talk of partner Hamiltonians in terms of D_i alone. We can define the usual creation and the anhilation operators, $a_i^\dagger$ and a_i, in case we are dealing with the rational-CSM and construct partner Hamiltonians. Unfortunately, this can not be done for the trigonometric models. On the other hand, as described in [7], we can indeed introduce partner Hamiltonians for A_n type of trignometric models provided the exchange operator formalism has not been used. The SI is present in this formalism also but the task of relating the eigenspectrum of the partner Hamiltonians is unknown as yet. We have shown in this paper that the SI is also present in the most general BC_N type of trignometric models. However, the problem again lies in our inability to relate the the spectrum of the partner Hamiltonians.

APPENDIX A: SI IN TRIGONOMETRIC BC_N CSM MODEL

In this Appendix, we present the SI conditions for the trigonometric BC_N models. The trigonometric BC_N Hamiltonian is given by,

$$H_{BC_N} = -\sum_i \partial_i^2 + l(l-1) \sum_{i,j}{}' \left[\frac{1}{\sin^2(x_i - x_j)} + \frac{1}{\sin^2(x_i + x_j)} \right]$$
$$+ \sum_i \frac{l_1(l_1 - 1)}{\sin^2 x_i} + l_2(l_2 - 1) \sum_i \frac{1}{\sin^2 2x_i} \ . \tag{A1}$$

This model reduces to B_N, C_N and D_N for (a) $l_2 = 0$, (b) $l_1 = 0$, (c) $l_1 = l_2 = 0$, respectively. We define a superpotential of the form,

$$W_i = l \sum_j \left[\cot(x_i - x_j) + \cot(x_i + x_j) \right] + l_1 \cot x_i + l_2 \cot 2x_i \ . \tag{A2}$$

Using this expression of W_i in the definition of the creation and the anhilation operators as defined in (1), one can construct partner Hamiltonians which are equivalent to H_{BC_N} up to an overall constant. The SI condition for these partner Hamiltonians is

$$H_2^{BC_N}(\{x_i\}, l, l_1, l_2) = H_1^{BC_N}(\{x_i\}, l', l'_1, l'_2) + R^{BC_N} \ , \tag{A3}$$

where

$$R^{BC_N} = 2(l'l'_1 - ll_1)N(N-1) + \frac{4}{3}N(N-1)(N-2)(l'^2 - l^2) + 4N(l'_2 l'_1 - l_2 l_1)$$
$$+ 4N(N-1)(l'_2 l' - l_2 l) + 4N(l'^2_2 - l_2) + N(l'^2_1 - l_1) + 2N(N-1)(l'^2 - l^2) \tag{A4}$$

and

$$l' = l - 1 \ , \quad l'_1 = l_1 - 1 \ , \quad l'_2 = l_2 - 1 \ . \tag{A5}$$

The SI condition for B_N, C_N and D_N can be obtained from the above equations by taking appropriate limits. In particular, by putting $l_2 = 0$, $l_1 = 0$ or $l_2 = l_1 = 0$, we obtain the corresponding results for the B_N, C_N and D_N type models respectively.

REFERENCES

[1] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. **251** (1995) 267.

[2] F. Calogero, J. Math. Phys. (N.Y.) **10** (1969) 2191; **10** (1969) 2197.

[3] B. Sutherland, J. Math. Phys.(N.Y.) **12** (1971) 246; **12** (1971) 251; Phys. Rev. **A 4** (1971) 2019.

[4] M. A. Olshanetsky and A. M. Perelomov, Phys. Rep. **71** (1981) 314; **94** (1983) 6.

[5] F. D. M. Haldane, Phys. Rev. Lett. **60** (1988) 635; B. S. Shastry, *ibid.* **60** (1988) 639;

[6] B. D. Simons, P. A. Lee and B. L. Altshuler, Phys. Rev. Lett. **72** (1994) 64.

[7] C. Efthimiou and H. Spector, Phys. Rev. **A 56** (1997) 208.

[8] A. P. Polychronakos, Phys. Rev. Lett. **69** (1992) 703.

[9] S.B. Isakov and J.M.Leinass, Nucl Phys **B463** (1996) 194.

[10] T.Yamamoto, cond-mat/9508012 YITP/K-1110.