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Relativistic particle interactions—A comparison of independent and collective variable models

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Abstract. We present a detailed comparison of two models for relativistic classical particle interactions recently discussed in the literature—one based on independent particle variables, and the other on centre of mass plus relative variables. Basic to a meaningful comparison is a reformulation of the latter model which shows that it makes essential use of the concept of invariant relations from constrained Hamiltonian theory. We conclude that these two models have very different physical and formal structures and cannot be thought of as two equivalent descriptions of the same physical theory.

Keywords. Hamiltonian relativistic mechanics; constrained Hamiltonian dynamics; relativistic particle mechanics; relativistic particle interactions.

1. Introduction

Considerable attention has been paid recently to the problem of describing interactions among classical relativistic point particles in the Hamiltonian formalism. The motivation behind much of this work was to find ways to overcome the well-known no-interaction theorem (Currie *et al* 1963; Cannon and Jordan 1964; Leutwyler 1965) which states: in the instant form of Dirac's relativistic Hamiltonian dynamics, there can be no interactions if one insists on the objective reality of particle world lines. It has been found possible to achieve this aim by exploiting the methods of constrained Hamiltonian dynamics, and at the same time by choosing the evolution parameter in a dynamical way (Todorov 1971, 1976; Komar 1978a, b, c; Dominici *et al* 1978a, b; Rohrlich 1979a, b; Mukunda and Sudarshan 1981; Sudarshan *et al* 1981; Balachandran *et al* 1982a). This means that the Hamiltonian equations of motion must be written not with respect to kinematical time but with respect to a dynamically chosen parameter. In most of the work in this direction, this is taken to be essentially the time in the overall rest frame of the system. As a result one works outside the framework of Dirac's instant form of relativistic dynamics (Dirac 1949). This increased flexibility permits the construction of relativistic particle models with non trivial interactions as well as invariant world lines.

It is a feature of the constraint formalism that one can handle the relativistic aspects of the problem in several apparently different ways. One may use the method of independent particle variables (IPV) involving four-vectors of position and momentum for each particle (Todorov 1971, 1976; Komar 1978a, b, c; Sudarshan *et al* 1981); or one may introduce collective centre of energy and total four-momentum supplemented by relative variables for the individual particles (CMV) (Rohrlich 1979a, b; Mukunda and Sudarshan 1981). One can even construct a third scheme in which the collective variables describe the overall space-time orientation and four-dimensional angular momentum of the entire system (Balachandran *et al* 1982a). What differs from case to case is the choice of initial variables and the pattern of constraints needed to ensure that ultimately one has the correct number of degrees of freedom—$6N$ for an N particle system.

The purpose of this paper is to examine in some detail the relationship between the IPV and the CMV approaches to this problem. Several other attempts in this direction have appeared, but each of them differs in its approach and its specific results from ours (King and Rohrlich 1980; Balachandran *et al* 1982b; Iranzo *et al* 1982). Our method is to bring the CMV theory into an intermediate form where it can be rather easily compared to the starting point of the IPV theory, and thereafter the development of the two theories can be followed rather closely. We find that the IPV and CMV models are indeed very different in their structures and cannot be simply viewed as two different ways of presenting the same physical theory. In particular it will turn out that the CMV model, when reformulated as we do, makes essential use of an aspect of constraint theory that had hitherto not appeared in the present context—this is the idea of invariant relations.

The use of rest frame time as the evolution parameter leads to a physical problem—the various models are invariably nonseparable (Balachandran *et al* 1982c). It is possible to alter the IPV models significantly and to avoid this problem (Samuel 1982a, b). We have in this paper nothing new to say on this aspect, and as in the early work on this subject will identify the evolution parameter with rest frame time in both IPV and CMV models. Since the criterion of objective reality of particle world lines will be met in the models we discuss and compare, this part of the subject and the so-called world-line-conditions (Currie *et al* 1963; Kihlberg *et al* 1981) will not be touched upon here.

The material of this paper is arranged as follows. In §2 we give brief resumes of the IPV and CMV models as originally presented in the literature. The latter refers to a specific sequence in which the constraints of the CMV model are imposed. For simplicity and because of a common physical interpretation, the individual particle positions and momenta are denoted as q_a^μ and p_a^μ, $a = 1, 2, \ldots, N$, in both models: while they are primitive quantities in the IPV method, they are derived quantities in the CMV method. Section 3 develops the CMV model in a different way, and in particular takes it through an intermediate stage at which point the q_a and p_a are the independent variables of the theory. It is this that makes possible a meaningful comparison of the two models, and of the forces or interaction potentials that arise in them. Both in §§2 and 3 the final form of the CMV theory is the same, involving exactly $6N$ independent degrees of freedom. But the presentation of §3 makes it possible to see how very differently the final stages of the reduction to $6N$ degrees of freedom are carried out in the two cases. Section 4 examines the final physical bracket structures on the $6N$-dimensional physical phase spaces in the IPV and CMV models—what is common is the occurrence of quantities q_a, p_a with a common physical interpretation, but the phase space structures show many differences. Section 5 summarizes the main points of the analysis of §§3 and 4, and includes some concluding remarks.

2. Resumé of the IPV and CMV models

In this section we review briefly the IPV and the CMV methods for constructing models of relativistic interacting point particles. This will help set up a suitable notation and also serve as the basis for the discussion of later sections.

2.1 *The* IPV *method*

We follow here the presentation of Sudarshan *et al* 1981. The starting point is an $8N$-dimensional phase space Γ_{8N} with canonical coordinates q_a^μ, $p_{a\mu}$, $a = 1, 2, \ldots, N$. The fundamental PB's are

$$\{q_{a\mu}, p_{b\nu}\} = \delta_{ab}\, g_{\mu\nu},$$

$$\{q_{a\mu}, q_{b\nu}\} = \{p_{a\mu}, p_{b\nu}\} = 0. \tag{1}$$

(The metric is $g_{00} = 1$). So Γ_{8N} is the usual phase space built on the $4N$ dimensional configuration space of the variables $q_{a\mu}$. When the model is completed, q_a^μ and p_a^μ will respectively be interpreted as the space-time position and the four momentum of particle number a. On Γ_{8N} one has a canonical realisation of the Poincaré group $\mathscr{P}$, with generators

$$\mathscr{J}_{\mu\nu} = \sum_{a=1}^{N} (q_{a\mu} p_{a\nu} - q_{a\nu} p_{a\mu}),$$

$$\mathscr{P}_\mu = \sum_{a=1}^{N} p_{a\mu}. \tag{2}$$

The PB's among $\mathscr{J}_{\mu\nu}$ and $\mathscr{P}_\mu$ have the standard values corresponding to the Lie algebra of $\mathscr{P}$.

One now imposes on Γ_{8N} a set of N independent constraints

$$K_a = p_a^2 - m_a^2 - V_a(q, p) \approx 0,\ a = 1, 2, \ldots, N, \tag{3}$$

with the "potentials" V_a subject to two conditions: (i) they must be Poincaré invariant, (ii) the K_a must be first class. These requirements may be expressed as

$$\{\mathscr{J}_{\mu\nu}, V_a\} = \{\mathscr{P}_\mu, V_a\} = 0, \tag{4}$$

where the generators (2) of $\mathscr{P}$ are to be used, and

$$\{K_a, K_b\} = \{V_b, p_a^2\} + \{p_b^2, V_a\} + \{V_a, V_b\} = 0. \tag{5}$$

For the case that the V_a are all equal to a common V, the general solution to (5) has been developed in Sudarshan *et al* (1981).

Conditions (3) determine a $7N$-dimensional region Σ_{7N} in Γ_{8N}. (The symbols Γ, Γ', ... are used for spaces of dimension of the order of $8N$, while Σ, Σ', ... are used for spaces of dimension of the order of $6N$ or $7N$). Due to the first class conditions (5), the original PB's (1) on Γ_{8N} do not lead to a natural system of PB's among functions on Σ_{7N}. This region is invariant under the realisation of $\mathscr{P}$ generated by $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$. Moreover, the K_a generate an N-parameter Abelian group of canonical transformations also carrying Σ_{7N} onto itself. In fact these transformations give rise to a foliation of Σ_{7N} into a $6N$-parameter family of leaves or "sheets", each of dimension N. The Poincaré invariance of the K_a now implies that under the canonical transformations generated by $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$ these sheets are mapped onto one another in their entirety. Stated in another way, the quotient of Σ_{7N} with respect to the above foliation is the space of leaves $\tilde{\Sigma}_{6N}$, of dimension $6N$. Both the PB structure (1) and the canonical realisation of $\mathscr{P}$ on Γ_{8N} project down in a natural way to a PB structure and a canonical action of $\mathscr{P}$ on $\tilde{\Sigma}_{6N}$.

The IPV model, leading to a physical system of N interacting point particles, is completed by adjoining to the first class constraints (3) the following system of N

additional constraints (no confusion is likely to arise by the use of $\mathscr{P}$ for Poincaré group and $\mathscr{P}_\mu$ for the generator of translations):

$$\chi_a = \mathscr{P}\cdot(q_a - q_{a+1}) \approx 0,\ a = 1,2,\ldots,N-1;$$
$$\chi_N = \mathscr{P}\cdot q_N - \tau \approx 0. \tag{6}$$

(It is important to stress that though $\tilde{\Sigma}_{6N}$ is of dimension $6N$ and carries a realisation of $\mathscr{P}$, it is not the appropriate space for defining the dynamics of an N-particle system.) The χ_a for $a=1,2,N-1$ are Poincaré invariant; χ_N is not Poincaré invariant, and moreover involves an evolution parameter τ. The K_a and χ_a taken together should form a second class system:

$$\det |\{\chi_a, K_b\}| \neq 0. \tag{7}$$

The purpose of the constraints $\chi_a \approx 0$ is to determine a one-dimensional curve, parametrised by τ, on each sheet in Σ_{7N}. These curves are the possible states of motion of the physical N-particle system. In a moment we shall explain how the original canonical realisation of $\mathscr{P}$ on Γ_{8N} is to be amended so as to act on these states of motion in a consistent way. For each τ the region in Σ_{7N} determined by $\chi_a \approx 0$ will be denoted as Σ^τ_{6N}: it consists of one point, that with parameter value τ, taken from the one-dimensional curve on each sheet. Alternatively we can think of Σ^τ_{6N} as providing a τ-dependent section from $\tilde{\Sigma}_{6N}$ into Σ_{7N}.

The general equation of motion for a function $f(q,p,\tau)$ has the Hamiltonian form

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} \approx \frac{\partial f}{\partial \tau} + v_a\{f, K_a\} \tag{8}$$

where the coefficients v_a must be chosen so as to maintain the conditions $\chi_a \approx 0$. In view of (7), we denote the inverse to the matrix $(\{\chi_a, K_b\})$ by $(\mathscr{A}_{ab})$:

$$\mathscr{A}_{ab}\{\chi_b, K_c\} = \delta_{ac}. \tag{9}$$

Then since only χ_N has an explicit τ-dependence, the coefficients v_a in (8) are

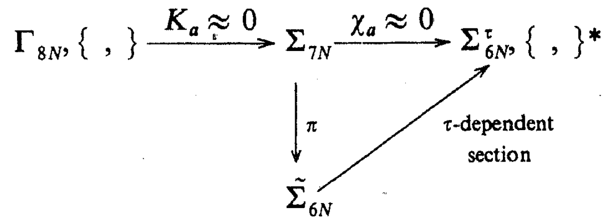
$$v_a = \mathscr{A}_{aN}. \tag{10}$$

The final physical bracket among dynamical variables is not the PB$\{\ ,\ \}$ originally defined on Γ_{8N} but rather the Dirac bracket (DB) $\{\ ,\ \}^*$ corresponding to elimination of all $2N$ constraints K, χ:

$$\{f,g\}^* = \{f,g\} - \mathscr{A}_{ab}(\{f,K_a\}\{\chi_b,g\} - \{f,\chi_b\}\{K_a,g\})$$
$$- \{f,K_a\}\,\mathscr{A}_{ab}\{\chi_b,\chi_{b'}\}\,\mathscr{A}_{a'b'}\{K_{a'},g\}. \tag{11}$$

Because of the vanishing of $\{\mathscr{J}_{\mu\nu}, K_a\}$ and $\{\mathscr{P}_\mu, K_a\}$, we see that the DB's among $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$ reproduce again the Lie algebra of $\mathscr{P}$. Thus these expressions generate a new realisation of $\mathscr{P}$, canonical with respect to the DB and distinct from the realisation *via* PB's used upto now. This new DB realisation of $\mathscr{P}$ is the physical one, and it does have the property of mapping each state of motion onto another one. Obviously so since the conditions $K_a \approx 0$, $\chi_a \approx 0$ are preserved.

The choice (6) for the χ_a identifies τ as the time in the centre-of-energy frame, apart from a factor $(\mathscr{P}^\mu \mathscr{P}_\mu)^{1/2}$. (It is implicitly assumed that in configurations of physical interest $\mathscr{P}^\mu$ is positive time-like). The fact that only χ_N is Poincaré-non-invariant (under the original PB realisation of $\mathscr{P}$) ensures that the world-lines of the N particles are objectively real—the so-called world line conditions are all obeyed. Since it is not of direct concern to us here, we avoid a discussion of these conditions, as also of the details of rewriting the equation of motion (8) in DB form using a suitable Hamiltonian. Suffice it to say that the physical system of N particles, realised *via* the IPV method here, has exactly $6N$ independent dynamical variables since all constraints $K_a \approx 0$, $\chi_a \approx 0$ must be obeyed. The scheme underlying the method can be depicted by a diagram, wherein the projection $\Sigma_{7N} \to \tilde{\Sigma}_{6N}$ is denoted by π:



2.2 *The* CMV *method*

We follow here the presentation of Mukunda and Sudarshan (1981). The symbols Γ', Σ' will be used for the various spaces that arise in this method. The starting point now is an $(8N+8)$-dimensional phase space Γ'_{8N+8} with canonical coordinates Q^μ, P_μ, ξ^μ_a, $\eta_{a\mu}$, $a = 1, 2, \ldots, N$. The basic non-vanishing PB's are taken to be

$$\{Q_\mu, P_\nu\} = g_{\mu\nu}, \quad \{\xi_{a\mu}, \eta_{b\nu}\} = \delta_{ab}\, g_{\mu\nu}. \tag{12}$$

(No confusion is likely to arise from the use of the same symbol $\{\ ,\ \}$ for the starting PB on Γ_{8N} in the IPV approach and for the PB on Γ'_{8N+8} here). Thus Γ'_{8N+8} is the usual phase space built on a $(4N+4)$-dimensional configuration space of variables Q_μ, $\xi_{a\mu}$. The final physical interpretation of Q^μ and P^μ is that they represent the centre-of-energy and the four momentum, respectively, of an N-particle system. On Γ'_{8N+8} we define a canonical realisation of $\mathscr{P}$ by taking the generators

$$\mathscr{J}_{\mu\nu} = Q_\mu P_\nu - Q_\nu P_\mu + \sum_{a=1}^{N} (\xi_{a\mu}\eta_{a\nu} - \xi_{a\nu}\eta_{a\mu}),$$

$$\mathscr{P}_\mu = P_\mu. \tag{13}$$

As in the IPV method, the final physical realisation of $\mathscr{P}$ will again be generated by these expressions but through a new bracket.

To identify the quantities q^μ_a, p^μ_a that will ultimately serve as the individual particle positions and momenta, we need to introduce a set of single-particle potentials U_a which are functions of differences of the ξ's, and of the η's, invariant under the homogeneous Lorentz group acting in the obvious way on the ξ's and η's:

$$U_a = U_a(\Delta\xi, \eta),$$

$$\{\mathscr{J}_{\mu\nu}, U_a\} = 0, \ a = 1, 2, \ldots, N. \tag{14}$$

In terms of U_a we define auxiliary expressions ω_a, ε_a as follows:

$$\omega_a(\Delta\xi, \eta) = (m_a^2 - \eta_a^2 + U_a(\Delta\xi, \eta))^{1/2},$$

$$\varepsilon_a(\Delta\xi, \eta) = \omega_a(\Delta\xi, \eta) / \sum_b \omega_b(\Delta\xi, \eta). \tag{15}$$

Like U_a, the ω_a and ε_a also have vanishing PB's with $\mathscr{J}_{\mu\nu}$; and it is assumed that $\omega_a > 0$ for physically relevant values of ξ, η. The individual particle positions and momenta are then defined by

$$\begin{aligned} q_a^\mu &= Q^\mu + \xi_a^\mu, \\ p_a^\mu &= \omega_a \hat{P}^\mu + \eta_a^\mu, \\ \hat{P}^\mu &= P^\mu/(P^2)^{1/2}. \end{aligned} \tag{16}$$

A system of $(2N+8)$ independent constraints needs to be imposed on Γ'_{8N+8} to lead to a space with just $6N$ dimensions, appropriate to an N-particle system. To do this efficiently, we introduce several useful objects at this point:

$$K_\mu = \sum_a \eta_{a\mu}^\perp; \tag{17a}$$

$$H_{\text{int}} = \sum_a \omega_a; \tag{17b}$$

$$K_4 = (P^2)^{1/2} - H_{\text{int}}; \tag{17c}$$

$$\chi_\mu = \sum_a \varepsilon_a \xi_{a\mu}^\perp; \tag{17d}$$

$$\chi_4 = P \cdot Q - \tau. \tag{17e}$$

The symbol $\perp$ denotes orthogonal projection with respect to P_μ. (It is again implicitly assumed that P^μ is time-like positive in all physically relevant configurations).

The objects defined above are all functions on Γ'_{8N+8}; and only the last of them, χ_4, involves an evolution parameter τ.

The first step in the reduction procedure is to impose on Γ'_{8N+8} a set of $2N$ second class constraints

$$P \cdot \xi_a \approx 0, \quad P \cdot \eta_a \approx 0. \tag{18}$$

These lead to a region Σ'_{6N+8} in Γ'_{8N+8} of the indicated dimension; and moreover among functions on Σ'_{6N+8} we can introduce a DB $\{\ ,\ \}'$ arising from the PB $\{\ ,\ \}$ on elimination of the constraints (18). The non-zero DB's among Q, P, ξ, η are

$$\begin{aligned} \{Q_\mu, Q_\nu\}' &= \frac{-1}{P^2} \sum_a (\xi_{a\mu}\eta_{a\nu} - \xi_{a\nu}\eta_{a\mu}); \\ \{Q_\mu, P_\nu\}' &= g_{\mu\nu}; \\ \{Q_\mu, \xi_{a\nu}\}' &= -\xi_{a\mu} P_\nu / P^2; \\ \{Q_\mu, \eta_{a\nu}\}' &= -\eta_{a\mu} P_\nu / P^2; \\ \{\xi_{a\mu}, \eta_{b\nu}\}' &= \delta_{ab}(g_{\mu\nu} - \hat{P}_\mu \hat{P}_\nu). \end{aligned} \tag{19}$$

The second step in the reduction procedure is similar in geometric terms to the passage from Γ_{8N} to Σ_{7N} in the IPV method. One imposes on Σ'_{6N+8} the four independent first class constraints contained in

$$K_\mu \approx 0, \ K_4 \approx 0, \tag{20}$$

and thereby arrives at a region Σ'_{6N+4} within Σ'_{6N+8}. (There are only four independent constraints here since $P^\mu K_\mu = 0$ identically). The first class condition here means that over Σ'_{6N+4}

$$\{K_\mu, K_\nu\}' \approx \{K_\mu, K_4\}' \approx 0. \tag{21}$$

We have also the Poincaré invariance of these constraints in the sense that, again over Σ'_{6N+4},

$$\{\mathcal{J}_{\mu\nu} \text{ or } \mathcal{P}_\mu, K_\lambda \text{ or } K_4\}' \approx 0. \tag{22}$$

Therefore the canonical transformations generated (through the DB $\{\ ,\ \}'$) by K_μ, K_4 map Σ'_{6N+4} onto itself, giving rise to a foliation of Σ'_{6N+4} into a $6N$ parameter family of four-dimensional leaves. This foliation is respected by the action of the Poincaré group (Note that whether we use $\mathcal{J}_{\mu\nu}$, $\mathcal{P}_\mu$ to generate a realisation of $\mathcal{P}$ *via* the PB $\{\ ,\ \}$ or the DB $\{\ ,\ \}'$ the effect on the basic variables is the same: Q_μ behaves as space-time position, P_μ and $\xi_{a\mu}$ and $\eta_{a\mu}$ as translation invariant four-vectors). If we pass to the quotient of Σ'_{6N+4} with respect to this foliation, we obtain a space $\tilde{\Sigma}'_{6N}$ which carries both a non-degenerate bracket structure and an action of $\mathcal{P}$. However, as in the IPV method, the dynamics of the N particle system is not seen on the space $\tilde{\Sigma}'_{6N}$; rather it involves choosing in a suitable way a one-dimensional curve on each leaf in Σ'_{6N+4}, with an evolution parameter τ, and then amending the action of $\mathcal{P}$ so as to map these curves onto one another in a consistent way. Before doing this, we record some useful DB relations which hold over Σ'_{6N+4}:

$$\{Q_\mu \text{ or } K_\mu, U_a \text{ or } \omega_a \text{ or } \varepsilon_a \text{ or } H_{\text{int}}\}' \approx 0; \tag{23a}$$

$$\{K_\mu, Q_\nu\}' \approx 0; \tag{23b}$$

$$\{K_\mu, \xi_{a\nu} \text{ or } \chi_\nu\}' \approx -g_{\mu\nu} + \hat{P}_\mu \hat{P}_\nu; \tag{23c}$$

$$\{P\cdot Q, \xi_{a\mu} \text{ or } \eta_{a\mu} \text{ or } K_\mu \text{ or } \chi_\mu\}' \approx 0; \tag{23d}$$

$$\{P\cdot Q, P_\mu\}' = P_\mu, \{P\cdot Q, K_4\}' \approx (P^2)^{1/2}; \tag{23e}$$

$$\{Q_\mu, (P^2)^{1/2}\}' \approx \hat{P}_\mu. \tag{23f}$$

The third and final step in the CMV method is to impose on Σ'_{6N+4} the four constraints

$$\chi_\mu \approx 0, \quad \chi_4 \approx 0. \tag{24}$$

These taken together with the earlier set (20) now form a second class system. We introduce at this point the following index conventions: $j, k, \ldots$ shall run over the values 1, 2, 3 while $r, s, \ldots$ run over 1, 2, 3, 4. Then 4×4 non-singular matrix of the DB's of the χ's with the K's can be exhibited as follows: If

$$\{\chi_\mu, K_4\}' = F_\mu, \quad P^\mu F_\mu \approx 0, \tag{25}$$

then

$$(\{\chi_r, K_s\}') \approx \begin{pmatrix} -\delta_{jk} - \hat{P}_j\hat{P}_k & F_j \\ 0 & (P^2)^{1/2} \end{pmatrix}. \tag{26}$$

(Note that in the limit $U_a = 0$, $F_\mu \approx 0$ as well). The inverse matrix is

$$\mathcal{A}_{rs}\{\chi_s, K_t\}' = \delta_{rt},$$

$$(\mathcal{A}_{rs}) = \begin{pmatrix} -\delta_{jk} + \dfrac{P_jP_k}{P_0^2} & G_j \\ 0 & (P^2)^{-1/2} \end{pmatrix}, \tag{27}$$

$$G_j = (F_j - P_j P_k F_k / P_0^2)/(P^2)^{1/2}.$$

The constraints (24) pick out a one-dimensional curve on each leaf in the foliation of Σ'_{6N+4}. These curves are parametrised by τ which appears in χ_4, and they correspond to states of motion of an interacting N-particle system. At a given value of τ, the constraints (24) determine a space Σ'^{τ}_{6N} in Σ'_{6N+4}. This appears as a τ-dependent section from $\tilde{\Sigma}'_{6N}$ into Σ'_{6N+4}. The general equation of motion for any function f of Q, P, ξ, η, τ is

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} \approx \frac{\partial f}{\partial \tau} + v^\mu \{f, K_\mu\}' + w\{f, K_4\}', \quad P^\mu v_\mu = 0, \tag{28}$$

where v^μ and w must be such that (24) are maintained in τ.

The solution is

$$w = (P^2)^{-1/2}, \quad v_\mu = -F_\mu/(P^2)^{1/2}, \tag{29}$$

so

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial \tau} + \{f, K_4 - F^\mu K_\mu\}'/(P^2)^{1/2}$$

$$\approx \frac{\partial f}{\partial \tau} + \left\{f, G_j K_j + \frac{K_4}{(P^2)^{1/2}}\right\}'. \tag{30}$$

One checks as a particular case that

$$\mathrm{d}Q^\mu/\mathrm{d}\tau = P^\mu/P^2. \tag{31}$$

The final physical bracket among dynamical variables is a DB $\{\ ,\ \}'^*$ arising from $\{\ ,\ \}'$ upon elimination of the K's and χ's:

$$\{f, g\}'^* = \{f, g\}' - \mathscr{A}_{rs}(\{f, K_r\}'\{\chi_s, g\}' - \{f, \chi_s\}'\{K_r, g\}')$$
$$- \{f, K_r\}' \mathscr{A}_{rs}\{\chi_s, \chi_{s'}\}' \mathscr{A}_{r's'}\{K_{r'}, g\}'. \tag{32}$$

In a "manifestly covariant form" this can be written as

$$\{f, g\}'^* \approx \{f, g\}' + [\{f, \chi^\mu\}'\{K_\mu, g\}'$$
$$+ \frac{1}{(P^2)^{1/2}}\{f, F^\mu K_\mu - K_4\}'\{P\cdot Q, g\}' - (f \leftrightarrow g)]$$
$$- \{f, K_\mu\}'\{\chi^\mu, \chi^\nu\}'\{K_\nu, g\}'. \tag{33}$$

Because of (22), we see that the DB's $\{\ ,\ \}'^*$ among $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$ reproduce the Lie algebra of $\mathscr{P}$. Thus they generate a new realisation of $\mathscr{P}$ canonical with respect to $\{\ ,\ \}'^*$, distinct from the original geometrical realisation on Γ'_{8N+8}. It is this new realisation of $\mathscr{P}$ that is the physical one, and that maps states of motion onto one another in a consistent way. After all the constraints of the CMV approach have been imposed one finds

$$\mathscr{J}_{\mu\nu} = \sum_a (q_{a\mu} p_{a\nu} - q_{a\nu} p_{a\mu}),$$

$$\mathscr{P}_\mu = \sum_a p_{a\mu},$$

$$Q_\mu = \sum_a \varepsilon_a q_{a\mu}, \quad P_\mu = \sum_a p_{a\mu}. \tag{34}$$

As in the IPV method, here again it is the fact that only χ_4 is τ-dependent and Poincaré-non-invariant (under the original geometrical action of $\mathscr{P}$) that allows the world line conditions for the individual particle positions q_a^μ as well as for Q^μ to be obeyed in a simple way. The parameter τ is once again essentially the time in the centre-of-energy frame, and one can put the equation of motion (30) into the Hamiltonian form using the physical DB $\{\ ,\ \}'^*$. The entire scheme can be depicted as follows:

$$\begin{array}{ccccccc} \Gamma'_{8N+8},\{\ ,\ \} & \xrightarrow{P\cdot\xi\approx P\cdot\eta\approx 0} & \Sigma'_{6N+8},\{\ ,\ \}' & \xrightarrow{K_r\approx 0} & \Sigma'_{6N+4} & \xrightarrow{\chi_r\approx 0} & \Sigma'^{\tau}_{6N},\{\ ,\ \}'^{*} \\ & & & & \Big\downarrow \pi & \nearrow \text{τ-dependent section} & \\ & & & & \tilde{\Sigma}'_{6N} & & \end{array}$$

3. Alternative form of the CMV model

The CMV model has a more intricate structure than the IPV model, though this is balanced by the fact that only very mild restrictions need to be imposed on the CMV potentials U_a. The full set of $(2N+8)$ constraints was imposed in the following sequence: First, $P\cdot\xi_a\approx P\cdot\eta_a\approx 0$; next $K_\mu\approx K_4\approx 0$; and last, $\chi_\mu\approx\chi_4\approx 0$. However the final result is independent of the sequence in which these $(2N+8)$ constraints are introduced, as long as all of them are included. Any other sequence is sure to lead to the same final physical N-particle system, physical brackets and realisation of $\mathscr{P}$. We now adopt the sequence

$$\Gamma'_{8N+8}\xrightarrow{K_\mu\approx K_4\approx 0}\Gamma'_{8N+4}\xrightarrow{\chi_\mu\approx\chi_4\approx 0}\Gamma'^{\tau}_{8N}\xrightarrow{P\cdot\xi\approx P\cdot\eta\approx 0}\Sigma'^{\tau}_{6N}. \tag{35}$$

In this form it becomes very easy to compare this model with the IPV model.

Over Γ'_{8N+8} the variables Q, P, ξ_a, η_a form a canonical coordinate system, with elementary values for their PB's $\{\ ,\ \}$. The variables q_a, p_a are defined by (16) all over Γ'_{8N+8}. We now show that we can use Q, P, q_a, p_a as a (non-canonical) coordinate system over Γ'_{8N+8}. Since, to begin with, it is not true that P_μ equals the sum of the $p_{a\mu}$ over a, we introduce as a definition over Γ'_{8N+8}:

$$P'_\mu=\sum_a p_{a\mu}. \tag{36}$$

The orthogonal projection with respect to P'_μ will be denoted by $\perp'$; it is distinct from the projection $\perp$ with respect to P_μ. (As with P_μ, we assume that P'_μ is positive timelike in all physically relevant configurations). From (16), the difference $\Delta\xi$ between any two ξ's equals the difference Δq between the corresponding q's: Symbolically,

$$\Delta\xi=\Delta q. \tag{37}$$

We can easily eliminate ξ_a in terms of q_a and Q,

$$\xi_a=q_a-Q. \tag{38}$$

The more difficult problem is to see how to eliminate η_a in favour of p_a; starting with the definitions of p_a in (16),

$$p_a=\omega_a\hat{P}+\eta_a, \tag{39}$$

we tentatively write

$$\eta_a = p_a - \alpha_a(\Delta q, p, \hat{P})\hat{P}. \tag{40}$$

We then develop the equation

$$\alpha_a^2 = \omega_a^2 = m_a^2 - \eta_a^2 + U_a(\Delta\xi, \eta) = m_a^2 + U_a - (p_a - \alpha_a \hat{P})^2,$$

i.e.

$$\alpha_a^2 - \alpha_a p_a \cdot \hat{P} = \tfrac{1}{2}(m_a^2 - p_a^2 + U_a(\Delta q, p - \alpha\hat{P})), \quad a = 1, 2, \ldots, N. \tag{41}$$

For the arguments of U_a, we have used (37) and (40). We will assume that these highly implicit equations for the α's can be solved to express each α_a as some function of Δq, p and $\hat{P}$: these are the only quantities appearing here apart from the α's. The solutions will for convenience be written in the form

$$\alpha_a(\Delta q, p, \hat{P}) = \tfrac{1}{2}\{p_a \cdot \hat{P} + [(p_a \cdot \hat{P})^2 + 2(m_a^2 - p_a^2 + \tilde{U}_a(\Delta q, p, \hat{P}))]^{1/2}\}, \tag{42}$$

thereby introducing the expressions $\tilde{U}_a$. It is important to stress that the $\tilde{U}_a$ are completely determined by the original potentials U_a of the CMV model. For orientation, we remark that $U_a = 0 \Rightarrow \tilde{U}_a = 0$, and $U_a = U_a(\Delta\xi) \Rightarrow \tilde{U}_a = U_a(\Delta q)$. With the solutions for α_a in hand, we have (38) and (40) valid all over Γ'_{8N+8}, and moreover

$$U_a(\Delta\xi, \eta) = \tilde{U}_a(\Delta q, p, \hat{P}),$$

$$\omega_a(\Delta\xi, \eta) = \alpha_a(\Delta q, p, \hat{P}), \tag{43}$$

also hold all over Γ'_{8N+8}. Thus we can use QPq_ap_a as a system of independent coordinates for Γ'_{8N+8}. In terms of them, we can write:

$$K_\mu = (P'^{\perp})_\mu, \quad K_4 = (P^2)^{1/2} - H_{\text{int}},$$

$$H_{\text{int}} = \sum_a \alpha_a(\Delta q, p, \hat{P}). \tag{44}$$

Now we take the first step of the new reduction procedure (35) and impose on Γ'_{8N+8} four independent first class constraints

$$K_\mu \approx 0, K_4 \approx 0. \tag{45}$$

The first class property, with respect to the PB $\{\ ,\ \}$ on Γ'_{8N+8}, is obvious from the previous section; it depends essentially on the fact that only the differences $\Delta\xi$ appear in the CMV potentials U_a. Denote by Γ'_{8N+4} the region determined by the constraints (45) in Γ'_{8N+8}. We can see that in this region P_μ is determined as a function of q_a and p_a. For, firstly,

$$K_\mu \approx 0 \Rightarrow \hat{P}_\mu \approx \hat{P}'_\mu, \quad \perp \approx \perp',$$

$$\hat{P}'_\mu = P'_\mu/(P'^2)^{1/2}. \tag{46}$$

Making use of this simplification, we shall set over Γ'_{8N+4}:

$$\alpha_a(\Delta q, p, \hat{P}') = \beta_a(\Delta q, p),$$

$$\tilde{U}_a(\Delta q, p, \hat{P}') = \bar{V}_a(\Delta q, p). \tag{47}$$

(It is these $\bar{V}_a$ that will later be compared to the IPV potentials V_a). Then the remaining constraint in (45) gives, over Γ'_{8N+4}:

$$P_\mu \approx \hat{P}'_\mu \sum_a \beta_a(\Delta q, p). \tag{48}$$

It follows that since $QPq_a p_a$ formed a coordinate system over Γ'_{8N+8}, now $Qq_a p_a$ form a coordinate system for Γ'_{8N+4}.

With the Poincaré generators of (13) and the PB$\{\ ,\ \}$ on Γ'_{8N+8}, it is obvious that on Γ'_{8N+4} we have

$$\{\mathscr{J}_{\mu\nu} \text{ or } \mathscr{P}_\mu,\ K_\lambda \text{ or } K_4\} \approx 0. \tag{49}$$

(This statement must be distinguished from (22).) Thus the canonical realisation of $\mathscr{P}$ on Γ'_{8N+8} leaves Γ'_{8N+4} invariant. Moreover by a now familiar argument we see that the canonical transformations generated by K_μ and K_4 foliate Γ'_{8N+4} giving a $8N$-parameter family of four-dimensional sheets, and this foliation is respected by the canonical realisation of $\mathscr{P}$. If we pass to the quotient of Γ'_{8N+4} with respect to this foliation, we get a space $\tilde{\Gamma}'_{8N}$ which carries both a bracket structure and a compatible action of $\mathscr{P}$.

In preparation for the second step of (35), we express χ_μ, χ_4 over Γ'_{8N+4} in terms of $Q\,q_a\,p_a$:

$$\chi_\mu \approx -Q_\mu^\perp + \Big(\sum_a \beta_a(\Delta q, p)\, q_{a\mu}^\perp\Big)\Big/\sum_b \beta_b(\Delta q, p),$$

$$\chi_4 \approx -\tau + \hat{P}'\cdot Q \sum_a \beta_a(\Delta q, p). \tag{50}$$

The second step in the reduction (35), namely imposing on Γ'_{8N+4} the four constraints

$$\chi_\mu \approx 0, \quad \chi_4 \approx 0, \tag{51}$$

can be viewed in two equivalent ways: either (i) as a way of choosing a one-dimensional curve parametrized by τ on each four-dimensional sheet in Γ'_{8N+4}, or (ii) as a choice of a τ-dependent section Γ'^{τ}_{8N} from $\tilde{\Gamma}'_{8N}$ into Γ'_{8N+4}. This already means that on Γ'^{τ}_{8N} we have a definite equation of motion, *i.e.* a definite dynamics, though it is for a system with $8N$ degrees of freedom and not for an N-particle system. First we note that the effect of the four constraints (51) is to determine Q in terms of $q_a\,p_a$ over Γ'^{τ}_{8N}. Remembering that $\perp \approx \perp'$ already on Γ'_{8N+4}, we get:

$$\chi_\mu \approx 0,\ \chi_4 \approx 0 \Rightarrow$$

$$Q_\mu \approx \Big(\tau \hat{P}'_\mu + \sum_a \beta_a(\Delta q, p)\, q_{a\mu}^\perp\Big)\Big/\sum_b \beta_b(\Delta q, p). \tag{52}$$

Therefore, we can say that $q_a\,p_a$ form a coordinate system over Γ'^{τ}_{8N}. As a result, from this point onwards a meaningful comparison with the IPV model becomes possible. As part of this comparison, we must pass from the PB$\{\ ,\ \}$ on Γ'_{8N+8} to a DB$\{\ ,\ \}^\dagger$ on Γ'^{τ}_{8N} by elimination of K_μ, K_4, χ_μ, χ_4 and express the Poincaré generators $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$ of (13) in terms of q_a, p_a. The latter task is easy; we find the expressions

$$\mathscr{P}_\mu = P_\mu = \hat{P}'_\mu \sum_a \beta_a(\Delta q, p),$$

$$\mathscr{J}_{\mu\nu} = (Q_\Lambda P + \sum_a \xi_{a\Lambda}\eta_a)_{\mu\nu}$$

$$= \left(\sum_a q_{a\Lambda} p_a + \left(\frac{1}{(P'^2)^{1/2}} - \frac{1}{\sum_a \beta_a} \right) \sum_b \beta_b q_{b\Lambda} P' \right)_{\mu\nu}. \tag{53}$$

As functions of $q_a p_a$, these expressions are strikingly different from those of the IPV model, equations (2). Nevertheless, our general procedure guarantees, because of (49), that with respect to the DB $\{\ ,\ \}^\dagger$ on Γ'^{τ}_{8N} these $\mathscr{J}_{\mu\nu}$ and $\mathscr{P}_\mu$ will reproduce the Lie algebra of $\mathscr{P}$. To determine $\{\ ,\ \}^\dagger$, we note some properties of the K's and χ's and introduce some definitions:

$$\{K_\mu, K_\nu \text{ or } K_4 \text{ or } \chi_4\} = \{\chi_\mu, \chi_4\} = 0; \tag{54a}$$

$$\{\chi_\mu, K_\nu\} = g_{\mu\nu} - \hat{P}_\mu \hat{P}_\nu,\ \{\chi_4, K_4\} = (P^2)^{1/2}; \tag{54b}$$

$$\{\chi_\mu, K_4\} = \zeta_\mu, \quad P^\mu \zeta_\mu = 0; \tag{54c}$$

$$\{\chi_\mu, \chi_\nu\} = \Omega_{\mu\nu}, \quad P^\mu \Omega_{\mu\nu} = 0. \tag{54d}$$

Then among the independent components K_r, χ_s, $rs = 1, 2, 3, 4$ of the constraint systems [(45) and (51)], we have the following matrices of PB's:

$$\{K_r, K_s\} = 0; \tag{55a}$$

$$(\{\chi_r, K_s\}) = \begin{pmatrix} -\delta_{jk} - \hat{P}_j \hat{P}_k & \zeta_j \\ 0 & (P^2)^{1/2} \end{pmatrix}; \tag{55b}$$

$$(\{\chi_r, \chi_s\}) = \begin{pmatrix} \Omega_{jk} & 0 \\ 0 & 0 \end{pmatrix}. \tag{55c}$$

The inverse to the matrix (55b) enters in the construction of the DB $\{\ ,\ \}^\dagger$:

$$\Delta_{rs}\{\chi_s, K_t\} = \delta_{rt},$$

$$(\Delta_{rs}) = \begin{pmatrix} -\delta_{jk} + \dfrac{P_j P_k}{P_0^2} & \rho_j \\ 0 & (P^2)^{-1/2} \end{pmatrix}, \tag{56}$$

$$\rho_j = (\zeta_j - P_j P_k \zeta_k / P_0^2)/(P^2)^{1/2}.$$

(Note that the matrices $(\{\chi_r, K_s\})$ and (Δ_{rs}) here are similar in structure to the matrices $(\{\chi_r, K_s\}')$ and $(\mathscr{A}_{rs})$ of (26) and (27) used in the previous reduction procedure: F_j and G_j are replaced by ζ_j and ρ_j respectively). On Γ'^{τ}_{8N} we now have the DB

$$\{f, g\}^\dagger = \{f, g\} - \Delta_{rs}(\{f, K_r\}\{\chi_s, g\} - \{f, \chi_s\}\{K_r, g\})$$
$$- \{f, K_r\}\Delta_{rs}\{\chi_s, \chi_{s'}\}\Delta_{r's'}\{K_{r'}, g\}. \tag{57}$$

This DB among the independent coordinates $q_a p_a$ of Γ'^{τ}_{8N} arising at this stage of the reduction of the CMV model, is to be compared with the elementary brackets (1) on Γ_{8N} which formed the starting point of the IPV model!

The equation of motion on Γ'^{τ}_{8N} takes the form

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} \approx \frac{\partial f}{\partial \tau} + u^{\mu}\{f, K_{\mu}\} + z\{f, K_4\},$$

$$P_{\mu}u^{\mu} = 0, \tag{58}$$

with u^{μ} and z chosen so as to maintain the constraints (51) in τ.

The solution is

$$z = (P^2)^{-1/2}, \quad u_{\mu} = -\zeta_{\mu}/(P^2)^{1/2}, \tag{59}$$

so

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} \approx \frac{\partial f}{\partial \tau} + \{f, K_4 - \zeta^{\mu}K_{\mu}\}/(P^2)^{1/2}$$

$$\approx \frac{\partial f}{\partial \tau} + \{f, \rho_j K_j + K_4/(P^2)^{1/2}\}. \tag{60}$$

Equations (58), (59) and (60) are to be compared with the previous equations (28), (29) and (30) respectively. The present set describes a Poincaré invariant dynamics for a system with $8N$ degrees of freedom; the Poincaré generators are as in (53) and they act through the DB $\{\ ,\ \}^{\dagger}$. The previous set described already the final form of the CMV model: a Poincaré invariant dynamics for a physical N-particle system with $6N$ degrees of freedom, the Poincaré generators taken as in (34) and acting through the final physical DB $\{\ ,\ \}'^{*}$ on Σ'^{τ}_{6N}.

For the last step of the reduction (35) to take us from Γ'^{τ}_{8N} to Σ'^{τ}_{6N}, we note that on Γ'^{τ}_{8N} we have

$$P\cdot\xi_a \approx P\cdot q_a - \tau; \tag{61a}$$

$$\hat{P}\cdot\eta_a \approx \hat{P}'\cdot(p_a - \beta_a\hat{P}')$$

$$\approx \tfrac{1}{2}\{p_a\cdot\hat{P}' - [(p_a\cdot\hat{P}')^2 + 2(m_a^2 - p_a^2 + \bar{V}_a(\Delta q, p))]^{1/2}\}. \tag{61b}$$

From the *numerical* point of view, therefore, on imposing the $2N$ second class constraints on Γ'^{τ}_{8N}

$$P\cdot\xi_a \approx P\cdot\eta_a \approx 0 \tag{62}$$

to arrive at Σ'^{τ}_{6N}, what we have are just the relations

$$P\cdot\xi_a \approx 0 \Rightarrow P\cdot q_a - \tau \approx 0; \tag{63a}$$

$$P\cdot\eta_a \approx 0 \Rightarrow \beta_a(\Delta q, p) \approx p_a\cdot\hat{P} \Rightarrow H_{\text{int}} \approx P'\cdot\hat{P} \Rightarrow P_{\mu} \approx P'_{\mu},$$

also $$\Rightarrow p_a^2 - m_a^2 - \bar{V}_a(\Delta q, p) \approx 0. \tag{63b}$$

These are similar in *appearance* to the K's and χ's that define the IPV model, but not in their *canonical* properties. (Incidentally we note that while on the spaces Γ'_{8N+4} and Γ'^{τ}_{8N} we have $\hat{P}_{\mu} = \hat{P}'_{\mu}$, on Σ'^{τ}_{6N} we have the complete equality $P_{\mu} = P'_{\mu}$.) From the *dynamical* point of view, with reference to the equation of motion (60) already existing on Γ'^{τ}_{8N}, we find

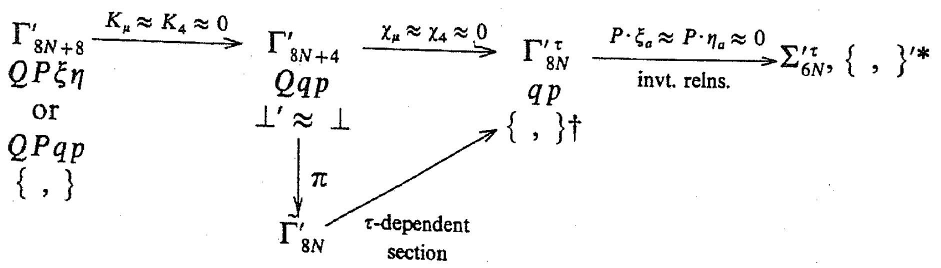
$$\frac{\mathrm{d}}{\mathrm{d}\tau}P\cdot\eta_a \approx \frac{1}{2(P^2)^{1/2}}\sum_b \frac{1}{\omega_b}P\cdot\frac{\partial}{\partial\xi_a}U_b(\Delta\xi, \eta),$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}P\cdot\xi_a \approx \frac{P\cdot\eta_a}{\omega_a(P^2)^{1/2}} - \frac{1}{2(P^2)^{1/2}}\sum_b \frac{1}{\omega_b}P\cdot\frac{\partial}{\partial\eta_a}U_b(\Delta\xi, \eta). \tag{64}$$

Since the U's are formed in a Lorentz invariant way out of $\Delta\xi$ and η, we assume that what appear in the U's are various scalar products among these four-vectors. Then the right sides in (64) are linear in $P\cdot\xi$ and $P\cdot\eta$. This means that with respect to the dynamics (60) on Γ'^{τ}_{8N}, the constraints (62) have the appearance of a system of *invariant relations*: if to begin with $P\cdot\xi_a \approx P\cdot\eta_a \approx 0$, they continue to vanish for all τ. The system (62) is therefore a way to reduce the number of degrees of freedom from $8N$ in Γ'^{τ}_{8N} to $6N$ in Σ'^{τ}_{6N}, consistent with the dynamics on the former space.

If from the DB $\{\ ,\ \}^{\dagger}$ on Γ'^{τ}_{8N} we form the final DB $\{\ ,\ \}'^{*}$ by elimination of $P\cdot\xi_a$ and $P\cdot\eta_a$, the result will be the same as in the previous section: we definitely recover (32) and (33). At the same time, making use of (63b) we easily see that the Poincaré generators $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_{\mu}$ (53) reduce to the forms (34), as they ought to.

Having completed this second way of presenting the CMV model, we can depict this scheme by adding the new important elements to the diagram (35):



4. Comparison on the physical phase spaces

It is possible to compare the two models at yet another level, by formulating each model on its true physical phase space. The idea here is to choose $6N$ variables u_α, $\alpha = 1, 2, \ldots, 6N$ with the property that their final physical Dirac brackets are independent of τ when expressed in terms of themselves:

$$\frac{\partial'}{\partial\tau}\{u_\alpha, u_\beta\}^{*} = 0. \tag{65}$$

($\partial'/\partial\tau$ denotes τ differentiation with the u_α's held constant; and the bracket here refers to the final bracket obtained by eliminating all the constraints of the theory). It then follows (Sudarshan *et al* 1981) that there exists a physical Hamiltonian function, $\mathscr{H}(u,\tau)$ on the physical phase space Σ_{6N} (we drop the superscript τ in view of (65)) spanned by these variables (u_α, $\alpha = 1, 2, \ldots, 6N$), which reproduces the equations of motion as given by (8), (10) or (28), (29) through the *Dirac* bracket

$$\mathrm{d}u_\alpha/\mathrm{d}\tau = \{u_\alpha, \mathscr{H}\}^{*}. \tag{66}$$

On Σ_{6N}, the constraint functions vanish strongly, so these relations on the original phase space can be used to express the physically significant functions (q^{μ}_{a}, $p_{a\mu}$, $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_{\mu}$) in terms of the u's and τ. $\mathscr{J}_{\mu\nu}(u,\tau)$ and $\mathscr{P}_{\mu}(u,\tau)$ provide a realisation of the Poincaré group on Σ_{6N} through the Dirac bracket. $q^{\mu}_{a}(u,\tau)$ and $p_{a\mu}(u,\tau)$ give the physical identification of the particle position and momentum variables on Σ_{6N}. $\mathscr{H}(u,\tau)$ is the Hamiltonian function and $\{u_\alpha, u_\beta\}^{*}$ provides Σ_{6N} with a (τ independent (65)) bracket structure. Thus it is possible to formulate each of the above models on a phase space with just the right number ($6N$) of degrees of freedom and no constraints.

In order to have a clear comparison of the two models, we will choose variables u with the same physical significance, so that the two theories resemble each other as closely as possible. It turns out that such a choice is indeed possible and consistent with the requirement (65). Once this is done, the differences between the two models are clearly visible and we will find (as we did in §3) that the IPV and CMV models represent entirely different physical theories.

IPV model: We choose for the u's the following $6N$ functions:

$$q_a^{\perp\mu} = O^\mu_\nu q_a^\nu,\ p_{a\mu}^\perp = O^\nu_\mu p_{a\nu},\quad \hat{\mathscr{P}}_\mu \tag{67}$$

where

$$O_{\mu\nu} = g_{\mu\nu} - \hat{\mathscr{P}}_\mu \hat{\mathscr{P}}_\nu \tag{68}$$

and

$$\hat{\mathscr{P}}_\mu = \mathscr{P}_\mu / \mathscr{P}, \quad \mathscr{P} = (\mathscr{P}_\mu \mathscr{P}^\mu)^{1/2} \tag{69}$$

For the physical Hamiltonian we guess the form

$$\mathscr{H} = -\ln \mathscr{P}. \tag{70}$$

(This functional form for $\mathscr{H}$ differs from that found by Hsu and Shi (1982). This is due to their use of a different set of functions for the u's). The equations of motion ((8) and (10)) for the τ evolution of any function give

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} = \{u_\alpha, K_a\}\mathscr{A}_{aN} = -\{u_\alpha, K_a\}\mathscr{A}_{ab}\frac{\partial \chi_b}{\partial \tau}. \tag{71}$$

If we notice that

$$\{K_a, \mathscr{H}\} = 0 \text{ (from (4))}, \quad \{u_\alpha, \mathscr{H}\} \approx 0, \quad \{\chi_b, \mathscr{H}\} \approx \partial\chi_b/\partial\tau, \tag{72}$$

we see from the definition of the DB (11) that

$$\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} \approx \{u_\alpha, \mathscr{H}\}^*, \tag{73}$$

and so $\mathscr{H}$ does reproduce the equations of motion (8) and (10) in Hamiltonian form with respect to the Dirac bracket. (It is now possible to reverse the arguments of Sudarshan *et al* (1981) and show that the existence of such an $\mathscr{H}$ in fact ensures that the Dirac brackets between the u's are τ independent (65) (Samuel 1984).

We now use the constraint functions (3), (6) to express the functions q_a^μ, $p_{a\mu}$ (and $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$) in terms of the u's. From (6) we have for the component of q_a along $\hat{\mathscr{P}}_\mu$:

$$q_a^\parallel \equiv q_a \cdot \hat{\mathscr{P}} = \tau / \mathscr{P}. \tag{74}$$

Next we suppose that the N constraints (3) are solved to expose $p_a^\parallel$ (the component of p_a along $\hat{\mathscr{P}}_\mu$) in terms of $q_a^\perp$, $p_a^\perp$, $\hat{\mathscr{P}}_\mu$ and τ:

$$p_a^\parallel = W_a(q_a^\perp, p_a^\perp, \hat{\mathscr{P}}_\mu, \tau). \tag{75}$$

It is clear that

$$\sum_a p_a^\parallel = \sum_a W_a = \mathscr{P}, \tag{76}$$

so that

$$q_a^\mu = q_a^{\perp\mu} + \tau\, \hat{\mathscr{P}}^\mu / \sum_b W_b, \tag{77}$$

$$p_{a\mu} = p^{\perp}_{a\mu} + W_a \hat{\mathscr{P}}_{\mu}. \tag{78}$$

The Poincaré generators (2) are written in terms of q_a, p_a and so they too can be expressed through (77) and (78) as functions of u and τ. Finally $\mathscr{H}$ is given by

$$\mathscr{H} = -\ln \sum_a W_a, \tag{79}$$

and so all the relevant variables have been expressed on the physical phase space Σ_{6N}.

CMV model: We again choose for the physical variables

$$\begin{aligned} q_a^{\perp\mu} &= Q^{\perp\mu} + \xi_a^{\mu}, \\ p_{a\mu}^{\perp} &= \eta_{a\mu}, \\ \hat{P}_{\mu}, & \end{aligned} \tag{80}$$

where $\perp$ now refers to orthogonal projection with respect to P_μ, and

$$\hat{P}_\mu = P_\mu/P, \quad P = (P_\mu P^\mu)^{1/2}. \tag{81}$$

Let us suppose that the first stage of reduction is done as discussed in §2 and the $2N$ constraints (18) eliminated. The present discussion starts with the space Σ'_{6N+8} endowed with the bracket $\{\ ,\ \}'$ given by (19). The equations of motion (28) and (29) on this space may be written

$$\mathrm{d}f/\mathrm{d}\tau \approx -\{f, K_r\}' \quad \mathscr{A}_{rs} \frac{\partial \chi_s}{\partial \tau}, \tag{82}$$

since the v^μ and w must be such that (24) are maintained in τ. We can once again confirm that the form of the physical Hamiltonian is

$$\mathscr{H}' = -\ln P. \tag{83}$$

By noticing that the u's (80), constraints (17) and $\mathscr{H}'$ (83) satisfy

$$\begin{aligned} &\{u_\alpha, \mathscr{H}'\}' \approx 0, \\ &\{\chi_s, \mathscr{H}'\}' \approx \partial\chi_s/\partial\tau, \\ &\{K_r, \mathscr{H}'\}' \approx 0 \text{ from (22)}, \end{aligned} \tag{84}$$

we see that

$$\mathrm{d}u_\alpha/\mathrm{d}\tau \approx \{u_\alpha, \mathscr{H}'\}'^{*}, \tag{85}$$

where the bracket $\{\ ,\ \}'^{*}$ is given by (32).

As before, we can express the physically significant functions q_a, p_a in terms of the u's (80) by using the constraint equations:

$$\begin{aligned} q_a^\mu &= Q^\mu + \xi_a^\mu = q_a^{\perp\mu} + \hat{P}^\mu \tau / \sum_b \omega_b, \\ p_{a\mu} &= p^{\perp}_{a\mu} + \omega_a \hat{P}_\mu, \end{aligned} \tag{86}$$

using (17e) and (24). ω_a in (15) is directly expressed as a function of $\Delta q^\perp$ and $p^\perp$ and so q_a^μ and $p_{a\mu}$ are given by (86) as functions of the u's (80). From (34), which holds after all the constraints have been imposed, the Poincaré generators too can be expressed as functions of the u's. From (17b, c) and (20) we see that

$$\mathscr{H}' = -\ln \sum_a \omega_a. \tag{87}$$

The two models have now been cast into a form where they can easily be compared. Note that in each case we have a $6N$ dimensional phase space spanned by $q_a^{\perp\mu}$, $p_{a\mu}^{\perp}$, $\hat{\mathscr{P}}_\mu$ and the expressions for the physically significant functions q_a, p_a ((77), (78) and (86)), $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$ ((2) and (34)) and $\mathscr{H}$ ((79) and (87)) are identical. The W_a (75) occurring in the IPV model have the same four-dimensional kinematic physical meaning as the ω_a (15) occurring in the CMV model and so, the correspondence is complete.

However, to discuss the equivalence of the two models it is also necessary to consider the bracket relations between the dynamical variables of the two theories: a correspondence at the algebraic level is not enough. It is in this respect that the two models differ. We will see below that although the bracket relations between some variables are identical in both models, they are completely different for some others. We discuss the similarities first and then the differences.

We know on general grounds that both the models are Poincaré invariant and so the bracket relations between the Poincaré generators (and functions of them) coincide:

$$\begin{aligned}
\{\mathscr{J}_{\mu\nu}, \mathscr{J}_{\rho\sigma}\}^* &= \{\mathscr{J}_{\mu\nu}, \mathscr{J}_{\rho\sigma}\}'^*,\\
\{\mathscr{J}_{\mu\nu}, \mathscr{P}_\rho\}^* &= \{\mathscr{J}_{\mu\nu}, \mathscr{P}_\rho\}'^*,\\
\{\mathscr{P}_\mu, \mathscr{P}_\nu\}^* &= \{\mathscr{P}_\mu, \mathscr{P}_\nu\}'^* = 0,\\
\{\hat{\mathscr{P}}_\mu, \mathscr{J}_{\rho\sigma}\}^* &= \{\hat{\mathscr{P}}_\mu, \mathscr{J}_{\rho\sigma}\}'^*,\\
\{\hat{\mathscr{P}}_\mu, \hat{\mathscr{P}}_\nu\}^* &= \{\hat{\mathscr{P}}_\mu, \hat{\mathscr{P}}_\nu\}'^* = 0,\\
\{\hat{\mathscr{P}}_\mu, \mathscr{P}_\nu\}^* &= \{\hat{\mathscr{P}}_\mu, \mathscr{P}_\nu\}'^* = 0.
\end{aligned} \tag{88}$$

Some algebra reveals that

$$\begin{aligned}
\{q_a^{\perp\mu}, \mathscr{J}_{\rho\sigma}\}^* &= \{q_a^{\perp\mu}, \mathscr{J}_{\rho\sigma}\}'^*,\\
\{p_{a\mu}^{\perp}, \mathscr{J}_{\rho\sigma}\}^* &= \{p_{a\mu}^{\perp}, \mathscr{J}_{\rho\sigma}\}'^*,\\
\{q_a^{\perp\mu}, \hat{\mathscr{P}}_\nu\}^* &= \{q_a^{\perp\mu}, \hat{\mathscr{P}}_\nu\}'^*,\\
\{p_{a\mu}^{\perp}, \hat{\mathscr{P}}_\nu\}^* &= \{p_{a\mu}^{\perp}, \hat{\mathscr{P}}_\nu\}'^*,\\
\{p_{a\mu}^{\perp}, p_{b\nu}^{\perp}\}^* &= \{p_{a\mu}^{\perp}, p_{b\nu}^{\perp}\}'^*.
\end{aligned} \tag{89}$$

The remaining brackets are quite different in the two models. As an example, we exhibit $\{u_\alpha, \mathscr{H}\}^*$ and $\{u_\alpha, \mathscr{H}'\}'^*$ which determine the τ development of the u's and hence the world lines.

IPV model:

$$\begin{aligned}
\{q_a^{\perp\mu}, \mathscr{H}\}^* &= 2p_a^{\perp\mu}\mathscr{A}_{aN} - O_\nu^\mu \sum_b \frac{\partial V_b}{\partial p_{a\nu}} \mathscr{A}_{bN},\\
\{p_{a\mu}^{\perp}, \mathscr{H}\}^* &= O_\mu^\nu \sum_b \frac{\partial V_b}{\partial q_a^\nu} \mathscr{A}_{bN}.
\end{aligned} \tag{90}$$

CMV model:

$$\{q_a^{\perp\mu}, \mathscr{H}'\}'^* = \frac{1}{\mathscr{P}}\left(\frac{\eta_a^\mu}{\omega_a} - O^\mu_\nu \sum_b \frac{\partial U_b}{\partial \eta_{a\nu}} \frac{1}{2\omega_b}\right) - O^{\mu\nu}\frac{F_\nu}{\mathscr{P}},$$

$$\{p_{a\mu}^\perp, \mathscr{H}'\}'^* = \frac{1}{\mathscr{P}} O^\nu_\mu \sum_b \frac{1}{2\omega_b}\frac{\partial U_b}{\partial \xi_a^\nu}. \tag{91}$$

It is easily seen that these expressions do not coincide, even in the case where the interaction potentials are small. (They of course coincide, as they must, in the free particle case). We therefore conclude that the two models have different dynamics and different particle world lines and so describe different physical systems.

5. Comparison and conclusions

The principal idea of §3 has been to realise that in the presentation of the CMV model, we may take the initial $8N+8$ independent variables to be Q_μ, P_μ, $q_{a\mu}$ and $p_{a\mu}$ rather than Q_μ, P_μ, $\xi_{a\mu}$, $\eta_{a\mu}$. Of course we realize that, unlike (12), the PB's $\{\ ,\ \}$ among q_a and p_a are not at all simple and kinematical even on Γ'_{8N+8}: while the PB's among the q's do vanish, both $\{q_a, p_b\}$ and $\{p_a, p_b\}$ are interaction-dependent. With the first stage of the reduction scheme (35), P_μ ceases to be an independent variable; and with the second stage Q_μ also gets determined, so that on Γ'^{τ}_{8N} just the $q_{a\mu}$ and $p_{a\mu}$ are independent.

The IPV model at the level of Γ_{8N} must now be compared to the CMV model at the level of Γ'^{τ}_{8N}. Both use $q_{a\mu}$, $p_{a\mu}$ with identical physical interpretations as independent variables. However, in the IPV case we have elementary PB's given by (1); while the DB's $\{\ ,\ \}^\dagger$ on Γ'^{τ}_{8N} among q's and p's are complicated and interaction-dependent. The potentials $V_a(q,p)$ of the IPV model are highly restricted by the first class requirements (5), to be contrasted with the CMV potentials $\bar{V}_a(\Delta q, p)$ which arise from the essentially unrestricted original CMV potentials $U_a(\Delta\xi, \eta)$. For example, the choice $U_a = U_a(\Delta\xi)$ is quite acceptable in the CMV approach; this leads to $\bar{V}_a = U_a(\Delta q)$ with no p-dependence at all, but such choices of V_a are completely disallowed in the IPV theory. In any case we must note that in general the functional forms of U_a and $\bar{V}_a$ are quite different, and the initial potentials U_a of the CMV method are not the objects to be compared to the V_a of the IPV method.

From the work of §4, we see that for a given IPV model with specified V_a, we can certainly choose a CMV model with suitable U_a such that q_a, p_a, the Poincaré generators, and physical Hamiltonian $\mathscr{H}$ are identical functions of variables u_α which share a common physical interpretation. To do this it is only necessary to determine W_a in (76) from V_a and choose U_a in the CMV model such that $\omega_a = W_a$. However, even after having achieved a similarity to this degree the difference between the models persists—being now isolated in the structure of the bracket relations. It is clear that there is far greater freedom in the CMV models, since the possible W_a in the IPV models are highly constrained by the first class conditions obeyed by the corresponding V_a. The ω_a of the CMV model are subject to no such restrictions. The extra freedom here is attained by trading the first class conditions (5) for invariant relations (64).

The Poincaré generators $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$ for the IPV theory have again the simple four-dimensional kinematic forms (2) on Γ_{8N}, making it obvious that their $\{\ ,\ \}$ brackets reproduce the Poincaré Lie algebra. For the CMV theory on Γ'^{τ}_{8N} on the other hand, we have highly non-trivial expressions (53) for $\mathscr{J}_{\mu\nu}$, $\mathscr{P}_\mu$: one would hardly believe that these expressions reproduce the Poincaré algebra through their DB's $\{\ ,\ \}^\dagger$, had one not known where these expressions and $\{\ ,\ \}^\dagger$ came from! Of course in their final forms on Σ^τ_{6N} and Σ'^τ_{6N} the IPV and CMV models use identical Poincaré generators—identical in appearance when expressed in terms of q_a and p_a as shown by (2) and (34). However, the final physical brackets are very different in the two cases, so the Poincaré realisations must be treated as distinct. This emerges clearly from §4 where the brackets $\{\ ,\ \}^*$ on Σ^τ_{6N} and $\{\ ,\ \}'^*$ on Σ'^τ_{6N} have been explicitly computed for pairs of quantities with the same physical meanings in the two cases, and the results have been quite different.

In the (final) reduction of the IPV model from Γ_{8N} to Σ^τ_{6N}, we have no dynamics to begin with, but end up with a definite dynamics. The reduction itself involves a foliation by first class constraints $K_a \approx 0$, followed by the choice of a τ-dependent section from Σ_{6N} into Σ_{7N}. On the other hand, the CMV model has a definite dynamics on Γ'^τ_{8N}; and the final reduction to Σ'^τ_{6N} uses a system of $2N$ second class constraints which are invariant relations with respect to this dynamics. This is a new and perhaps unexpected feature that has emerged from the particular way we cast the CMV theory in §3, and is added proof of the great degree of flexibility available with constraint methods. It also shows that the appearance of a large number of first class constraints, characteristic of the IPV model, is not an unavoidable feature of such constrained Hamiltonian theories of interacting relativistic particles—a fact realised and exploited elsewhere to solve the separability problem (Samuel 1982a, b).

It would be interesting to carry out a similar comparison of the IPV model and the model of Balachandran *et al* (1982a) wherein the collective variables consist of a Lorentz matrix describing the spacetime orientation of the entire system, and its conjugate four-dimensional angular momentum. One would need a reformulation of this model similar in spirit to §3 of this paper. We are then likely to find that the concept of invariant relations plays again, an essential role in the model.

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