

THE PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 178, No. 5

25 FEBRUARY 1969

Master Equations in Phase-Space Formulation of Quantum Optics*

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(Received 12 August 1968; revised manuscript received 17 October 1968)

Using the recently discussed quantum dynamics in phase space, we derive a master equation, starting from the phase-space equivalent to the Schrödinger equation of motion for the density operator. Use is made of Zwanzig's projection-operator techniques and some explicit realizations of the projection operators are given. The master equation is then applied to show that the time-correlation functions, as defined in the text, satisfy an integral equation of the Volterra type. Next, a master equation for a system interacting with a large system is derived. As an illustration, we determine the lowest-order Born approximation and carry out a short-memory-approximation calculation for an oscillator coupled to a reservoir and for a two-level system interacting with an oscillator heat bath; we obtain equations of the Fokker-Planck type. Some physical implications of these equations are also discussed.

1. INTRODUCTION

IN the last several years, there has been considerable interest in deriving the so-called "master equations"¹⁻⁴ for closed^{1,2} as well as open^{3,4} systems. One starts from the quantum-mechanical Liouville equation for the density operator and by suitable techniques one derives from it the equation of motion for the relevant part of the density operator. Using master equations, one can obtain equations of motion for the macroscopic observables and can study the evolution of macroscopic observables towards thermodynamic equilibrium.

Recently, a general method⁵ of transcribing the quantum-mechanical equations of motion into an equivalent equation of motion in the associated phase space was found.^{6,7} When this is done, the equation of

motion for the density operator becomes a c -number differential equation. In the present investigation, this equation forms the basis for the derivation of the master equation. We use the projection operator techniques as pioneered by Zwanzig^{1,8} and subsequently used by many authors.^{2-4,9} Once the phase-space form of the master equations are derived, they can be solved as if they were classical equations and in this manner many problems of nonequilibrium statistical mechanics can be solved.

We also consider the interaction of the system with another (large) system which acts as a heat bath. Again using Zwanzig's projection-operator technique, we obtain a master equation for the phase-space distribution function^{6,7} of the system of interest. In our analysis the bath and the system are assumed to form an isolated system described by the equivalent of a quantum-mechanical density-operator equation in phase space. We only assume that at some initial time the bath and the system are statistically independent and that the bath is in thermal equilibrium. Subject to these assumptions, the equation so derived is exact.

In Sec. 5, we consider the application of the formalism to two systems, namely, a harmonic oscillator and a two-level system interacting with a heat bath which consists of harmonic oscillators. We carry out the Born-approximation calculation and also make use of the

* Research supported by the U. S. Air Force Office of Scientific Research.

¹ R. Zwanzig, *Physica* **30**, 1109 (1964).

² G. G. Emch, in *Lectures in Theoretical Physics, 1965 Summer Institute for Theoretical Physics*, edited by W. E. Brittin (University of Colorado Press, Boulder, 1966), Vol. VIII A, p. 65.

³ G. G. Emch and G. L. Sewell, *J. Math. Phys.* **9**, 946 (1968).

⁴ P. N. Argyres, in *Lectures in Theoretical Physics, 1965 Summer Institute for Theoretical Physics*, edited by W. E. Brittin (University of Colorado Press, Boulder, 1966), Vol. VIII A, p. 183, and some of the references therein.

⁵ G. S. Agarwal and E. Wolf, *Phys. Rev. Letters* **21**, 180 (1968). Since this paper was written, a generalization of these results was obtained to a much wider class of associations; this will be discussed in a forthcoming paper.

⁶ G. S. Agarwal and E. Wolf, *Phys. Letters* **26A**, 485 (1968).

⁷ E. Wolf and G. S. Agarwal, in *Polarisation, Matière et Rayonnement* (Presses Universitaires de France, Paris, 1968), pp. 543-558.

⁸ R. W. Zwanzig, in *Lectures in Theoretical Physics*, edited by W. E. Brittin, B. W. Downs, and J. Downs (Interscience Publishers, Inc., New York, 1961), Vol. 3, p. 106.

⁹ P. N. Argyres and P. L. Kelley, *Phys. Rev.* **134**, A98 (1964).

short-memory approximation⁴ and obtain equations of the Fokker-Planck type. Some of the implications of these equations are also discussed.

2. MASTER EQUATION FOR A GENERAL SYSTEM

The density operator $\rho(t)$ of a quantum-mechanical system satisfies the equation of motion

$$i\partial\rho/\partial t = +[H, \rho] \quad (\hbar = 1), \quad (2.1)$$

where H is the Hamiltonian of the system. We have shown elsewhere^{5,7} that the density operator [Eq. (2.1)] is equivalent to the following equation in phase space:¹⁰

$$i\partial F_\rho/\partial t = F_H \exp(\vec{\Lambda}_1) [\exp(\vec{\Lambda}_2) - \exp(-\vec{\Lambda}_2)] F_\rho, \quad (2.2)$$

where F_ρ and F_H are¹¹ the Ω equivalents of the density operator $\rho(t)$ and of the Hamiltonian, respectively. Note that the relation between the Ω equivalent $F_G(z, z^*)$ of an operator G and the operator $G(a, a^\dagger)$ is given by⁶

$$F_G(z, z^*) = \pi \text{Tr}\{G(a, a^\dagger) \Delta^{(\bar{\Omega})}(z-a, z^*-a^\dagger)\}, \quad (2.3)$$

where

$$\Delta^{(\bar{\Omega})}(z-a, z^*-a^\dagger) = \frac{1}{\pi^2} \int d^2\alpha [\Omega(\alpha, \alpha^*)]^{-1} \times \exp[\alpha(z^*-a^\dagger) - \alpha^*(z-a)]. \quad (2.4)$$

The function $\Omega(\alpha, \alpha^*)$, which appears in (2.4), characterizes the rule of association which is adopted for the ordering of functions of noncommuting operators a and a^\dagger . For a wide class of associations of interest, $\Omega(\alpha, \alpha^*)$ is of the form^{5,7}

$$\Omega(\alpha, \alpha^*) = \exp(\mu\alpha^2 + \nu\alpha^{*2} + \lambda\alpha\alpha^*), \quad (2.5)$$

where μ , ν , and λ are parameters. The operators $\vec{\Lambda}_1$ and $\vec{\Lambda}_2$, which appear in (2.2), are given by

$$\vec{\Lambda}_1 = -2\nu \frac{\vec{\partial}}{\partial z} \frac{\vec{\partial}}{\partial z} - 2\mu \frac{\vec{\partial}}{\partial z^*} \frac{\vec{\partial}}{\partial z^*} + \lambda \left(\frac{\vec{\partial}}{\partial z} \frac{\vec{\partial}}{\partial z^*} + \frac{\vec{\partial}}{\partial z^*} \frac{\vec{\partial}}{\partial z} \right), \quad (2.6a)$$

$$\vec{\Lambda}_2 = \frac{1}{2} \left(\frac{\vec{\partial}}{\partial z} \frac{\vec{\partial}}{\partial z^*} - \frac{\vec{\partial}}{\partial z^*} \frac{\vec{\partial}}{\partial z} \right). \quad (2.6b)$$

The generalization of (2.2) to systems with many degrees of freedom is straightforward. We see that Eq. (2.2) is of the form

$$\partial F_\rho/\partial t = -i\mathcal{L}F_\rho, \quad (2.7)$$

¹⁰ Since we are mainly interested in applying the formalism to problems in quantum optics, we shall throughout this paper consider operator functions which depend on the boson annihilation and creating operators only. One can formulate a similar theory in which the operator functions depend explicitly on position and momentum operators so that other than boson systems could be dealt with.

¹¹ The Ω -ordered equivalents F_ρ and F_H are functions of c numbers z and z^* . Since the explicit dependence of the Ω -ordered equivalents on z and z^* will always be clear from the context in which they are used, we shall often suppress this dependence.

where the Liouville operator \mathcal{L} is defined by

$$\mathcal{L} = F_H \exp(\vec{\Lambda}_1) [\exp(\vec{\Lambda}_2) - \exp(-\vec{\Lambda}_2)]. \quad (2.8)$$

We decompose F_ρ in the form

$$F_\rho = \mathcal{P}F_\rho + (1-\mathcal{P})F_\rho, \quad (2.9)$$

where \mathcal{P} is the projection operator associated with the part of the system that is of interest. Our aim is to derive an equation of motion for $\mathcal{P}F_\rho$. Let $C(\beta)$ be the Laplace transform of $F_\rho(t)$, i.e.,

$$C(\beta) = \int_0^\infty e^{-i\beta t} F_\rho(t) dt \quad (\text{Re}\beta \geq 0). \quad (2.10)$$

On taking the Laplace transform of (2.7), we obtain the equation

$$\beta C(\beta) - F_\rho(0) = -i\mathcal{L}C(\beta). \quad (2.11)$$

Next we multiply (2.11) by \mathcal{P} and by $(1-\mathcal{P})$, respectively, and obtain the equations

$$\mathcal{P}[\beta C(\beta) - F_\rho(0)] = -i\mathcal{P}\mathcal{L}\mathcal{P}C(\beta) - i\mathcal{P}\mathcal{L}(1-\mathcal{P})C(\beta), \quad (2.12)$$

$$(1-\mathcal{P})[\beta C(\beta) - F_\rho(0)] = -i(1-\mathcal{P})\mathcal{L}\mathcal{P}C(\beta) - i(1-\mathcal{P})\mathcal{L}(1-\mathcal{P})C(\beta). \quad (2.13)$$

Equation (2.13) can be rearranged to give

$$(1-\mathcal{P})C(\beta) = [\beta + i(1-\mathcal{P})\mathcal{L}]^{-1} \times [-i(1-\mathcal{P})\mathcal{L}\mathcal{P}C(\beta) + (1-\mathcal{P})F_\rho(0)]. \quad (2.14)$$

If we substitute (2.14) in (2.12), we obtain the equation

$$\beta\mathcal{P}C(\beta) - \mathcal{P}F_\rho(0) = \{-i\mathcal{P}\mathcal{L}\mathcal{P}C(\beta) - i\mathcal{P}\mathcal{L}[\beta + i(1-\mathcal{P})\mathcal{L}]^{-1}[-i(1-\mathcal{P})\mathcal{L}\mathcal{P}C(\beta) + (1-\mathcal{P})F_\rho(0)]\}. \quad (2.15)$$

Taking the inverse Laplace transform of (2.15), we find that

$$\begin{aligned} & \frac{\partial}{\partial t} [\mathcal{P}F_\rho(t)] + i\mathcal{P}\mathcal{L}[\mathcal{P}F_\rho(t)] \\ & + i\mathcal{P}\mathcal{L}\{\exp[-i(1-\mathcal{P})\mathcal{L}t]\}[(1-\mathcal{P})F_\rho(0)] \\ & + \int_0^t d\tau \mathcal{P}\mathcal{L}\{\exp[-i\tau(1-\mathcal{P})\mathcal{L}]\}(1-\mathcal{P}) \\ & \times \mathcal{L}[\mathcal{P}F_\rho(t-\tau)] = 0. \quad (2.16) \end{aligned}$$

Equation (2.16) is the master equation in phase space or the relevant part of the distribution function. This equation is an exact consequence of the dynamics of our system. Further simplification can be made if we know the form of the Hamiltonian of the system. In the following sections we shall give some explicit realizations of the projection operator and discuss in detail the master equation for a system interacting with a reservoir.

3. THEOREM ON TIME-CORRELATION FUNCTIONS

As a first application of the master equation (2.16), we shall show that the quantum-mechanically defined time-correlation functions obey an equation of the Volterra type. Consider the correlation function defined by

$$\Gamma(t) \equiv \text{Tr}\{\rho(0)G(0)G(t)\}, \quad (3.1)$$

where $G(t)$ is a Heisenberg operator. As noted elsewhere,^{6,7} the correlation function $\Gamma(t)$ can be expressed in the form of an integral over phase space

$$\Gamma(t) = \frac{1}{\pi} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) F_G^{(\Omega)}(z, z^*, t), \quad (3.2)$$

where $F_{\rho G(0)}(\bar{\Omega})$ is the $\bar{\Omega}$ equivalent of $\rho G(0)$ and $F_G^{(\Omega)}(z, z^*, t)$ is the Ω equivalent of $G(t)$ as defined by equations of the type (2.3). The superscript on F indicates the rule of association which is adopted for mapping the operators onto c -number functions. $F_G^{(\Omega)}(z, z^*, t)$ satisfies the equation of motion⁵

$$dF_G^{(\Omega)}(z, z^*, t)/dt = i\mathcal{L}F_G^{(\Omega)}(z, z^*, t). \quad (3.3)$$

Now we introduce the projection operator \mathcal{P} such that

$$\begin{aligned} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \mathcal{P}\psi(z, z^*) \\ = \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \psi(z, z^*), \end{aligned} \quad (3.4)$$

$$\mathcal{P}F_G^{(\Omega)}(z, z^*, 0) = F_G^{(\Omega)}(z, z^*, 0), \quad (3.5)$$

where $\psi(z, z^*)$ is an arbitrary function of z and z^* . It can easily be verified that an explicit form of the projection operator which satisfies the requirements (3.4) and (3.5) is

$$\begin{aligned} \mathcal{P}\psi(z, z^*) = F_G^{(\Omega)}(z, z^*, 0) \int d^2z' F_{\rho G(0)}(\bar{\Omega})(z', z'^*) \psi(z', z'^*) \\ \int d^2z' F_{\rho G(0)}(\bar{\Omega})(z', z'^*) F_G^{(\Omega)}(z', z'^*, 0). \end{aligned} \quad (3.6)$$

Following the method of Sec. 2 one can derive the following equation for $\mathcal{P}F_G^{(\Omega)}(z, z^*, t)$:

$$\begin{aligned} i \frac{\partial}{\partial t} [\mathcal{P}F_G^{(\Omega)}(z, z^*, t)] + \mathcal{P}\mathcal{L}\mathcal{P}F_G^{(\Omega)}(z, z^*, t) \\ + i \int_0^t d\tau \mathcal{P}\mathcal{L}[\exp i\tau(1-\mathcal{P})\mathcal{L}](1-\mathcal{P}) \\ \times \mathcal{L}\mathcal{P}F_G^{(\Omega)}(z, z^*, t-\tau) = 0, \end{aligned} \quad (3.7)$$

where we have made use of the fact that

$$(1-\mathcal{P})F_G^{(\Omega)}(z, z^*, 0) = 0. \quad (3.8)$$

It then follows from (3.2), (3.4), and (3.7) that

$$\begin{aligned} i \frac{\partial \Gamma(t)}{\partial t} = -\frac{1}{\pi} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \mathcal{P}\mathcal{L}\mathcal{P}F_G^{(\Omega)}(z, z^*, t) \\ - \frac{i}{\pi} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \int_0^t d\tau \mathcal{P}\mathcal{L}[\exp i\tau(1-\mathcal{P})\mathcal{L}] \\ \times (1-\mathcal{P})\mathcal{L}\mathcal{P}F_G^{(\Omega)}(z, z^*, t-\tau). \end{aligned} \quad (3.9)$$

On using the property (3.4) of the projection operator, the first term on the right-hand side of (3.9) can be written as

$$\begin{aligned} -\frac{1}{\pi} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \mathcal{L}\mathcal{P}F_G^{(\Omega)}(z, z^*, t) \\ = -\frac{1}{\pi} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \mathcal{L}F_G^{(\Omega)}(z, z^*, 0) \\ \times \int d^2z' F_{\rho G(0)}(\bar{\Omega})(z', z'^*) F_G^{(\Omega)}(z', z'^*, t) \\ \times \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) F_G^{(\Omega)}(z, z^*, 0), \end{aligned} \quad (3.10)$$

where we have made use of the form (3.6) of the projection operator. On using the definition (3.2) of the correlation function $\Gamma(t)$, Eq. (3.10) becomes

$$\begin{aligned} -\frac{1}{\pi} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \mathcal{P}\mathcal{L}\mathcal{P}F_G^{(\Omega)}(z, z^*, t) \\ = i \frac{\partial \Gamma(t)}{\partial t} \Big|_{t=0} \frac{\Gamma(t)}{\Gamma(0)}. \end{aligned} \quad (3.11)$$

The contribution of the second term on the right-hand side of (3.9) can be shown to be equal to

$$-i \int_0^t d\tau K(\tau) \Gamma(t-\tau),$$

where

$$\begin{aligned} K(\tau) = \left\{ \frac{1}{\pi \Gamma(0)} \int d^2z F_{\rho G(0)}(\bar{\Omega})(z, z^*) \mathcal{L}[\exp i\tau(1-\mathcal{P})\mathcal{L}] \right. \\ \times \mathcal{L}F_G^{(\Omega)}(z, z^*, 0) + \frac{i}{\pi \Gamma(0)} \left[\frac{\partial \Gamma(t)}{\partial t} \Big|_{t=0} \frac{1}{\Gamma(0)} \right] \int d^2z \\ \times F_{\rho G(0)}(\bar{\Omega})(z, z^*) \mathcal{L}[\exp i\tau(1-\mathcal{P})\mathcal{L}] \\ \left. \times F_G^{(\Omega)}(z, z^*, 0) \right\}. \end{aligned} \quad (3.12)$$

On combining (3.9)–(3.11), one finds

$$\frac{\partial \Gamma(t)}{\partial t} = \frac{\partial \Gamma(t)}{\partial t} \Big|_{t=0} \frac{\Gamma(t)}{\Gamma(0)} - \int_0^t d\tau K(\tau) \Gamma(t-\tau). \quad (3.13)$$

This equation shows that the quantum-mechanically defined time-correlation functions (3.1) obey an equation of the Volterra type. A corresponding result for the classically defined time-correlation functions has also been established by Zwanzig.⁸ The kernel appearing in the integral equation can, in principle, be evaluated if one knows $F_G^{(0)}(z, z^*, 0)$ and $F_{\rho G^{(0)}}^{(0)}(z, z^*)$. This result appears to be of importance in the study of transport coefficients.

4. MASTER EQUATION FOR OPEN SYSTEMS

In this section we derive the phase-space form of the master equation of an open system. Following Emch and Sewell,³ we refer to a system as being "open" if it is coupled to a large system R , usually called reservoir. The reservoir is assumed to be in thermal equilibrium at some initial time. The Hamiltonian of the total system ($S+R$) can be written as

$$H = H_R + H_S + H_{RS}, \quad (4.1)$$

where H_S and H_R are the Hamiltonians the system and the reservoir, respectively, and H_{RS} is the Hamiltonian characterizing the interaction between the system and the reservoir. It is obvious that the total Liouville operator can be written as

$$\mathcal{L} = \mathcal{L}_R + \mathcal{L}_S + \mathcal{L}_{RS}. \quad (4.2)$$

Let $F(z, z^*, \{z_R\}, \{z_R^*\}, t)$ be the distribution of function for the total system ($S+R$) and let $F_S(z, z^*, t)$ be the distribution function of the system S alone. Then $F_S(z, z^*, t)$ is obtained from $F(z, z^*, \{z_R\}, \{z_R^*\}, t)$ by integration over the reservoir variables $\{z_R\}$, i.e.,

$$F_S(z, z^*, t) = \int d^2\{z_R\} F(z, z^*, \{z_R\}, \{z_R^*\}, t). \quad (4.3)$$

Let us express F in the form

$$F(z, z^*, \{z_R\}, \{z_R^*\}, t) = F_S(z, z^*, t) F_R(\{z_R\}, \{z_R^*\}) + R(z, z^*, \{z_R\}, \{z_R^*\}, t), \quad (4.4)$$

where $F_R(\{z_R\}, \{z_R^*\})$ is the initial thermal-equilibrium distribution function for the reservoir. It is then evident that for this system the projection operator \mathcal{P} is given by (cf. Ref. 9)

$$\mathcal{P} = F_R(\{z_R\}, \{z_R^*\}) \int d^2\{z_R\}. \quad (4.5)$$

We now introduce certain assumptions about the initial condition of the total system. We assume that at time $t=0$ the system S and the reservoir R are statistically independent. This assumption is equivalent to the following condition on the distribution function $F(z, z^*, \{z_R\}, \{z_R^*\}, t)$:

$$(1 - \mathcal{P})F(z, z^*, \{z_R\}, \{z_R^*\}, 0) = 0. \quad (4.6)$$

It further follows, from the definition (4.5) of the projection operator, that

$$\mathcal{P}\mathcal{L}_S = \mathcal{L}_S\mathcal{P}, \quad (4.7)$$

$$\mathcal{P}\mathcal{L}_R = \mathcal{L}_R\mathcal{P} = 0. \quad (4.8)$$

Using (4.6)–(4.8), the master equation (2.16) can be expressed in the form

$$\begin{aligned} \frac{\partial}{\partial t} [\mathcal{P}F(t)] + i(\mathcal{L}_S + \mathcal{P}\mathcal{L}_{RS}\mathcal{P})[\mathcal{P}F(t)] \\ + \int_0^t d\tau \mathcal{P}\mathcal{L} \{ \exp[-i\tau(1-\mathcal{P})\mathcal{L}] (1-\mathcal{P}) \\ \times \mathcal{L}[\mathcal{P}F(t-\tau)] \} = 0. \end{aligned} \quad (4.9)$$

It is easily verified from series expansion that

$$\begin{aligned} \{ \exp[-i\tau(1-\mathcal{P})\mathcal{L}] (1-\mathcal{P}) \\ = (1-\mathcal{P}) \{ \exp[-i\tau(1-\mathcal{P})\mathcal{L}(1-\mathcal{P})] \} (1-\mathcal{P}). \end{aligned} \quad (4.10)$$

One also has

$$(1-\mathcal{P})\mathcal{L}_S[\mathcal{P}F(t)] = \mathcal{L}_S(1-\mathcal{P})[\mathcal{P}F(t)] = 0. \quad (4.11)$$

Using (4.10) and (4.11), one can simplify the master equation (4.9):

$$\begin{aligned} \frac{\partial}{\partial t} [\mathcal{P}F(t)] + i(\mathcal{L}_S + \mathcal{P}\mathcal{L}_{RS}\mathcal{P})[\mathcal{P}F(t)] \\ + \int_0^t d\tau \mathcal{P}\mathcal{L}_{RS}(1-\mathcal{P})U'(t-\tau)(1-\mathcal{P}) \\ \times \mathcal{L}_{RS}[\mathcal{P}F(\tau)] = 0, \end{aligned} \quad (4.12)$$

where

$$\begin{aligned} U'(t-\tau) = \exp[-i(t-\tau)(1-\mathcal{P}) \\ \times (\mathcal{L}_R + \mathcal{L}_S + \mathcal{L}_{RS})(1-\mathcal{P})]. \end{aligned} \quad (4.13)$$

Equation (4.12) may be simplified by employing the analysis of Emch and Sewell.³ As is well known, $U'(\tau)$ has the expansion

$$\begin{aligned} U'(\tau) = U_0'(\tau) \\ \times T \exp \left[-i \int_0^\tau dt' U_0'(-t') (1-\mathcal{P}) \mathcal{L}_{RS} (1-\mathcal{P}) U_0'(t') \right], \end{aligned} \quad (4.14)$$

where T is Dyson's time-ordering operator and

$$U_0'(\tau) = \exp[-i\tau(1-\mathcal{P})(\mathcal{L}_R + \mathcal{L}_S)(1-\mathcal{P})]. \quad (4.15)$$

Using the properties (4.7) and (4.8) of the projection operator \mathcal{P} , one can show that $U_0'(\tau)$ can be expressed in the form

$$\begin{aligned} U_0'(\tau) = \exp(-i\mathcal{L}_R\tau)(1-\mathcal{P}) \\ \times [\exp(-i\mathcal{L}_S\tau) - 1] + \exp(-i\mathcal{L}_R\tau) \\ = \mathcal{P} + (1-\mathcal{P})U_0(\tau)(1-\mathcal{P}), \end{aligned} \quad (4.16)$$

where

$$U_0(\tau) = \exp[-i\tau(\mathcal{L}_R + \mathcal{L}_S)]. \quad (4.17)$$

Using the relation (4.16), we find that the time-ordered product appearing in (4.14) simplifies to

$$T \exp \left[-i \int_0^t dt' (1-\mathcal{P}) U_0(-t') \mathcal{L}_{RS} U_0(t') (1-\mathcal{P}) \right] \equiv U(t). \quad (4.18)$$

On substituting (4.16) and (4.18) in (4.12), we find that the master equation may be expressed in the following simpler form:

$$\begin{aligned} \frac{\partial}{\partial t} [\mathcal{P}F(t)] + i(\mathcal{L}_S + \mathcal{P}\mathcal{L}_{RS}\mathcal{P})[\mathcal{P}F(t)] \\ + \int_0^t d\tau K(t-\tau)[\mathcal{P}F(\tau)] = 0, \end{aligned} \quad (4.19)$$

where the kernel is given by

$$K(\tau) = \mathcal{P}\mathcal{L}_{RS}U_0(\tau)(1-\mathcal{P})U(\tau)\mathcal{L}_{RS}. \quad (4.20)$$

It should be noted that Eq. (4.19) is of the same form as the one derived by Emch and Sewell³; however, our equation is for the phase-space distribution function and is therefore a *c*-number differential equation. As we will see later, the master equation in our formalism is useful in studying the connection with the Fokker-Planck equations of stochastic processes.

In applications, it is preferable to employ the master equation for the phase-space distribution function which corresponds to the density operator in the interaction picture. It can be obtained either by transforming Eq. (4.19) or in the following more direct manner: The density operator in the interaction picture satisfies the following equation:

$$\partial \rho^I / \partial t = -i[H_{RS}^I(t), \rho^I(t)] \quad (\hbar = 1), \quad (4.21)$$

where $H_{RS}^I(t)$ is the interaction Hamiltonian in the interaction picture. The phase-space equivalent of Eq. (4.21) is

$$\partial F(t) / \partial t = -i\mathcal{L}(t)F(t), \quad (4.22)$$

where the Liouville operator $\mathcal{L}(t)$ is defined by an expression analogous to (2.5) and $F(t)$ is the Ω equivalent of the density operator in the interaction picture. It is understood that all the quantities from now on refer to the interaction picture; hence, we have dropped the superscript *I*. We obtain from (4.22) the following two equations:

$$\begin{aligned} \frac{\partial}{\partial t} [\mathcal{P}F(t)] = -i\mathcal{P}\mathcal{L}(t)\mathcal{P}F(t) \\ - i\mathcal{P}\mathcal{L}(t)(1-\mathcal{P})F(t), \end{aligned} \quad (4.23)$$

$$\begin{aligned} \frac{\partial}{\partial t} [(1-\mathcal{P})F(t)] = -i(1-\mathcal{P})\mathcal{L}(t)\mathcal{P}F(t) \\ - i(1-\mathcal{P})\mathcal{L}(t)(1-\mathcal{P})F(t). \end{aligned} \quad (4.24)$$

Here the projection operator \mathcal{P} is again defined by the

equation

$$\mathcal{P} = F_R \int d^2\{z_R\}. \quad (4.25)$$

We solve equation (4.24) for $(1-\mathcal{P})F(t)$ and substitute in (4.23). It is easy to verify that a solution of (4.24) is given by

$$(1-\mathcal{P})F(t) = \mathfrak{u}(t,0)[(1-\mathcal{P})F(0)]$$

$$-i \int_0^t d\tau \mathfrak{u}(t,\tau)(1-\mathcal{P})\mathcal{L}(\tau)[\mathcal{P}F(\tau)], \quad (4.26)$$

where

$$\mathfrak{u}(t,\tau) = T \exp \left[-i \int_\tau^t (1-\mathcal{P})\mathcal{L}(t') dt' \right]. \quad (4.27)$$

On substituting (4.26) in (4.23) and making use of the initial condition $(1-\mathcal{P})F(0) = 0$, one finds that

$$\begin{aligned} \frac{\partial}{\partial t} [\mathcal{P}F(t)] + i\mathcal{P}\mathcal{L}(t)[\mathcal{P}F(t)] \\ + \int_0^t d\tau \mathcal{P}\mathcal{L}(t)\mathfrak{u}(t,\tau)(1-\mathcal{P})\mathcal{L}(\tau)[\mathcal{P}F(\tau)] = 0. \end{aligned} \quad (4.28)$$

The exact master equation (4.28) in phase space is quite hard to employ for actual calculations. However, it provides a rigorous starting point for the derivation of simpler approximate equations for $[\mathcal{P}F(t)]$ and a basis for the development of systematic approximate procedures.

Let us now analyze the effects of interaction with the bath, in the lowest order of perturbation, i.e., in the Born approximation.⁴ In this approximation one replaces $\mathfrak{u}(t,\tau)$ by unity. We further assume that the properties of the reservoir are such that

$$\mathcal{P}\mathcal{L}(\tau)\mathcal{P}F(t) = 0. \quad (4.29)$$

In all the applications which we will encounter below, the condition (4.29) will be satisfied. Under these approximations the master equation (4.28) reduces to

$$\frac{\partial}{\partial t} \{\mathcal{P}F(t)\} + \int_0^t \mathcal{P}\mathcal{L}(t)\mathcal{L}(\tau)[\mathcal{P}F(\tau)] d\tau = 0. \quad (4.30)$$

We now write the interaction Hamiltonian as

$$H_{RS} = \sum_{\mathbf{k}} [G_{\mathbf{k}}^R G_{\mathbf{k}}^{S\dagger} + G_{\mathbf{k}}^{R\dagger} G_{\mathbf{k}}^S], \quad (4.31)$$

where $G_{\mathbf{k}}^R$ ($G_{\mathbf{k}}^S$) is an operator acting on the reservoir (system) variables alone. Thus, the interaction Hamiltonian in the interaction picture takes the form

$$H_{RS}^I(t) = \sum_{\mathbf{k}} [G_{\mathbf{k}}^R(t) G_{\mathbf{k}}^{S\dagger}(t) + G_{\mathbf{k}}^{R\dagger}(t) G_{\mathbf{k}}^S(t)]. \quad (4.32)$$

In practical applications, one is usually concerned with rules of association which are such that our Hermitian operator function is mapped onto a real function. Obviously Weyl, normal, and antinormal rules of associa-

tion^{5,12} belong to this class. For the sake of simplicity, we will from now on consider only this class. The phase-space equivalent F_{RS} of H_{RS}^I ($H_{RS}^I = \Omega F_{RS}$) is then of the form

$$F_{RS} = \sum_k [\psi_k^R(t) \psi_k^{S*}(t) + \psi_k^{R*}(t) \psi_k^S(t)], \quad (4.33)$$

where $\psi_k^R(t)$ is the Ω equivalent of $G_k^R(t)$ and $\psi_k^S(t)$ is the Ω equivalent of $G_k^S(t)$. It is now evident that

$$\begin{aligned} \mathcal{L}(\tau)[\mathcal{P}F(t)] = & \sum_k [(\psi_k^R(\tau) \mathcal{D}_R^{(1)} F_R)(\psi_k^{S*}(\tau) \mathcal{D}_S^{(1)} F_S(t)) \\ & + (\psi_k^{R*}(\tau) \mathcal{D}_R^{(1)} F_R)(\psi_k^S(\tau) \mathcal{D}_S^{(1)} F_S(t)) \\ & - (\psi_k^R(\tau) \mathcal{D}_R^{(2)} F_R)(\psi_k^{S*}(\tau) \mathcal{D}_S^{(2)} F_S(t)) \\ & - (\psi_k^{R*}(\tau) \mathcal{D}_R^{(2)} F_R)(\psi_k^S(\tau) \mathcal{D}_S^{(2)} F_S(t))], \quad (4.34) \end{aligned}$$

$$K(t, \tau) = \sum_{k, l} \{ \Gamma_{11R}^{lk}(t, \tau) [\psi_l^{S*}(t) \mathcal{D}_S^{(1)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] \}$$

$$\begin{aligned} & + \Gamma_{12R}^{lk}(t, \tau) [\psi_l^{S*}(t) \mathcal{D}_S^{(1)} (\psi_k^S(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] - \Gamma_{11R}^{kl}(\tau, t) [\psi_l^{S*}(t) \mathcal{D}_S^{(1)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] \\ & - \Gamma_{21R}^{kl}(\tau, t) [\psi_l^{S*}(t) \mathcal{D}_S^{(1)} (\psi_k^S(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] + \Gamma_{21R}^{lk}(t, \tau) [\psi_l^S(t) \mathcal{D}_S^{(1)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] \\ & + \Gamma_{22R}^{lk}(t, \tau) [\psi_l^S(t) \mathcal{D}_S^{(1)} (\psi_k^S(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] - \Gamma_{12R}^{kl}(\tau, t) [\psi_l^S(t) \mathcal{D}_S^{(1)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] \\ & - \Gamma_{22R}^{kl}(\tau, t) [\psi_l^S(t) \mathcal{D}_S^{(1)} (\psi_k^S(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] - \Gamma_{11R}^{lk}(t, \tau) [\psi_l^{S*}(t) \mathcal{D}_S^{(2)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] \\ & - \Gamma_{12R}^{lk}(t, \tau) [\psi_l^{S*}(t) \mathcal{D}_S^{(2)} (\psi_k^S(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] + \Gamma_{11R}^{kl}(\tau, t) [\psi_l^{S*}(t) \mathcal{D}_S^{(2)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] \\ & + \Gamma_{21R}^{kl}(\tau, t) [\psi_l^{S*}(t) \mathcal{D}_S^{(2)} (\psi_k^S(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] - \Gamma_{21R}^{lk}(t, \tau) [\psi_l^S(t) \mathcal{D}_S^{(2)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] \\ & - \Gamma_{22R}^{lk}(t, \tau) [\psi_l^S(t) \mathcal{D}_S^{(2)} (\psi_k^S(\tau) \mathcal{D}_S^{(1)} F_S(\tau))] + \Gamma_{12R}^{kl}(\tau, t) [\psi_l^S(t) \mathcal{D}_S^{(2)} (\psi_k^{S*}(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] \\ & + \Gamma_{22R}^{kl}(\tau, t) [\psi_l^S(t) \mathcal{D}_S^{(2)} (\psi_k^S(\tau) \mathcal{D}_S^{(2)} F_S(\tau))] \}, \quad (4.37a) \end{aligned}$$

and where $\Gamma_{ijR}^{kl}(t, \tau)$ are the elements of the matrix

$$\begin{pmatrix} \text{tr}(\rho_R G_R^k(t) G_R^l(\tau)) & \text{tr}(\rho_R G_R^k(t) G_R^{l\dagger}(\tau)) \\ \text{tr}(\rho_R G_R^{k\dagger}(t) G_R^l(\tau)) & \text{tr}(\rho_R G_R^{k\dagger}(t) G_R^{l\dagger}(\tau)) \end{pmatrix}. \quad (4.37b)$$

Here, ρ_R is the equilibrium density operator of the reservoir at the initial time. It should be noted that for quadratic Hamiltonians, the Eq. (4.36) is a second-order differential equation with respect to z and z^* , and an integral equation with respect to the time variable.

Suppose that our system, interacting with the reservoir, is driven by an external field whose effect is represented by an interaction Hamiltonian $E(t)$. Then the density operator in the interaction picture satisfies the equation of motion

$$\partial \rho^I / \partial t = -i [H_{RS}^I(t) + E^I(t), \rho^I(t)]. \quad (4.38)$$

Usually one is interested in the linear response⁴ of the system. When the contribution of the external field is

¹² C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965). In this reference, the terms normally and anti-normally ordered distribution functions are used in the opposite sense than in the present paper. The reason for this difference in terminology is explained in footnote 35 of Ref. 7.

where

$$\mathcal{D}^{(1)} = \exp(\vec{\Lambda}_1 + \vec{\Lambda}_2), \quad (4.35a)$$

$$\mathcal{D}^{(2)} = \exp(\vec{\Lambda}_1 - \vec{\Lambda}_2), \quad (4.35b)$$

$\vec{\Lambda}_1$ and $\vec{\Lambda}_2$ being the operators defined by Eqs. (2.6). The operators \mathcal{D}_R and \mathcal{D}_S act on the reservoir and on the system variables, respectively. On substituting (4.34) in (4.30) and after some simplifications, one finds that the approximate master equation (4.30) reduces to

$$\frac{\partial F_S(t)}{\partial t} + \int_0^t d\tau K(t, \tau) F_S(\tau) = 0, \quad (4.36)$$

where the kernel $K(t, \tau)$ is defined by the formula

taken into account in linear approximation, the master equation (4.36) reduces to

$$\begin{aligned} \frac{\partial F_S(t)}{\partial t} + \int_0^t d\tau K(t, \tau) F_S(\tau) \\ + i F_E [\exp(\vec{\Lambda}_1 + \vec{\Lambda}_2) - \exp(\vec{\Lambda}_1 - \vec{\Lambda}_2)] F_S(t) = 0, \quad (4.39) \end{aligned}$$

where F_E is the Ω equivalent of $E^I(t)$.

5. APPLICATIONS

A. Harmonic Oscillator Coupled to a Reservoir

As a first application of the theory, we consider a harmonic oscillator which is coupled to a reservoir. The reservoir is assumed to consist of harmonic oscillators and to be weakly coupled to the oscillator (the oscillator corresponds to the system S). The Hamiltonian of the total system is given by

$$\begin{aligned} H = \omega a^\dagger a + \sum_j \omega_j a_j^\dagger a_j + \sum_j g_j a_j^\dagger (a + q^* a^\dagger) \\ + \sum_j g_j a_j (a^\dagger + qa) \quad (\hbar = 1). \quad (5.1) \end{aligned}$$

The operators a , a^\dagger , a_j , and a_j^\dagger satisfy the following commutation relations:

$$[a, a^\dagger] = 1, \quad [a_j, a_j^\dagger] = 1. \quad (5.2)$$

All other commutators vanish. The Hamiltonian of the type (5.1) is important in many physical problems such as those encountered in Brownian motion of an oscillator,¹³ and in the quantum theory of field attenuation.¹⁴ For the sake of simplicity, we first consider the case when $q=0$, and later on we shall give the result for $q \neq 0$. The interaction Hamiltonian (with $q=0$) in the interaction picture is then given by

$$H_{RS}^I(t) = \sum_j g_j [a^\dagger a_j \exp i v_j t + a a_j^\dagger \exp(-i v_j t)], \quad (5.3)$$

where

$$v_j = (\omega - \omega_j). \quad (5.4)$$

Now for the class of associations for which Hermitian operators are mapped onto real functions, the Ω equivalent of (5.3), such that $H_{RS}^I = \Omega F_{RS}$, is given by

$$F_{RS} = \sum_j g_j [z^* z_j \exp i v_j t + z z_j^* \exp(-i v_j t)]. \quad (5.5)$$

Comparing (5.5) with (4.33), we may make the identification

$$\psi_j^R(t) = (g_j)^{1/2} z_j e^{-i \omega_j t}, \quad (5.6a)$$

$$\psi_j^S(t) = (g_j)^{1/2} z_j e^{-i \omega t}. \quad (5.6b)$$

The thermal-equilibrium density operator for the reservoir is

$$\rho_R = \exp(-\beta \sum_j \omega_j a_j^\dagger a_j) / \text{tr} \exp(-\beta \sum_j \omega_j a_j^\dagger a_j). \quad (5.7)$$

Using (5.3), (5.6), and (5.7) one can show that the matrix defined by (4.37b) is given by¹⁵

$$(g_k g_l)^{1/2} \begin{pmatrix} 0 & \delta_{kl} (\langle n_k \rangle + 1) e^{-i \omega_k (t-\tau)} \\ \langle n_k \rangle e^{i \omega_k (t-\tau)} \delta_{kl} & 0 \end{pmatrix}, \quad (5.8)$$

where

$$\langle n_k \rangle = (e^{\beta \omega_k} - 1)^{-1}. \quad (5.9)$$

We shall now form the normally ordered equivalent of the density-operator equation. In this case it is seen from Eqs. (2.3a) and (2.3b) that the operators $\mathfrak{D}_S^{(1)}$

¹³ See, for example, P. Ullersma, *Physica* **32**, 27 (1966); **32**, 56 (1966). Similar phase-space techniques have also been used by McKenna and Frisch [J. McKenna and H. L. Frisch, *Phys. Rev.* **145**, 93 (1966)] to study Brownian motion.

¹⁴ B. R. Mollow, *Phys. Rev.* **168**, 1896 (1968).

¹⁵ This calculation can easily be done by using the phase-space distribution functions. See, for example, L. Mandel and E. Wolf, *Rev. Mod. Phys.* **37**, 231 (1965); R. J. Glauber, in *Quantum Optics and Electronics, Les Houches, 1964*, edited by C. deWitt, A. Blandon, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, Inc., New York, 1965), p. 65.

and $\mathfrak{D}_S^{(2)}$ are given by

$$\mathfrak{D}_S^{(1)} = \exp\left(\frac{\overleftarrow{\partial}}{\partial z} \frac{\overrightarrow{\partial}}{\partial z^*}\right), \quad (5.10)$$

$$\mathfrak{D}_S^{(2)} = \exp\left(+\frac{\overleftarrow{\partial}}{\partial z^*} \frac{\overrightarrow{\partial}}{\partial z}\right). \quad (5.11)$$

On substituting (5.6), (5.8), (5.10), and (5.11) in (4.36), one finds that

$$\begin{aligned} \frac{\partial F_S^{(N)}(t)}{\partial t} = & \int_0^t d\tau \left\{ \sum_k g_k^2 e^{-i \omega_k \tau + i \omega \tau} \right. \\ & \times \left[(1 + \langle n_k \rangle) \frac{\partial^2 F_S^{(N)}(t-\tau)}{\partial z \partial z^*} + \frac{\partial}{\partial z} [z F_S^{(N)}(t-\tau)] \right] + \text{c.c.} \left. \right\}. \end{aligned} \quad (5.12)$$

The superscript (N) indicates that the phase-space representation corresponds to association by means of normal ordering. Now it can be shown that the memory effects are of order higher than second in perturbation and hence one can replace $F_S^{(N)}(t-\tau)$ by $F_S^{(N)}(t)$ in the integrand. We assume that the bath oscillators are closely spaced in frequency so that summation over k can be replaced by integration, i.e.,

$$\sum_k \rightarrow \int d\omega_k h(\omega_k), \quad (5.13)$$

where $h(\omega_k) d\omega_k$ is the number of oscillators with frequencies lying between ω_k and $\omega_k + d\omega_k$. We further assume that our system possesses short memory, i.e., there exists a correlation time τ_c such that

$$\int d\omega_k h(\omega_k) g^2(\omega_k) e^{i \tau (\omega - \omega_k)} = 0 \quad \text{for } |\tau| > \tau_c. \quad (5.14)$$

Therefore for times $|\tau| > \tau_c$ one can replace the upper limit on time integration by ∞ and one obtains the following equation of motion for the phase distribution function $F_S^{(N)}(t)$:

$$\begin{aligned} \frac{\partial F_S^{(N)}}{\partial t} = & \int d\omega_k h(\omega_k) g^2(\omega_k) \\ & \times \left\{ \delta_{(+)}(\omega - \omega_k) \left[[1 + \langle n(\omega_k) \rangle] \frac{\partial^2 F_S^{(N)}(t)}{\partial z \partial z^*} \right. \right. \\ & \left. \left. + \frac{\partial}{\partial z} [z F_S^{(N)}(t)] \right] + \text{c.c.} \right\}, \end{aligned} \quad (5.15)$$

where

$$\delta_{(\pm)}(\omega) = \pi \delta(\omega) \pm i P(1/\omega), \quad (5.16)$$

with P denoting the Cauchy principal part. Neglecting the contribution from the principal part, which only

causes a small frequency shift, we find that

$$\frac{\partial F_S^{(N)}}{\partial t} = \pi h(\omega) g^2(\omega) \left\{ 2[1 + \langle n(\omega) \rangle] \frac{\partial^2 F_S^{(N)}}{\partial z \partial z^*} + \frac{\partial}{\partial z} (z F_S^{(N)}) + \frac{\partial}{\partial z^*} (z^* F_S^{(N)}) \right\}. \quad (5.17)$$

We wish to emphasize the fact that we derived an equation for the normally ordered equivalent and not for the antinormally ordered equivalent of the density operator. This choice was made, because as is well known, the normally ordered equivalent of any density operator is a well-behaved function, whereas the antinormally ordered equivalent is not well behaved in some cases.¹² It is to be noted that as far as the calculations of normally ordered time-ordered correlation functions is concerned, it does not matter which equation of motion we use. It is to be also noted that for a quadratic Hamiltonian, our master equation, under the approximations made in this and the preceding sections, reduces to an equation of the Fokker-Planck type.

Let us obtain an equation for the Sudarshan-Glauber phase-space distribution function¹⁶ (which in our terminology is the antinormally ordered equivalent of the density operator). For this purpose we may either use the general formalism of Sec. 4 or we can transform the result (5.17) by the use of the following relation¹²:

$$F_S^{(N)} = \exp\left(\frac{\partial^2}{\partial z \partial z^*}\right) F_S^{(A)}. \quad (5.18)$$

It is readily found that $F_S^{(A)}$ satisfies the following equation:

$$\frac{\partial F_S^{(A)}}{\partial t} = \pi h(\omega) g^2(\omega) \left[2\langle n(\omega) \rangle \frac{\partial^2 F_S^{(A)}}{\partial z \partial z^*} + \frac{\partial}{\partial z} (z F_S^{(A)}) + \frac{\partial}{\partial z^*} (z^* F_S^{(A)}) \right]. \quad (5.19)$$

Equations of the type (5.19) have been derived by many other authors using different techniques.¹⁷ One can recast Eq. (5.19) into the following stochastically equivalent Langevin equation^{18,19}:

$$\dot{z} = -\pi h(\omega) g^2(\omega) z + \xi(t), \quad (5.20)$$

$$\begin{aligned} \frac{\partial F_S^{(N)}}{\partial t} = \pi h(\omega) g^2(\omega) \left\{ 2[1 + \langle n(\omega) \rangle] \frac{\partial^2 F_S^{(N)}}{\partial z \partial z^*} + \frac{\partial}{\partial z} (z F_S^{(N)}) + \frac{\partial}{\partial z^*} (z^* F_S^{(N)}) \right\} \\ - \pi h(\omega) g^2(\omega) \left\{ q e^{-2i\omega t} \frac{\partial F_S^{(N)}}{\partial z^*} + \text{c.c.} \right\} - \pi h(\omega) g^2(\omega) [1 + \langle n(\omega) \rangle] \left\{ q e^{-2i\omega t} \frac{\partial^2 F_S^{(N)}}{\partial z^*{}^2} + \text{c.c.} \right\}. \quad (5.26) \end{aligned}$$

where the Langevin force $\xi(t)$ has the following properties:

$$\langle \xi(t) \rangle = 0, \quad \langle \xi(t) \xi(t') \rangle = 0, \quad (5.21)$$

$$\langle \xi(t) \xi^*(t') \rangle = 2\pi h(\omega) g^2(\omega) \langle n(\omega) \rangle \delta(t-t'). \quad (5.22)$$

Let us finally obtain an equation satisfied by the diagonal matrix elements $\rho_{n,n}$ of the density operator. One can readily show that¹⁵

$$\rho_{n,n} = \frac{1}{\pi} \int F_S^{(A)} \frac{|z|^{2n} e^{-|z|^2}}{n!} d^2z. \quad (5.23)$$

Then

$$\begin{aligned} \frac{\partial \rho_{n,n}}{\partial t} = \frac{1}{\pi} \int d^2z \frac{|z|^{2n} e^{-|z|^2}}{n!} \\ \times \left[2\pi h(\omega) g^2(\omega) \langle n(\omega) \rangle \frac{\partial^2 F_S^{(A)}}{\partial z \partial z^*} + \pi h(\omega) g^2(\omega) \frac{\partial}{\partial z} (z F_S^{(A)}) + \pi h(\omega) g^2(\omega) \frac{\partial}{\partial z^*} (z^* F_S^{(A)}) \right], \end{aligned}$$

which on integration by parts leads to the equation

$$\begin{aligned} \partial \rho_{n,n} / \partial t + 2\pi h(\omega) g^2(\omega) n [1 + 2\langle n(\omega) \rangle] \rho_{n,n} \\ + 2\pi h(\omega) g^2(\omega) \langle n(\omega) \rangle \rho_{n,n} \\ = 2\pi h(\omega) g^2(\omega) \langle n(\omega) \rangle n \rho_{n-1, n-1} \\ + (n+1) 2\pi h(\omega) g^2(\omega) [1 + \langle n(\omega) \rangle] \rho_{n+1, n+1}. \quad (5.24) \end{aligned}$$

Some physical consequences of Eq. (5.24) are discussed in the Appendix. At zero temperature, Eq. (5.24) reduces to

$$\partial \rho_{n,n} / \partial t = 2\pi h(\omega) g^2(\omega) [(n+1) \rho_{n+1, n+1} - n \rho_{n,n}]. \quad (5.25)$$

This equation has been discussed by Mollow¹⁴ in the context of the quantum theory of field attenuation. Note that the Eqs. (5.24) and (5.25) have the form of the Pauli equation.

Returning to the original Hamiltonian (5.1) with $q \neq 0$, a similar procedure leads to the following equation of motion:

¹⁶ For a discussion of the Sudarshan-Glauber phase-space distribution function see, for example, J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, 1968), Chap. VIII.
¹⁷ See, for example, W. H. Louisell and J. H. Marburger, *IEEE J. Quant. Electron.* **QE-3**, 348 (1967).
¹⁸ R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach Science Publishers, Inc., New York, 1963), Vol. I; see also M. Lax, *Rev. Mod. Phys.* **38**, 541 (1966).
¹⁹ M. Lax, *Phys. Rev.* **145**, 110 (1966).

This equation, which can be treated by perturbation technique, shows some of the interesting features of the coupling. The solution of this differential equation to first order in the parameter q will contain terms at the driving frequencies $+2\omega$ and -2ω and can be used to discuss the statistical properties of the field; e.g., one can obtain the spectrum of the amplitude and of the intensity fluctuations.²⁰ In particular one can obtain the rate of change of the average numbers of photons at the frequency 2ω .

B. Two-Level System Interacting with the Reservoir

We shall now consider a two-level system which is interacting weakly with a reservoir assumed to be made up of harmonic oscillators. The Hamiltonian of such a system is given by²¹

$$H = \omega_0 S_z + \sum_j \omega_j a_j^\dagger a_j + \sum_j g_j [a_j^\dagger S^{(-)} + a_j S^{(+)}] \quad (\hbar = 1), \quad (5.27)$$

where $S^{(\pm)}$ are the spin-flip operators, and S_z is the z component of the spin, angular momentum, and a_j and a_j^\dagger are the usual boson annihilation and creation operators. We will now use Schwinger's²² coupled-boson representation to express $S^{(\pm)}$ and S_z in terms of the

$$\begin{aligned} \frac{\partial F_S^{(N)}}{\partial t} + \pi \hbar (\omega_0) g^2 (\omega_0) \left[(2\alpha\alpha^* - 2\beta\beta^*) + \left\{ \alpha [\langle n(\omega_0) \rangle + 1] - \beta^* \beta \alpha \right\} \frac{\partial}{\partial \alpha} \right. \\ \left. + [\alpha\alpha^* \beta + \beta \langle n(\omega_0) \rangle] \frac{\partial}{\partial \beta} + \text{c.c.} \right] - 2 [\langle n(\omega_0) \rangle + 1] \beta \beta^* \frac{\partial^2}{\partial \alpha \partial \alpha^*} - 2 \langle n(\omega_0) \rangle \alpha \alpha^* \frac{\partial^2}{\partial \beta \partial \beta^*} \\ + [2 \langle n(\omega_0) \rangle + 1] \beta^* \alpha^* \frac{\partial^2}{\partial \alpha^* \partial \beta^*} + [2 \langle n(\omega_0) \rangle + 1] \beta \alpha \frac{\partial^2}{\partial \alpha \partial \beta} \Big] F_S^{(N)} = 0. \quad (5.33) \end{aligned}$$

Let us now calculate the average polarization of this system. In order to do this, we make the following ansatz:

$$\rho(t) = \frac{1}{2} (1 - \langle \sigma^z \rangle) |0\rangle_S \langle 0|_S + \frac{1}{2} (1 + \langle \sigma^z \rangle) |1\rangle_S \langle 1|_S + \frac{1}{2} \langle \sigma^{(+)} \rangle |0\rangle_S \langle 1|_S + \frac{1}{2} \langle \sigma^{(-)} \rangle |1\rangle_S \langle 0|_S, \quad (5.34)$$

where $|0\rangle_S$ and $|1\rangle_S$ are the ground state and the first excited state of the system, respectively. We assume that the ground state is one with spin down. We can express these states in terms of the states of the har-

²⁰ See, for example, H. Risken, Z. Physik **191**, 302 (1966); R. D. Hempstead and M. Lax, Phys. Rev. **161**, 350 (1967).

²¹ C. R. Willis, J. Math. Phys. **5**, 1241 (1964); **6**, 1984 (1965).

²² See, for example, Y.-L. Wang and H. B. Callen, Phys. Rev. **148**, 433 (1966).

boson creating and annihilation operators, i.e.,

$$S_z = \frac{1}{2} (v^\dagger v - u^\dagger u), \quad S^{(-)} = u^\dagger v, \quad S^{(+)} = v^\dagger u; \quad (5.28)$$

where u , v and their adjoints satisfy the following commutation relations:

$$[u, u^\dagger] = 1, \quad [v, v^\dagger] = 1; \quad (5.29)$$

all other commutators vanish. Substituting (5.28) in (5.27), we find that the interaction Hamiltonian in the interaction picture can be written as

$$H_{RS}^I(t) = \sum_j g_j (e^{-i\omega_j t + i\omega_0 t} a_j v^\dagger u + e^{i\omega_j t - i\omega_0 t} a_j u^\dagger v). \quad (5.30)$$

The normally ordered equivalent of (5.30), such that $H_{RS}^I = \Omega_N F_{RS}$, is given by

$$F_{RS} = \sum_j g_j (e^{-i\omega_j t + i\omega_0 t} z_j \beta \alpha^* + \text{c.c.}), \quad (5.31)$$

where we use the correspondence

$$u \rightarrow \beta, \quad v \rightarrow \alpha. \quad (5.32)$$

Now we proceed in a strictly similar manner as in Secs. 4 and 5 A to write down the master equation for the normally ordered equivalent of the density operator for the system of interest. We make similar approximations as in Sec. 5 A and obtain the following Fokker-Planck equation:

monic-oscillator system in the following manner:

$$|0\rangle_S = u^\dagger |0\rangle, \quad |1\rangle_S = v^\dagger |0\rangle, \quad (5.35)$$

where $|0\rangle$ is the vacuum state of the oscillator system. Note also that

$$|0\rangle \langle 0| = \Omega_N \exp(-\alpha^* \alpha - \beta \beta^*), \quad (5.36)$$

where Ω_N denotes normal ordering operator. It then follows from (5.34)–(5.36) that the normally ordered equivalent of the density operator in interaction picture is given by

$$F_S^{(N)} = \left[\frac{1}{2} (1 - \langle \sigma^z \rangle) \beta^* \beta + \frac{1}{2} (1 + \langle \sigma^z \rangle) \alpha^* \alpha + \frac{1}{2} \langle \sigma^{(+)} \rangle \beta^* \alpha e^{-i\omega_0 t} + \frac{1}{2} \langle \sigma^{(-)} \rangle \beta \alpha^* e^{i\omega_0 t} \right] e^{-\beta \beta^* - \alpha \alpha^*}. \quad (5.37)$$

We substitute expression (5.37) in (5.33) and equate the coefficient of equal powers of α and β on both sides.

This leads to the equations

$$\frac{\partial}{\partial t} \langle \sigma^z \rangle = -[1 + \langle 2n(\omega_0) \rangle] 2\pi h(\omega_0) g^2(\omega_0) \times \left[\langle \sigma^z \rangle + \frac{1}{1 + \langle 2n(\omega_0) \rangle} \right], \quad (5.38)$$

$$\frac{\partial}{\partial t} \langle \sigma^{(\pm)} \rangle = \langle \sigma^{(\pm)} \rangle \{ -\pi h(\omega_0) g^2(\omega_0) \times [1 + \langle 2n(\omega_0) \rangle] - i\omega_0 \}. \quad (5.39)$$

Equations (5.38) and (5.39) give the rate of change of the average polarization of our system. In particular, for a steady state one has

$$\langle \sigma^{(\pm)} \rangle = \langle \sigma^{(\mp)} \rangle = 0, \quad \langle \sigma^z \rangle = -\tanh\left(\frac{1}{2}\beta\omega_0\right). \quad (5.40)$$

One would have of course expected the results (5.40) from general principles.

ACKNOWLEDGMENTS

It is a great pleasure to thank Professor E. Wolf and Professor G. G. Emch for helpful discussions. The author is also indebted to Professor G. G. Emch for making available a report of his paper.

APPENDIX: PHYSICAL CONSEQUENCES OF EQ. (5.24)

Let us write Eq. (5.24) in the form

$$\frac{\partial p_n(t)}{\partial t} = \delta \langle n(\omega) \rangle n p_{n-1}(t) + \delta(n+1) [1 + \langle n(\omega) \rangle] p_{n+1}(t) - \delta [\langle n(\omega) \rangle (2n+1) + n] p_n(t), \quad (A1)$$

where

$$\delta = 2\pi h(\omega) g^2(\omega), \quad p_n(t) = \rho_{n,n}(t). \quad (A2)$$

Note that $p_n(t)$ gives the probability of finding n photons in the field at time t . In order to solve (A1), we introduce the generating function $Q(x,t)$ for $p_n(t)$, defined by

$$Q(x,t) = \sum_0^\infty (1-x)^n p_n(t). \quad (A3)$$

The relation (A3) can be inverted for $p_n(t)$ and one obtains

$$p_n(t) = \frac{(-1)^n}{n!} \left. \frac{\partial^n Q(x,t)}{\partial x^n} \right|_{x=1}. \quad (A4)$$

It then follows from (A1) and (A3) that $Q(x,t)$ satisfies the following differential equation:

$$\frac{\partial Q}{\partial t} = -\delta \langle n(\omega) \rangle x Q - \delta [1 + \langle n(\omega) \rangle x] x \frac{\partial Q}{\partial x}. \quad (A5)$$

Equation (A5), subject to initial condition on $Q(x,t)$, can be solved by the method of characteristics.²³ We shall now give the solution of (A5) subject to various initial conditions.

(a) *Initial condition—thermal equilibrium.* In this case $p_n(0)$ and $Q(x,0)$ are given by

$$p_n(0) = \bar{n}^n / (1 + \bar{n})^{n+1}, \quad (A6a)$$

$$Q(x,0) = (1 + \bar{n}x)^{-1}. \quad (A6b)$$

Note that $p_n(0)$ is the Bose-Einstein distribution for one cell of phase space. Under the initial condition (A6b), the solution of (A5) can be shown to be

$$Q(x,t) = [1 + \bar{n}(t)x]^{-1}, \quad (A7)$$

where

$$\bar{n}(t) = \langle n(\omega) \rangle + e^{-\delta t} [\bar{n} - \langle n(\omega) \rangle]. \quad (A8)$$

We therefore conclude that the initial distribution of n remains Bose-Einstein, i.e.,

$$p_n(t) = [\bar{n}(t)]^n / [1 + \bar{n}(t)]^{n+1}. \quad (A9)$$

(b) *Initial condition—Poisson distribution.* In this case we have for $t=0$

$$p_n(0) = (\bar{n}^n / n!) e^{-\bar{n}}, \quad (A10a)$$

$$Q(x,0) = e^{-\bar{n}x}, \quad (A10b)$$

where \bar{n} represents initial average number of photons. Then the solution of (A5), subject to initial condition (A10b), is

$$Q(x,t) = [1 + x \langle n(\omega) \rangle (1 - e^{-\delta t})]^{-1} \times \exp \left[-\frac{\bar{n}x e^{-\delta t}}{1 + x \langle n(\omega) \rangle (1 - e^{-\delta t})} \right]. \quad (A11)$$

Using (A11) and (A4), we find that $p_n(t)$ can be written in terms of Laguerre polynomials as follows²³:

$$p_n(t) = \frac{[\langle n(\omega) \rangle (1 - e^{-\delta t})]^n}{[1 + \langle n(\omega) \rangle (1 - e^{-\delta t})]^{n+1}} \times \exp \left[-\frac{\bar{n} e^{-\delta t}}{1 + \langle n(\omega) \rangle (1 - e^{-\delta t})} \right] \times L_n \left(\frac{-\bar{n} e^{-\delta t}}{\langle n(\omega) \rangle (1 - e^{-\delta t}) [1 + \langle n(\omega) \rangle (1 - e^{-\delta t})]} \right). \quad (A12)$$

Using (A12) one can obtain the moments of the number

²³I. N. Sneddon, *Elements of Partial Differential Equations* (McGraw-Hill Book Co., New York, 1957), Chap. II. Similar methods have also been used by Narducci [L. M. Narducci (to be published)].

operator and in particular one can show that

$$\langle a^\dagger(t)a(t) \rangle = \bar{n}e^{-\delta t} + \langle n(\omega) \rangle (1 - e^{-\delta t}), \quad (A13)$$

$$\begin{aligned} \langle [a^\dagger(t)a(t)]^2 \rangle - \langle a^\dagger(t)a(t) \rangle^2 &= \bar{n}e^{-\delta t} + 2\langle n(\omega) \rangle \bar{n} \\ &\times (1 - e^{-\delta t})e^{-\delta t} + \langle n(\omega) \rangle (1 - e^{-\delta t}) [1 + \langle n(\omega) \rangle (1 - e^{-\delta t})]. \end{aligned} \quad (A14)$$

Note that the distribution (A12) is the one which would be obtained by the superposition of coherent and thermal fields. It should also be noted that the first term in the right-hand side of (A13) is characteristic of a Poisson distribution, whereas the second term is dependent on the initial properties of the reservoir.

(c) *Initial condition—m-quantum state.* In this case we have for $t=0$

$$p_n(0) = \delta_{n,m}, \quad (A15a)$$

$$Q(x,0) = (1-x)^m. \quad (A15b)$$

From (A5) and (A15b), one can easily show that

$$\begin{aligned} Q(x,t) &= [1 + \langle n(\omega) \rangle (1 - e^{-\delta t})x]^{-1} \\ &\times \left[1 - \frac{x e^{-\delta t}}{1 + x \langle n(\omega) \rangle (1 - e^{-\delta t})} \right]^m. \end{aligned} \quad (A16)$$

Using (A4) and (A16), one can prove that $p_n(t)$ may be written in terms of the hypergeometric function as

follows:

$$\begin{aligned} p_n(t) &= \frac{(1 - e^{-\delta t})^{n+m}}{(e^{\beta\omega} - e^{-\delta t})^{n+m}} \left[\frac{e^{m\beta\omega}(e^{\beta\omega} - 1)}{e^{\beta\omega} - e^{-\delta t}} \right] \\ &\times F \left[-n, -m, 1; \left(\frac{\sinh \frac{1}{2} \beta\omega}{\sinh \frac{1}{2} \delta t} \right)^2 \right], \end{aligned} \quad (A17)$$

where we have used Eq. (5.9) and the relation²⁴

$$(1-y)^{\alpha-1} (1-y+\gamma y)^{-\alpha} = \sum_0^\infty y^n F(-n, \alpha, 1; \gamma). \quad (A18)$$

The relation (A18) is valid for $|y| < 1$ and $|y(1-\gamma)| < 1$.

It is worth noting that our results for $p_n(t)$ constitute a generalization of Mollow's results¹⁴ on the quantum theory of field attenuation in the sense that our results are valid for finite temperatures. Mollow's results can be obtained from our results in the limit of zero temperature, or, equivalently, by letting $\langle n(\omega) \rangle \rightarrow 0$. We also mention that equations of the type (A1) may be used to study the relaxation of the moments of the number operator.²⁵

²⁴ A. Erdélyi, W. Magnus, F. Oberhettinger, and F. C. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Co., New York, 1953), Vol. I, p. 82.

²⁵ K. E. Shuler, G. H. Weiss, and K. Anderson, *J. Math. Phys.* **3**, 550 (1963).