1967ApJ...147..334C

THE POST-NEWTONIAN EFFECTS OF GENERAL RELATIVITY ON THE EQUILIBRIUM OF UNIFORMLY ROTATING BODIES

II. THE DEFORMED FIGURES OF THE MACLAURIN SPHEROIDS

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Received June 22, 1966

ABSTRACT

The equations of post-Newtonian hydrodynamics are solved appropriately for a uniformly rotating homogeneous mass with symmetry about the axis of rotation. The post-Newtonian figure is obtained as a deformation of the Newtonian Maclaurin spheroid (with semi-axes a_1 and a_3, say) by a Lagrangian displacement proportional to

$$\xi^{(2)} = \frac{1}{a_1^2}(\varpi^2 x_1, \varpi^2 x_2, -4\varpi^2 z),$$

where ϖ denotes the distance from the axis of rotation and x_1 and x_2 are the Cartesian coordinates in the equatorial plane. It is shown that the equation defining the boundary of the post-Newtonian configuration is of the form

$$\frac{\varpi^2}{a_1^2} + \frac{z^2}{a_3^2} - 1 - 2S_2^{\dagger}(e)\frac{R_{Sc}}{a_1}\left(\frac{\varpi^4}{a_1^4} - \frac{4\varpi^2 z^2}{a_1^2 a_3^2}\right) = 0,$$

where $S_2{\dagger}(e)$ is a determinate function of the eccentricity e of the Maclaurin spheroid and R_{Sc} ($= 2GM/c^2$) is the Schwarzschild radius. The function $S_2{\dagger}(e)$ is tabulated in the paper. Further, the angular velocity of rotation of the post-Newtonian configuration differs from that of the Maclaurin spheroid by an amount which is also tabulated.

The solution of the post-Newtonian equations exhibits a singularity at a certain eccentricity e^*($=$ 0.985226) of the Maclaurin spheroid. The origin of this singularity is that at e^* the Maclaurin spheroid allows an infinitesimal neutral deformation by a displacement proportional to $\xi^{(2)}$; and the Newtonian instability of the Maclaurin spheroid at e^* is excited by the post-Newtonian effects of general relativity.

I. INTRODUCTION

In a recent paper (Chandrasekhar 1965; this paper will be referred to hereafter as "Paper I") the effect of general relativity, in the post-Newtonian approximation, on the equilibrium of uniformly rotating homogeneous masses was considered in terms of a suitably generalized form of the Newtonian tensor virial theorem. The principal result of that paper was the establishment of the relation

$$\frac{\Omega^2}{\pi G\rho} = \frac{\mathfrak{W}_{33} - \mathfrak{W}_{11}}{(\pi G\rho) I_{11}} + \frac{R_{Sc}}{a_1} E(e), \tag{1}$$

where $\mathfrak{W}_{ij}$ and I_{ij} denote the potential-energy and the moment of inertia tensors, $E(e)$ is a certain function (defined in that paper; see also eq. [30] below) of the eccentricity of the Maclaurin spheroid (whose semi-major axis is a_1) and R_{Sc}($= 2GM/c^2$) is the Schwarzschild radius.

As was stated in Paper I, the derivation of equation (1) does not, by any means, solve the problem of the equilibrium: the complete specification of the figure must inevitably depend on the explicit solution of the equations governing the equilibrium in the post-Newtonian approximation. It is the object of this paper to provide that solution.

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1967ApJ...147..334C

II. THE BASIC EQUATIONS

The equations governing the equilibrium of a configuration rotating uniformly with an angular velocity Ω about the x_3-axis, in the post-Newtonian approximation, are

$$\frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu)+\frac{\partial}{\partial x_\alpha}\left[\left(1+\frac{2U}{c^2}\right)p\right]-\rho\frac{\partial U}{\partial x_\alpha}-\frac{2}{c^2}\rho\left(\phi\frac{\partial U}{\partial x_\alpha}+\frac{\partial\Phi}{\partial x_\alpha}\right)$$
$$+\frac{4}{c^2}\rho\left[v_\mu\frac{\partial}{\partial x_\mu}(v_\alpha U-U_\alpha)+v_\mu\frac{\partial U_\mu}{\partial x_\alpha}\right]=0\,, \tag{2}$$

where

$$v_1=-\Omega x_2\,,\qquad v_2=+\Omega x_1\,,\qquad v_3=0\,,\qquad v^2=\Omega^2(x_1^2+x_2^2)\,, \tag{3}$$

$$\sigma=\rho\left[1+\frac{1}{c^2}\left(v^2+2U+\Pi+\frac{p}{\rho}\right)\right], \tag{4}$$

$$\phi=v^2+U+\tfrac{1}{2}\Pi+\tfrac{3}{2}\frac{p}{\rho}\,, \tag{5}$$

and U_α and Φ are, in general, solutions of the equations

$$\nabla^2 U_\alpha=-4\pi G\rho v_\alpha\qquad\text{and}\qquad\nabla^2\Phi=-4\pi G\rho\phi\,. \tag{6}$$

For the case we are presently considering, we may write

$$U_1=-\Omega\mathfrak{D}_2\qquad\text{and}\qquad U_2=+\Omega\mathfrak{D}_1\,, \tag{7}$$

where

$$\mathfrak{D}_\alpha(x)=G\int_V\frac{\rho(x')x_\alpha'}{|x-x'|}dx'\,, \tag{8}$$

and the integration is effected over the entire volume V occupied by the fluid.

In our further discussion of the foregoing equations, we shall restrict ourselves to the case when the system has symmetry about the axis of rotation. Then all scalar quantities such as p, ρ, U, σ, etc., will be functions of

$$\varpi=\sqrt{(x_1^2+x_2^2)}\qquad\text{and}\qquad z\equiv x_3 \tag{9}$$

only. It is also clear that, in the case of axisymmetry, we can write

$$\mathfrak{D}_1=x_1\mathfrak{D}(\varpi,z)\qquad\text{and}\qquad\mathfrak{D}_2=x_2\mathfrak{D}(\varpi,z)\,, \tag{10}$$

where $\mathfrak{D}(\varpi,z)$ is a solution of the equation

$$\frac{\partial^2\mathfrak{D}}{\partial\varpi^2}+\frac{3}{\varpi}\frac{\partial\mathfrak{D}}{\partial\varpi}+\frac{\partial^2\mathfrak{D}}{\partial z^2}=-4\pi G\rho\,. \tag{11}$$

Now Krefetz (1965) has shown that under circumstances of axisymmetry equation (2) governing the equilibrium can be brought to the form

$$\left[1-\frac{1}{c^2}\left(\Pi+\frac{p}{\rho}\right)\right]\frac{\partial p}{\partial x_\alpha}=\rho\frac{\partial}{\partial x_\alpha}\left[U+\tfrac{1}{2}\Omega^2\varpi^2+\frac{1}{c^2}\left(\tfrac{1}{4}\Omega^4\varpi^4+2U\Omega^2\varpi^2+2\Phi-4\Omega^2\varpi^2\mathfrak{D}\right)\right]. \tag{12}$$

As in Paper I we shall consider a configuration in which the energy-density $\epsilon(=\rho c^2+\Pi)$ is a constant throughout. This assumption, that ϵ is a constant, is *formally* equivalent to the assumption $\rho=$ constant and $\Pi=0$ and the assignment to ρ the meaning

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1967ApJ...147..334C

of ϵ/c^2. We shall continue our consideration of equation (12) explicitly under the latter assumption, namely,

$$\rho = \text{constant and } \Pi = 0 . \tag{13}$$

Equation (12) then allows of immediate integration in the form

$$\frac{p}{\rho} = U + \tfrac{1}{2}\Omega^2\varpi^2 + \frac{1}{c^2}\left[\tfrac{1}{2}\left(\frac{p}{\rho}\right)^2 + \Omega^2\varpi^2(\tfrac{1}{4}\Omega^2\varpi^2 + 2U - 4\mathfrak{D}) + 2\Phi\right] + \text{constant} . \tag{14}$$

The problem now is to determine the equation governing the surface of the equilibrium figure such that the pressure given by equation (14) vanishes identically on it.

III. THE TERM THAT IS EXPLICITLY POST-NEWTONIAN IN THE PRESSURE INTEGRAL

In evaluating the term that is explicitly post-Newtonian in equation (14), we may legitimately use relations and equations that are valid in the Newtonian limit when the equilibrium figure is a Maclaurin spheroid. The relevant relations and equations are (cf. Paper I, eqs. [15]–[18] and [25])

$$\frac{p}{\rho} = \pi G\rho a_3^2 A_3\left(1 - \frac{\varpi^2}{a_1^2} - \frac{z^2}{a_3^2}\right), \tag{15}$$

$$U = \pi G\rho(I - A_1\varpi^2 - A_3 z^2) \qquad (I = 2a_1^2A_1 + a_3^2A_3) , \tag{16}$$

$$\mathfrak{D} = \pi G\rho a_1^2(A_1 - A_{11}\varpi^2 - A_{13}z^2) , \tag{17}$$

and

$$\phi = \pi G\rho\left[(I + \tfrac{3}{2}a_3^2A_3) + \left(\frac{7\Omega^2}{4\pi G\rho} - \tfrac{5}{2}A_1\right)\varpi^2 - \tfrac{5}{2}A_3z^2\right], \tag{18}$$

where the index symbols A_α, $A_{\alpha\beta}$, etc., are so normalized that $\Sigma A_\alpha = 2$. (Note that in the case we are presently considering, namely, when $a_1 = a_2$, the value of any index symbol is unaltered if the subscript 2 is replaced by 1 wherever it may occur.)

In terms of the potential U and

$$\mathfrak{D}_{\alpha\beta}(x) = G\int_V \frac{\rho(x')x_\alpha' x_\beta'}{|x - x'|}dx' , \tag{19}$$

the solution for Φ can be explicitly written down. Thus,

$$\frac{\Phi}{\pi G\rho} = (I + \tfrac{3}{2}a_3^2A_3)U + \left(\frac{7\Omega^2}{4\pi G\rho} - \tfrac{5}{2}A_1\right)(\mathfrak{D}_{11} + \mathfrak{D}_{22}) - \tfrac{5}{2}A_3\mathfrak{D}_{33} . \tag{20}$$

It is known that (cf. Chandrasekhar and Lebovitz 1962, eqs. [68] and [70])

$$\frac{\mathfrak{D}_{\alpha\beta}}{\pi G\rho} = a_\alpha^2 a_\beta^2\left(A_{\alpha\beta} - \sum_{\mu=1}^{3} A_{\alpha\beta\mu}x_\mu^2\right)x_\alpha x_\beta + \tfrac{1}{4}a_\alpha^2\delta_{\alpha\beta}\left(B_\alpha - 2\sum_{\mu=1}^{3}B_{\alpha\mu}x_\mu^2 + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3}B_{\alpha\mu\nu}x_\mu^2x_\nu^2\right) \tag{21}$$

(summation *only* over the indices explicitly noted).

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1967ApJ...147..334C

Now inserting for U and $\mathfrak{D}_{\alpha\beta}$ in accordance with equations (16) and (21), we obtain for Φ the solution

$$
\begin{aligned}
\frac{\Phi}{(\pi G\rho)^2} = {} & (I + \tfrac{3}{2} a_3^2 A_3)(I - A_1 \varpi^2 - A_3 z^2) \\
& + \left(\frac{7\Omega^2}{4\pi G\rho} - \tfrac{5}{2} A_1\right)[a_1^4 (A_{11} - A_{111}\varpi^2 - A_{113} z^2)\varpi^2 \\
& + \tfrac{1}{2} a_1^2 (B_1 - 2B_{11}\varpi^2 - 2B_{13} z^2 + 2B_{113}\varpi^2 z^2 + B_{111}\varpi^4 + B_{133} z^4)] \\
& - \tfrac{5}{2} A_3 [a_3^4 (A_{33} - A_{331}\varpi^2 - A_{333} z^2) z^2 \\
& + \tfrac{1}{4} a_3^2 (B_3 - 2B_{31}\varpi^2 - 2B_{33} z^2 + 2B_{331}\varpi^2 z^2 + B_{311}\varpi^4 + B_{333} z^4)] .
\end{aligned} \tag{22}
$$

Finally, making use of equations (15)–(17) and (22), we find that equation (14) takes the form

$$
\frac{p}{\rho} = U + \tfrac{1}{2}\Omega^2\varpi^2 + \frac{(\pi G\rho)^2}{c^2}(\alpha_0 + \alpha_1\varpi^2 + \alpha_3 z^2 + \alpha_{11}\varpi^4 + \alpha_{13}\varpi^2 z^2 + \alpha_{33} z^4) + \text{constant} , \tag{23}
$$

where

$$
\begin{aligned}
\alpha_0 = {} & \tfrac{1}{2} a_3^4 A_3 + (2I + 3a_3^2 A_3) I + \tfrac{1}{2} a_1^2 B_1 (\tfrac{7}{2}\Omega^2 - 5A_1) - \tfrac{5}{4} a_3^2 A_3 B_3 , \\
\alpha_1 = {} & -\frac{a_3^4}{a_1^2} A_3^2 + 2a_3^2 A_3 \Omega^2 - (2I + 3a_3^2 A_3) A_1 \\
& + (\tfrac{7}{2}\Omega^2 - 5A_1)(a_1^4 A_{11} - a_1^2 B_{11}) + \tfrac{5}{2} a_3^2 A_3 B_{13} , \\
\alpha_3 = {} & -a_3^2 A_3^2 - (2I + 3a_3^2 A_3) A_3 - (\tfrac{7}{2}\Omega^2 - 5A_1) a_1^2 B_{13} \\
& - 5A_3 (a_3^4 A_{33} - \tfrac{1}{2} a_3^2 B_{33}) , \\
\alpha_{11} = {} & \tfrac{1}{2} \frac{a_3^4}{a_1^4} A_3^2 + \Omega^2 (\tfrac{1}{4}\Omega^2 - 2A_1 + 4a_1^2 A_{11}) \\
& + (\tfrac{7}{2}\Omega^2 - 5A_1)(-a_1^4 A_{111} + \tfrac{1}{2} a_1^2 B_{111}) - \tfrac{5}{4} a_3^2 A_3 B_{113} , \\
\alpha_{13} = {} & \frac{a_3^2}{a_1^2} A_3^2 + \Omega^2 (4a_1^2 A_{13} - 2A_3) + (\tfrac{7}{2}\Omega^2 - 5A_1)(-a_1^4 A_{113} + a_1^2 B_{113}) \\
& - 5A_3 (-a_3^4 A_{331} + \tfrac{1}{2} a_3^2 B_{331}) , \\
\alpha_{33} = {} & \tfrac{1}{2} A_3^2 + \tfrac{1}{2} (\tfrac{7}{2}\Omega^2 - 5A_1) a_1^2 B_{133} - 5A_3 (-a_3^4 A_{333} + \tfrac{1}{4} a_3^2 B_{333}) .
\end{aligned} \tag{24}
$$

(Ω^2 is measured in the unit $\pi G\rho$ in the terms on the right-hand sides of the foregoing equations.)

It should be noted that, in equation (23), U is the *Newtonian* gravitational potential of the *deformed* (as at present, unknown) figure which the body assumes in the post-Newtonian approximation. Similarly Ω (in the centrifugal potential $\frac{1}{2}\Omega^2\varpi^2$) is not necessarily the same (to order $1/c^2$) as that of the Maclaurin spheroid which was used in the evaluation of the terms that are *explicitly* post-Newtonian.

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1967ApJ...147..334C

The Virial Relation

With the expression for p/ρ given by equation (23), the equations of hydrostatic equilibrium (for the case that is being considered) are

$$\frac{\partial p}{\partial x_1}=\rho\left[\frac{\partial U}{\partial x_1}+\Omega^2 x_1+\frac{(\pi G\rho)^2}{c^2}(2\alpha_1 x_1+4\alpha_{11}\varpi^2 x_1+2\alpha_{13}z^2 x_1)\right] \tag{25}$$

and

$$\frac{\partial p}{\partial z}=\rho\left[\frac{\partial U}{\partial x_3}+\frac{(\pi G\rho)^2}{c^2}(2\alpha_3 z+2\alpha_{13}\varpi^2 z+4\alpha_{33}z^3)\right]. \tag{26}$$

Multiplying equations (25) and (26) by x_1 and by z, respectively, and integrating over the volume of the configuration, we obtain

$$\begin{aligned}-\int_V p\,dx&=\mathfrak{W}_{11}+\Omega^2 I_{11}+\frac{(4\pi a_1^2 a_3)(\pi G\rho)^2\rho}{c^2}\left(\frac{2a_1^2}{15}\alpha_1+\frac{16a_1^4}{105}\alpha_{11}+\frac{2a_1^2a_3^2}{105}\alpha_{13}\right)\\&=\mathfrak{W}_{33}+\frac{(4\pi a_1^2 a_3)(\pi G\rho)^2\rho}{c^2}\left(\frac{2a_3^2}{15}\alpha_3+\frac{4a_3^4}{35}\alpha_{33}+\frac{4a_1^2a_3^2}{105}\alpha_{13}\right).\end{aligned} \tag{27}$$

From equations (27) we obtain the eliminant relation

$$\Omega^2=\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{11}}+\frac{(\pi G\rho)^2}{a_1^2c^2}(2a_3^2\alpha_3+\tfrac{2}{7}a_1^2a_3^2\alpha_{13}+\tfrac{12}{7}a_3^4\alpha_{33}-2a_1^2\alpha_1-\tfrac{16}{7}a_1^4\alpha_{11}). \tag{28}$$

Letting

$$E_{13}=\frac{2}{a_1^2}(a_3^2\alpha_3-a_1^2\alpha_1+\tfrac{1}{7}a_1^2a_3^2\alpha_{13}+\tfrac{6}{7}a_3^4\alpha_{33}-\tfrac{8}{7}a_1^4\alpha_{11}), \tag{29}$$

we can rewrite equation (28) in the form (cf. eq. [1])

$$\Omega^2=\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{11}}+\frac{(\pi G\rho)^2}{c^2}E_{13}. \tag{30}$$ [1]

IV. THE NATURE OF THE POST-NEWTONIAN DEFORMATION

We shall suppose that the post-Newtonian figure is obtained by a deformation of the classical Maclaurin spheroid by the application of a suitable Lagrangian displacement at each point of its interior and boundary. Since the density has to remain constant, before and after the deformation, the displacement must, in all events, be divergence free. For the sake of convenience and also for emphasizing that we are considering an effect of order $1/c^2$, we shall denote the required Lagrangian displacement by

$$\frac{\pi G\rho a_1^2}{c^2}\xi(x). \tag{31}$$

In virtue of this displacement, the bounding surface,

$$S_{\text{Mc}}=\frac{\varpi^2}{a_1^2}+\frac{z^2}{a_3^2}-1=0, \tag{32}$$

[1] We readily verify that E_{13} as defined in eq. (29) is related to $E(e)$ as introduced in eq. (1) by

$$E=3E_{13}/8a_1a_3.$$

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1967ApJ...147..334C

of the Maclaurin spheroid becomes

$$S = S_{\mathrm{Mc}} - \frac{\pi G \rho a_1^2}{c^2} \xi_\mu \frac{\partial S_{\mathrm{Mc}}}{\partial x_\mu} = 0 \,, \tag{33}$$

or explicitly

$$S(x) = \frac{\varpi^2}{a_1^2} + \frac{z^2}{a_3^2} - 1 - \frac{2\pi G \rho a_1^2}{c^2} \left(\frac{\xi_1 x_1 + \xi_2 x_2}{a_1^2} + \frac{\xi_3 z}{a_3^2} \right) = 0 \,. \tag{34}$$

On consideration, it appears that the *requirement* that $S(x)$ be axisymmetric and the *fact* that the post-Newtonian term, already present in the expression (23) for p/ρ, is quadratic in ϖ^2 and z^2 restrict us to displacements ξ which are linearly dependent on the following three:

$$\xi^{(1)} = (x_1, x_2, -2z) \,, \tag{35}$$

$$\xi^{(2)} = \frac{1}{a_1^2} (\varpi^2 x_1, \varpi^2 x_2, -4\varpi^2 z) \,, \tag{36}$$

$$\xi^{(3)} = \frac{1}{a_1^2} (z^2 x_1, z^2 x_2, -\tfrac{2}{3} z^3) \,. \tag{37}$$

We shall suppose then that

$$\xi = S_1 \xi^{(1)} + S_2 \xi^{(2)} + S_3 \xi^{(3)} \,, \tag{38}$$

where S_1, S_2, and S_3 are certain constants. For a displacement of this form, the equation for the bounding surface is

$$S(x) = \frac{\varpi^2}{a_1^2} + \frac{z^2}{a_3^2} - 1 - \frac{2\pi G \rho}{c^2} \left[a_1^2 S_1 \left(\frac{\varpi^2}{a_1^2} - \frac{2z^2}{a_3^2} \right) + S_2 \left(\frac{\varpi^4}{a_1^2} - \frac{4\varpi^2 z^2}{a_3^2} \right) + S_3 \left(\frac{\varpi^2 z^2}{a_1^2} - \frac{2z^4}{3a_3^2} \right) \right] = 0 \,. \tag{39}$$

While equation (39) determining $S(x)$ involves the three constants S_1, S_2, and S_3, only two of them are independent *modulo* the spheroid S_{Mc}. To see that this is the case, we first observe that the terms in S_2 and S_3 in equation (39) can be combined in the manner

$$\begin{aligned} &\left(S_2 - \frac{1}{3} \frac{a_3^2}{a_1^2} S_3 \right) \left(\frac{\varpi^4}{a_1^2} - \frac{4\varpi^2 z^2}{a_3^2} \right) + \tfrac{1}{3} a_3^2 S_3 \left(\frac{\varpi^4}{a_1^4} - \frac{\varpi^2 z^2}{a_1^2 a_3^2} - 2 \frac{z^4}{a_3^4} \right) \\ &= \left(S_2 - \frac{1}{3} \frac{a_3^2}{a_1^2} S_3 \right) \left(\frac{\varpi^4}{a_1^2} - \frac{4\varpi^2 z^2}{a_3^2} \right) + \tfrac{1}{3} a_3^2 S_3 \left(\frac{\varpi^2}{a_1^2} + \frac{z^2}{a_3^2} \right) \left(\frac{\varpi^2}{a_1^2} - \frac{2z^2}{a_3^2} \right) . \end{aligned} \tag{40}$$

An equivalent form of equation (39) is, therefore,

$$\begin{aligned} \left(\frac{\varpi^2}{a_1^2} + \frac{z^2}{a_3^2} \right) \left[1 - \frac{2\pi G \rho}{3c^2} a_3^2 S_3 \left(\frac{\varpi^2}{a_1^2} - \frac{2z^2}{a_3^2} \right) \right] = 1 \\ + \frac{2\pi G \rho}{c^2} \left[a_1^2 S_1 \left(\frac{\varpi^2}{a_1^2} - \frac{2z^2}{a_3^2} \right) + \left(S_2 - \frac{1}{3} \frac{a_3^2}{a_1^2} S_3 \right) \left(\frac{\varpi^4}{a_1^2} - \frac{4\varpi^2 z^2}{a_3^2} \right) \right] , \end{aligned} \tag{41}$$

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1967ApJ...147..334C

or alternatively (correctly to order $1/c^2$),

$$\frac{\varpi^2}{a_1^2}+\frac{z^2}{a_3^2}=1+\frac{2\pi G\rho}{c^2}\left[a_1^2\left(S_1+\tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)\left(\frac{\varpi^2}{a_1^2}-\frac{2z^2}{a_3^2}\right)+\left(S_2-\tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)\left(\frac{\varpi^4}{a_1^2}-\frac{4\varpi^2z^2}{a_3^2}\right)\right]. \tag{42}$$

A comparison of this last equation with equation (39) leads to the following theorem: the *displacements*

$$S_1\xi^{(1)}+S_2\xi^{(2)}+S_3\xi^{(3)}$$

and (43)

$$\left(S_1+\tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)\xi^{(1)}+\left(S_2-\tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)\xi^{(2)}$$

are, modulo the spheroid $S_{\rm Mc}$, entirely equivalent. The displacements $\xi^{(1)}$, $\xi^{(2)}$, and $\xi^{(3)}$ are, therefore, not linearly independent modulo the spheroid.

From the foregoing theorem we may conclude that *all the physically significant quantities must depend on S_1, S_2, and S_3 only through the linear combinations*

$$S_1+\tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3 \quad\text{and}\quad S_2-\tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\,. \tag{44}$$

We shall verify that this is indeed the case.

And finally a remark on the choice of the spheroid $S_{\rm Mc}$ as the figure for comparison with the post-Newtonian configuration: it is clear that as far as the terms that are explicitly post-Newtonian in equations (14) and (23) are concerned, any spheroid with axes which differ from those of $S_{\rm Mc}$ by amounts of order $1/c^2$ would have served just as well. To that extent the choice of S_1 is arbitrary since the displacement $\xi^{(1)}$ simply deforms the spheroid $S_{\rm Mc}$ into a neighboring one with axes which are slightly different (by amounts of order $1/c^2$); but it *is* relevant for fixing the angular velocity of the post-Newtonian configuration *relative* to that of the comparison spheroid. In § V we shall derive an equation which relates the angular velocity of the Maclaurin spheroid with that of the post-Newtonian configuration derived from it by the application of the displacement (38).

V. THE RELATION BETWEEN THE ANGULAR VELOCITIES OF THE NEWTONIAN AND THE POST-NEWTONIAN CONFIGURATIONS

Since the post-Newtonian configuration is to be obtained by the deformation of a Maclaurin spheroid by the application of a Lagrangian displacement, we can write

$$\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{11}}=\frac{\mathfrak{W}_{33}(\mathrm{Mc})-\mathfrak{W}_{11}(\mathrm{Mc})}{I_{11}(\mathrm{Mc})}+\frac{\delta\mathfrak{W}_{33}-\delta\mathfrak{W}_{11}}{I_{11}(\mathrm{Mc})}-\frac{\mathfrak{W}_{33}(\mathrm{Mc})-\mathfrak{W}_{11}(\mathrm{Mc})}{[I_{11}(\mathrm{Mc})]^2}\delta I_{11}, \tag{45}$$

where $\mathfrak{W}_{33}$ and $\mathfrak{W}_{11}$ refer to the post-Newtonian figure and $\delta\mathfrak{W}_{ij}$ and δI_{ij} are the first variations in $\mathfrak{W}_{ij}(\mathrm{Mc})$ and $I_{ij}(\mathrm{Mc})$ caused by the deformation. Remembering that

$$\Omega_{\rm Mc}^2=\frac{\mathfrak{W}_{33}(\mathrm{Mc})-\mathfrak{W}_{11}(\mathrm{Mc})}{I_{11}(\mathrm{Mc})}, \tag{46}$$

we can rewrite equation (45) in the form

$$\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{11}}=\Omega_{\rm Mc}^2+\frac{1}{I_{11}}(\delta\mathfrak{W}_{33}-\delta\mathfrak{W}_{11}-\Omega_{\rm Mc}^2\delta I_{11}), \tag{47}$$

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1967ApJ...147..334C

where in the "correction term" we have not distinguished between I_{11} and $I_{11}(\mathrm{Mc})$ as clearly not necessary (though the distinction Ω_{Mc} has been retained as a matter of definition).

Comparison of equation (47) with the virial relation (30) gives

$$\delta\Omega^2 = \Omega^2 - \Omega_{\mathrm{Mc}}^2 = \frac{(\pi G\rho)^2}{c^2}E_{13} - \frac{1}{I_{11}}(\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33} + \Omega_{\mathrm{Mc}}^2\delta I_{11}). \tag{48}$$

Now it is known quite generally that (cf. Chandrasekhar and Lebovitz 1963, eq. [48])

$$\frac{\delta\mathfrak{W}_{\alpha\alpha}}{\pi G\rho} = -(2B_{\alpha\alpha} - a_\alpha^2 A_{\alpha\alpha})V_{\alpha\alpha} + a_\alpha^2\sum_{\mu\neq\alpha}A_{\alpha\mu}V_{\mu\mu}, \tag{49}$$

where

$$V_{\alpha\alpha} = 2\int_V \rho\,\xi_\alpha x_\alpha dx = \delta I_{\alpha\alpha} \tag{50}$$

(no summation over repeated indices in eqs. [49] and [50] except those explicitly indicated).

Making use of equations (49) and (50) we readily find that

$$\frac{1}{I_{11}}(\delta\mathfrak{W}_{11} - \delta\mathfrak{W}_{33} + \Omega_{\mathrm{Mc}}^2\delta I_{11}) = 4\pi G\rho(a_1^2 A_{11} - 2a_3^2 A_{13})\frac{V_{11}}{I_{11}}, \tag{51}$$

if appropriate use is made of the condition

$$V_{11} = -\frac{a_1^2}{2a_3^2}V_{33} \tag{52}$$

(satisfied in virtue of the solenoidal character of $\boldsymbol{\xi}$ and the axisymmetry of the spheroid), the relation (cf. eq. [46])

$$\Omega_{\mathrm{Mc}}^2 = 2B_{13}(1 - a_3^2/a_1^2)\pi G\rho, \tag{53}$$

and the various identities among the index symbols.

For the chosen form of $\boldsymbol{\xi}$, namely, that given by equations (35)–(38), it can be verified that

$$\frac{V_{11}}{I_{11}} = \frac{2\pi G\rho a_1^2}{7c^2}\left(7S_1 + 4S_2 + \frac{a_3^2}{a_1^2}S_3\right). \tag{54}$$

Equation (48) now becomes

$$\delta\Omega^2 = \frac{(\pi G\rho)^2}{c^2}\Big\{E_{13} - \tfrac{8}{7}a_1^2(a_1^2A_{11} - 2a_3^2A_{13})\Big[7\Big(S_1 + \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\Big) + 4\Big(S_2 - \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\Big)\Big]\Big\}; \tag{55}$$

and this is the desired relation. Notice that in this relation S_1, S_2, and S_3 occur only in the combinations (44).

The principal conclusion to be drawn from the discussions of this and the preceding section is that, as far as the solution of the *physical* problem is concerned, no generality *need* be lost by setting $S_1 = S_3 = 0$. We shall, nevertheless, retain the terms in S_1 and S_3 to see how "mathematical analysis brings us back to the common sense view" (Jeans).

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1967ApJ...147..334C

VI. THE CHANGE IN THE GRAVITATIONAL POTENTIAL CAUSED BY THE DEFORMATION

The deformation of the Maclaurin spheroid by the displacement (38) will change the gravitational potential U by the amount (see eq. [31])

$$\delta U = \frac{\pi G \rho a_1^2}{c^2}[S_1 \delta U^{(1)} + S_2 \delta U^{(2)} + S_3 \delta U^{(3)}], \tag{56}$$

where

$$\delta U^{(i)} = G\delta \int_V \frac{\rho(x')}{|x-x'|}dx' = G\int_V dx' \rho(x')\xi_l^{(i)}(x')\frac{\partial}{\partial x_l'}\frac{1}{|x-x'|}, \tag{57}$$

or, equivalently,

$$\delta U^{(i)} = -G\frac{\partial}{\partial x_l}\int_V \frac{\rho(x')\xi_l^{(i)}(x')}{|x-x'|}dx'. \tag{58}$$

For the particular displacements $\xi^{(i)}$ specified in equations (35)–(37), the required variations $\delta U^{(i)}$ can be expressed in terms of the potentials $\mathfrak{D}_\alpha$ (cf. eq. [8]) and

$$\mathfrak{D}_{\alpha\beta\gamma}(x) = G\int_V \frac{\rho(x')x_\alpha' x_\beta' x_\gamma'}{|x-x'|}dx'. \tag{59}$$

Thus

$$\delta U^{(1)} = -\frac{\partial \mathfrak{D}_1}{\partial x_1} - \frac{\partial \mathfrak{D}_2}{\partial x_2} + 2\frac{\partial \mathfrak{D}_3}{\partial x_3}, \tag{60}$$

$$a_1^2 \delta U^{(2)} = -\frac{\partial}{\partial x_1}(\mathfrak{D}_{111} + \mathfrak{D}_{122}) - \frac{\partial}{\partial x_2}(\mathfrak{D}_{112} + \mathfrak{D}_{222}) + 4\frac{\partial}{\partial x_3}(\mathfrak{D}_{113} + \mathfrak{D}_{223}), \tag{61}$$

and

$$a_1^2 \delta U^{(3)} = -\frac{\partial \mathfrak{D}_{331}}{\partial x_1} - \frac{\partial \mathfrak{D}_{332}}{\partial x_2} + \tfrac{2}{3}\frac{\partial \mathfrak{D}_{333}}{\partial x_3}. \tag{62}$$

We already have formulae for $\mathfrak{D}_\alpha$. And by a method described in Chandrasekhar (1964; see also Routh 1892) it can be shown that

$$\begin{aligned}\frac{\mathfrak{D}_{\alpha\beta\gamma}}{\pi G\rho} = {} & a_\alpha^2 a_\beta^2 a_\gamma^2\left(A_{\alpha\beta\gamma} - \sum_{\mu=1}^{3} A_{\alpha\beta\gamma\mu}x_\mu^2\right)x_\alpha x_\beta x_\gamma \\ & + \tfrac{1}{4}a_\alpha^2 a_\beta^2 \delta_{\beta\gamma}\left(B_{\alpha\beta} - 2\sum_{\mu=1}^{3} B_{\alpha\beta\mu}x_\mu^2 + \sum_{\mu=1}^{3}\sum_{\nu=2}^{3} B_{\alpha\beta\mu\nu}x_\mu^2 x_\nu^2\right)x_\alpha \\ & + \tfrac{1}{4}a_\beta^2 a_\gamma^2 \delta_{\gamma\alpha}\left(B_{\beta\gamma} - 2\sum_{\mu=1}^{3} B_{\beta\gamma\mu}x_\mu^2 + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} B_{\beta\gamma\mu\nu}x_\mu^2 x_\nu^2\right)x_\beta \\ & + \tfrac{1}{4}a_\gamma^2 a_\alpha^2 \delta_{\alpha\beta}\left(B_{\gamma\alpha} - 2\sum_{\mu=1}^{3} B_{\gamma\alpha\mu}x_\mu^2 + \sum_{\mu=1}^{3}\sum_{\nu=1}^{3} B_{\gamma\alpha\mu\nu}x_\mu^2 x_\nu^2\right)x_\gamma.\end{aligned} \tag{63}$$

And on evaluating $\delta U^{(i)}$ with the aid of the formulae for $\mathfrak{D}_\alpha$ and $\mathfrak{D}_{\alpha\beta\gamma}$ and remembering that we are now considering the deformation of a spheroid for which $a_1 = a_2$, we find that they are of the forms

$$\delta U^{(1)} = u_0^{(1)} + u_1^{(1)}\varpi^2 + u_3^{(1)}z^2, \tag{64}$$

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1967ApJ...147..334C

and

$$a_1^2\delta U^{(i)} = u_0^{(i)} + u_1^{(i)}\varpi^2 + u_3^{(i)}z^2 + u_{11}^{(i)}\varpi^4 + u_{13}^{(i)}\varpi^2z^2 + u_{33}^{(i)}z^4 \quad (i = 1,3)\,, \tag{65}$$

where

$$\begin{aligned} u_0^{(1)} &= 2(a_3^2A_3 - a_1^2A_1)\,, \\ u_1^{(1)} &= 2(2a_1^2A_{11} - a_3^2A_{13})\,, \\ u_3^{(1)} &= 2(a_1^2A_{13} - 3a_3^2A_{33})\,; \end{aligned} \tag{66}$$

$$\begin{aligned} u_0^{(2)} &= 2a_1^2(a_3^2B_{13} - a_1^2B_{11})\,, \\ u_1^{(2)} &= 4a_1^2(-a_1^4A_{111} + a_1^2a_3^2A_{113} + 2a_1^2B_{111} - a_3^2B_{113})\,, \\ u_3^{(2)} &= 4a_1^2(a_1^2B_{113} - 3a_3^2B_{133})\,, \\ u_{11}^{(2)} &= 2a_1^2(3a_1^4A_{1111} - 2a_1^2a_3^2A_{1113} - 3a_1^2B_{1111} + a_3^2B_{1113})\,, \\ u_{13}^{(2)} &= 4a_1^2(a_1^4A_{1113} - 3a_1^2a_3^2A_{1133} - 2a_1^2B_{1113} + 3a_3^2B_{1133})\,, \\ u_{33}^{(2)} &= 2a_1^2(5a_3^2B_{1333} - a_1^2B_{1133})\,; \end{aligned} \tag{67}$$

and

$$\begin{aligned} u_0^{(3)} &= \tfrac{1}{2}a_3^2(a_3^2B_{33} - a_1^2B_{13})\,, \\ u_1^{(3)} &= a_3^2(2a_1^2B_{113} - a_3^2B_{133})\,, \\ u_3^{(3)} &= a_3^2(-2a_3^2a_1^2A_{133} + a_1^2B_{133} + 2a_3^4A_{333} - 3a_3^2B_{333})\,, \\ u_{11}^{(3)} &= \tfrac{1}{2}a_3^2(a_3^2B_{1133} - 3a_1^2B_{1113})\,, \\ u_{13}^{(3)} &= a_3^2(4a_3^2a_1^2A_{1133} - 2a_1^2B_{1133} - 2a_3^4A_{1333} + 3a_3^2B_{1333})\,, \\ u_{33}^{(2)} &= a_3^2(2a_3^2a_1^2A_{1333} - \tfrac{1}{2}a_1^2B_{1333} - \tfrac{10}{3}a_3^4A_{3333} + \tfrac{5}{2}a_3^2B_{3333})\,. \end{aligned} \tag{68}$$

Combining the foregoing results, we can write

$$\delta U = \frac{(\pi G\rho)^2}{c^2}\Big\{a_1^2S_1[u_0^{(1)} + u_1^{(1)}\varpi^2 + u_3^{(1)}z^2] + \sum_{n=1}^{2} S_n[u_0^{(n)} + u_1^{(n)}\varpi^2 + u_3^{(n)}z^2 + u_{11}^{(n)}\varpi^4 + u_{13}^{(n)}\varpi^2z^2 + u_{33}^{(n)}z^4]\Big\}\,. \tag{69}$$

VII. THE DETERMINATION OF THE POST-NEWTONIAN FIGURE

Returning to equation (23), we shall now rewrite it in the form

$$\frac{p}{\rho} = U_{\rm Mc} + \tfrac{1}{2}\Omega_{\rm Mc}^2\varpi^2 + \delta U + \tfrac{1}{2}\delta\Omega^2\varpi^2 + \frac{(\pi G\rho)^2}{c^2}(\alpha_0 + \alpha_1\varpi^2 + \alpha_3z^2 + \alpha_{11}\varpi^4 + \alpha_{13}\varpi^2z^2 + \alpha_{33}z^4) + \text{constant}\,, \tag{70}$$

where $\delta\Omega^2$ and δU are given by equations (55) and (69). The first two terms on the right-hand side of equation (70), together with a suitably chosen additive constant, can be

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1967ApJ...147..334C

combined to give the expression (15) appropriate to a Maclaurin spheroid. We may, therefore, write

$$\frac{p}{\rho} = (\pi G\rho)a_3^2 A_3\left(1 - \frac{\varpi^2}{a_1^2} - \frac{z^2}{a_3^2}\right) + \frac{(\pi G\rho)^2}{c^2}(P_1\varpi^2 + P_3 z^2 + P_{11}\varpi^4 + P_{13}\varpi^2 z^2 + P_{33}z^4 + \text{constant}), \tag{71}$$

where

$$P_1 = \alpha_1 + a_1^2 S_1 u_1^{(1)} + S_2 u_1^{(2)} + S_3 u_1^{(3)} + \tfrac{1}{2}\delta\omega^2, \tag{72}$$

$$P_3 = \alpha_3 + a_1^2 S_1 u_3^{(1)} + S_2 u_3^{(2)} + S_3 u_3^{(3)}, \tag{73}$$

and

$$P_{ij} = \alpha_{ij} + S_2 u_{ij}^{(2)} + S_3 u_{ij}^{(3)} \qquad (ij = 11, 13, \text{ and } 33). \tag{74}$$

The additive constant in equation (71) is now of order $1/c^2$; and in equation (72) $\delta\omega^2$ is the quantity in braces in equation (55), *i.e.*,

$$\delta\Omega^2 = \frac{(\pi G\rho)^2}{c^2}\delta\omega^2. \tag{75}$$

Now it is shown in Appendix I that among the coefficients $u_i^{(n)}$ and $u_{ij}^{(n)}$ the following relations obtain:

$$u_i^{(3)} = \tfrac{1}{3}a_3^2\left[u_i^{(1)} - \frac{1}{a_1^2}u_i^{(2)}\right] \qquad (i = 1, 3)$$

and

$$u_{ij}^{(3)} = -\tfrac{1}{3}\frac{a_3^2}{a_1^2}u_{ij}^{(2)} \qquad (ij = 11, 13, \text{ and } 33). \tag{76}$$

In view of these relations the expressions for P_i and P_{ij} can be written in the forms

$$P_1 = \alpha_1 + a_1^2\left(S_1 + \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)u_1^{(1)} + \left(S_2 - \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)u_1^{(2)} + \tfrac{1}{2}\left\{E_{13} - \tfrac{8}{7}a_1^2(a_1^2 A_{11} - 2a_3^2 A_{13})\left[7\left(S_1 + \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right) + 4\left(S_2 - \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)\right]\right\}, \tag{77}$$

$$P_3 = \alpha_3 + a_1^2\left(S_1 + \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)u_3^{(1)} + \left(S_2 - \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)u_3^{(2)}, \tag{78}$$

and

$$P_{ij} = \alpha_{ij} + \left(S_2 - \tfrac{1}{3}\frac{a_3^2}{a_1^2}S_3\right)u_{ij}^{(2)} \qquad (ij = 11, 13, \text{ and } 33). \tag{79}$$

The facts, that the coefficients P_i and P_{ij} in the solution for the pressure involve the constants S_1, S_2, and S_3 only in the combinations (45) and further that these combinations occur precisely in the places of S_1 and S_2 (corresponding to the two displacements [43] being equivalent modulo the spheroid), imply that no loss of generality is entailed by setting

$$S_3 = 0. \tag{80}$$

Accordingly, setting $S_3 = 0$ in equations (77)–(79), we have

$$P_1 = \alpha_1 + a_1^2 S_1 u_1^{(1)} + S_2 u_1^{(2)} + \tfrac{1}{2}[E_{13} - \tfrac{8}{7}a_1^2(a_1^2 A_{11} - 2a_3^2 A_{13})(7S_1 + 4S_2)], \tag{81}$$

$$P_3 = \alpha_3 + a_1^2 S_1 u_3^{(1)} + S_2 u_3^{(2)}, \tag{82}$$

and

$$P_{ij} = \alpha_{ij} + S_2 u_{ij}^{(2)} \qquad (ij = 11, 13, \text{ and } 33). \tag{83}$$

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1967ApJ...147..334C

It remains to apply the proper boundary condition to the solution (71). The boundary condition is that on the surface (cf. eq. [39] with S_3 set equal to zero),

$$S=\frac{\varpi^2}{a_1^2}+\frac{z^2}{a_3^2}-1-\frac{2\pi G\rho}{c^2}\left[a_1^2S_1\left(\frac{\varpi^2}{a_1^2}-\frac{2z^2}{a_3^2}\right)+S_2\left(\frac{\varpi^4}{a_1^2}-\frac{4\varpi^2z^2}{a_3^2}\right)\right]=0\,, \tag{84}$$

the pressure, as given by equation (71), must vanish identically.

In view of equation (84), the value of p/ρ on S is given by

$$\left(\frac{p}{\rho}\right)_S=\frac{(\pi G\rho)^2}{c^2}\left\{-2a_3^2A_3\left[a_1^2S_1\left(\frac{\varpi^2}{a_1^2}-\frac{2z^2}{a_3^2}\right)+S_2\left(\frac{\varpi^4}{a_1^2}-\frac{4\varpi^2z^2}{a_3^2}\right)\right]\right.$$
$$\left.+P_1\varpi^2+P_3z^2+P_{11}\varpi^4+P_{13}\varpi^2z^2+P_{33}z^4+\text{constant}\right\}, \tag{85}$$

or

$$\left(\frac{p}{\rho}\right)_S=\frac{(\pi G\rho)^2}{c^2}(Q_1\varpi^2+Q_3z^2+Q_{11}\varpi^4+Q_{13}\varpi^2z^2+Q_{33}z^4+\text{constant})\,, \tag{86}$$

where

$$Q_1=P_1-2a_3^2A_3S_1\,,\qquad Q_3=P_3+4a_1^2A_3S_1\,,$$
$$Q_{11}=P_{11}-2\frac{a_3^2}{a_1^2}A_3S_2\,,\qquad Q_{13}=P_{13}+8A_3S_2\,,\qquad\text{and}\qquad Q_{33}=P_{33}\,. \tag{87}$$

Since $(p/\rho)_S$ is of order $1/c^2$, it will clearly suffice if we can arrange for it to vanish on the *original* spheroid S_{Mc}. And we can arrange for it in two steps: *first* we require that the expression

$$Q_1\varpi^2+Q_3z^2+Q_{11}\varpi^4+Q_{13}\varpi^2z^2+Q_{33}z^4 \tag{88}$$

remains constant on S_{Mc}; *then*, we determine the additive constant in the solution (86) so that $(p/\rho)_S$ does vanish.

We readily verify that *in order that the expression (88) remains constant on S_{Mc}, it is necessary and sufficient that*

$$a_1^4Q_{11}+a_3^4Q_{33}-a_1^2a_3^2Q_{13}=0$$
and
$$a_1^4Q_{11}-a_3^4Q_{33}+a_1^2Q_1-a_3^2Q_3=0\,. \tag{89}$$

We are, therefore, provided with two equations; and it might be assumed that the two equations will exactly suffice to determine the two "unknown" constants S_1 and S_2. However, we shall show that *neither of the two equations (89) involves S_1; that the two equations are simply multiples of one another; and that, as a result, while S_2 is uniquely determined, S_1 is left undetermined.*

Writing out the conditions (89) explicitly, we have

$$a_1^4\left[\alpha_{11}+u_{11}^{(2)}S_2-2\frac{a_3^2}{a_1^2}A_3S_2\right]+a_3^4[\alpha_{33}+u_{33}^{(2)}S_2]$$
$$-a_1^2a_3^2[\alpha_{13}+u_{13}^{(2)}S_2+8A_3S_2]=0 \tag{90}$$

and

$$a_1^4\left[\alpha_{11}+u_{11}^{(2)}S_2-2\frac{a_3^2}{a_1^2}S_2\right]-a_3^4[\alpha_{33}+u_{33}^{(2)}S_2]$$
$$+a_1^2\{\alpha_1+a_1^2u_1^{(1)}S_1+u_1^{(2)}S_2-2a_3^2A_3S_1$$
$$+\tfrac{1}{2}[E_{13}-\tfrac{8}{7}a_1^2(a_1^2A_{11}-2a_3^2A_{13})(7S_1+4S_2)]\}$$
$$-a_3^2[\alpha_3+a_1^2u_3^{(1)}S_1+u_3^{(2)}S_2+4a_1^2A_3S_1]=0\,. \tag{91}$$

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1967ApJ...147..334C

Equation (90) contains no terms in S_1. Equation (91) does contain terms in S_1; but the coefficient with which it occurs is

$$a_1^2[a_1^2u_1^{(1)} - 2a_3^2A_3 - 4a_1^2(a_1^2A_{11} - 2a_3^2A_{13})] - a_3^2[a_1^2u_3^{(1)} + 4a_1^2A_3]. \quad (92)$$

On substituting for $u_1^{(1)}$ and $u_3^{(1)}$ from equations (66), we find that the expression (92) becomes

$$\begin{aligned} a_1^2[2a_1^2(2a_1^2A_{11} - a_3^2A_{13}) - 2a_3^2A_3 - 4a_1^2(a_1^2A_{11} - 2a_3^2A_{13})] \\ - a_3^2[2a_1^2(a_1^2A_{13} - 3a_3^2A_{33}) + 4a_1^2A_3]; \end{aligned} \quad (93)$$

and on simplification this last expression is found to vanish.[2] Thus, as stated neither, of the two equations (89) contains terms in S_1; it is accordingly left undetermined. This last fact is in agreement with the view arrived at earlier in § V on "common sense." We shall return to the meaning of this indeterminancy in the solution later in this section.

Equations (90) and (91) are thus seen to become two equations for determining the single constant S_2. Therefore, in order to be consistent, the two equations must be simple multiples of one another. To show that this is indeed the case we shall first rewrite equations (90) and (91) (after suppressing in the latter the terms in S_1) in the forms

$$\begin{aligned} (a_1^4\alpha_{11} + a_3^4\alpha_{33} - a_1^2a_3^2\alpha_{13}) + [a_1^4u_{11}^{(2)} + a_3^4u_{33}^{(2)} \\ - a_1^2a_3^2u_{13}^{(2)} - 10a_1^2a_3^2A_3]S_2 = 0 \end{aligned} \quad (94)$$

and

$$\begin{aligned} (a_1^4\alpha_{11} - a_3^4\alpha_{33} + a_1^2\alpha_1 - a_3^2\alpha_3 + \tfrac{1}{2}a_1^2E_{13}) + [a_1^4u_{11}^{(2)} - a_3^4u_{33}^{(2)} \\ + a_1^2u_1^{(2)} - a_3^2u_3^{(2)} - 2a_1^2a_3^2A_3 - \tfrac{16}{7}a_1^4(a_1^2A_{11} - 2a_3^2A_{13})]S_2 = 0. \end{aligned} \quad (95)$$

On substituting for E_{13} from equation (29), we find that the constant term in equation (95) is

$$-\tfrac{1}{7}(a_1^4\alpha_{11} + a_3^4\alpha_{33} - a_1^2a_3^2\alpha_{13}); \quad (96)$$

and this, we observe, is exactly $-\frac{1}{7}$ of the constant term in equation (94). Therefore, in order that equations (94) and (95) may be mutually consistent, it is necessary that

$$\begin{aligned} a_1^4u_{11}^{(2)} + a_3^4u_{33}^{(2)} - a_1^2a_3^2u_{13}^{(2)} - 10a_1^2a_3^2A_3 = -7[a_1^4u_{11}^{(2)} - a_3^4u_{33}^{(2)} \\ + a_1^2u_1^{(2)} - a_3^2u_3^{(2)} - 2a_1^2a_3^2A_3] + 16a_1^4(a_1^2A_{11} - 2a_3^2A_{13}), \end{aligned} \quad (97)$$

or, alternatively,

$$\begin{aligned} 8a_1^4u_{11}^{(2)} - 6a_3^4u_{33}^{(2)} - a_1^2a_3^2u_{13}^{(2)} + 7[a_1^2u_1^{(2)} - a_3^2u_3^{(2)}] \\ - 24a_1^2a_3^2A_3 - 16a_1^6A_{11} + 32a_1^4a_3^2A_{13} = 0. \end{aligned} \quad (98)$$

It is shown in Appendix II that this last condition is satisfied. Accordingly, we may write as the solution for S_2,

$$S_2 = -\frac{a_1^4\alpha_{11} + a_3^4\alpha_{33} - a_1^2a_3^2\alpha_{13}}{a_1^4u_{11}^{(2)} + a_3^4u_{33}^{(2)} - a_1^2a_3^2u_{13}^{(2)} - 10a_1^2a_3^2A_3}. \quad (99)$$

[2] By virtue of the identity (cf. Chandrasekhar and Lebovitz 1962, eq. [20])

$$2a_1^2A_{13} + 3a_3^2A_{33} - 3A_3 = 0.$$

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1967ApJ...147..334C

We now return to the meaning of the constant S_1 having been left undetermined. As we have already noted in § IV, the displacement $\xi^{(1)}$ deforms the Maclaurin spheroid ($S_{\rm Mc}$) into a neighboring *Maclaurin spheroid.* The angular velocity of this neighboring Maclaurin spheroid differs from that of $S_{\rm Mc}$ by an amount determined by the equation (cf. eq. [55] after setting $S_3 = 0$ and ignoring the terms in E_{13} and S_2)

$$\delta\Omega_{\rm Mc}^2 = -\frac{(\pi G\rho)^2}{c^2} 8a_1^2(a_1^2 A_{11} - 2a_3^2 A_{13}) . \tag{100}$$

Accordingly, the free choice we have in selecting the amplitude of the displacement $\xi^{(1)}$ corresponds to the free choice we have in selecting a particular Maclaurin spheroid (among neighboring ones) for comparison with the post-Newtonian configuration; and the availability of this choice has clearly no physical content. Therefore, for the purposes of delineating the post-Newtonian Maclaurin sequence, we may, without any loss of generality, set

$$S_1 = 0 . \tag{101}$$

With this understanding, the figure of the post-Newtonian configuration, neighboring a *given* Maclaurin spheroid, is determined by the equation

$$S(x) = \frac{\varpi^2}{a_1^2} + \frac{z^2}{a_3^2} - 1 - \frac{2\pi G\rho a_1^2}{c^2} S_2 \left(\frac{\varpi^4}{a_1^4} - \frac{4\varpi^2 z^2}{a_1^2 a_3^2}\right) = 0 , \tag{102}$$

where S_2 is given by equation (99); and the corresponding angular velocity of rotation is determined by the equation (eq. [55] with S_1 and S_3 set equal to zero)

$$\Omega^2 = \Omega_{\rm Mc}^2 + \frac{(\pi G\rho)^2}{c^2}\left[E_{13} - \frac{32 a_1^2}{7} S_2 (a_1^2 A_{11} - 2a_3^2 A_{13})\right] . \tag{103}$$

The solution of the problem is now completed.

VIII. NUMERICAL RESULTS

The post-Newtonian effects of general relativity are most conveniently expressed in terms of the parameter

$$\frac{R_{\rm Sc}}{a_1} = \frac{2GM}{a_1 c^2} = \frac{8\pi G\rho a_1 a_3}{3c^2} . \tag{104}$$

In terms of this parameter, the solutions (102) and (103) take the forms

$$S(x) = \frac{\varpi^2}{a_1^2} + \frac{z^2}{a_3^2} - 1 - 2S_2^\dagger \frac{R_{\rm Sc}}{a_1}\left(\frac{\varpi^4}{a_1^4} - \frac{4\varpi^2 z^2}{a_1^2 a_3^2}\right) = 0 , \tag{105}$$

and

$$\frac{\delta\Omega^2}{\pi G\rho} = \frac{R_{\rm Sc}}{a_1}\left[E(e) - \tfrac{32}{7} S_2^\dagger (a_1^2 A_{11} - 2a_3^2 A_{13})\right] , \tag{106}$$

where (cf. footnote on p. 338)

$$S_2^\dagger = \frac{3a_1}{8a_3} S_2 \quad \text{and} \quad E(e) = \frac{3E_{13}}{8a_1 a_3} . \tag{107}$$

In Table 1 we list the values S_2, $S_2^\dagger$, and $a_1\delta\Omega^2/(\pi G\rho R_{\rm Sc})$ for various values of the eccentricity of the comparison Maclaurin spheroid; and in Table 2 we similarly list the values of the coefficients P_1, P_3, etc., which occur in the solution (71) for the pressure.

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1967ApJ...147..334C

From Tables 1 and 2 it is manifest that the solution is singular at

$$e^* \simeq 0.985226 . \tag{108}$$

The origin of this singularity is explained in § IX.

IX. THE ORIGIN OF THE SINGULARITY AT e^*

It is apparent from equation (99) for S_2 that the solution of the post-Newtonian configuration will have a singularity along the Maclaurin sequence where

$$a_1^4 u_{11}^{(2)} + a_3^4 u_{33}^{(2)} - a_1^2 a_3^2 u_{13}^{(2)} - 10 a_1^2 a_3^2 A_3 = 0 . \tag{109}$$

TABLE 1

THE VALUES OF S_2, $S_2\dagger$, AND $a_1\delta\Omega^2/\pi G\rho R_{S_0}$ WHICH DESCRIBE THE POST-NEWTONIAN CONFIGURATIONS DERIVED FROM MACLAURIN SPHEROIDS OF VARIOUS ECCENTRICITIES

($e_J = 0.8126700$; $e^* = 0.985226$)

e	S_2	$S_2\dagger$	$a_1\delta\Omega^2/\pi G\rho R_{S_0}$	e	S_2	$S_2\dagger$	$a_1\delta\Omega^2/\pi G\rho R_{S_0}$
0 ..	0	0	0	0 92	+ 0 08505	+ 0 08138	+ 0 28091
0 20	0 00003	0 00001	0 00935	e_{max} .	+ 0 09977	+ 0 10176	+ 0 27855
35	00032	00013	03001	94	+ 0 1198	+ 0 1317	+ 0 27087
40	00057	00023	04005	95	+ 0 1489	+ 0 1788	+ 0 25411
45	00098	00041	05196	96	+ 0 1965	+ 0 2632	+ 0 21949
50	00160	00069	06598	97	+ 0 2966	+ 0 4575	+ 0 13890
55	00254	00114	08238	975	+ 0 4122	+ 0 6957	+ 0 04554
60	00394	00185	10149	980	+ 0 7351	+ 1 3852	− 0 20639
65	00606	00299	12369	981	+ 0 8888	+ 1 7180	− 0 32372
70	00928	00488	14938	982	+ 1 1366	+ 2 2566	− 0 51117
75	01428	00810	17886	983	+ 1 6051	+ 3 2783	− 0 86278
80	02236	01397	21203	984	+ 2 8344	+ 5 9658	− 1 7795
e_J	02517	01620	22089	985	+14 943	+32 474	−10 771
82	02699	01768	22608	e^* ..	$\pm\infty$	$\pm\infty$	$\pm\infty$
84	03284	02270	24026	986	− 4 2106	− 9 4695	+ 3 4395
85	03636	02588	24728	987	− 1 7729	− 4 1366	+ 1 6245
86	04039	02968	25414	988	− 1 0901	− 2 6466	+ 1 1114
87	04502	03424	26072	989	− 0 7672	− 1 9450	+ 0 86470
88	05042	03981	26688	990	− 0 5779	− 1 5362	+ 0 71634
89	05677	04669	27238	995	− 0 1955	− 0 7342	+ 0 37418
90	06435	05536	27690	0 999	− 0 0572	− 0 4794	+ 0 16084
0 91	0 07357	0 06654	0 27999				

As we have seen this condition is met at e^* where the singularity occurs. We shall now show that the condition (109) is precisely the same as for the occurrence of a neutral point along the Maclaurin sequence for a deformation by a Lagrangian displacement proportional to $\xi^{(2)}$ (cf. eq. [36]).

Consider then the deformation of a Maclaurin spheroid by the displacement

$$\xi^{(2)} = \frac{S_2}{a_1^2}(\varpi^2 x_1, \varpi^2 x_2, -4\varpi^2 z) , \tag{110}$$

where S_2 is now an infinitesimal constant. We shall suppose that the deformation leaves the equilibrium unaffected (as is appropriate for a neutral point); and we shall obtain the condition for this to be the case.

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1967ApJ...147..334C

TABLE 2

THE VALUES OF THE VARIOUS COEFFICIENTS IN THE SOLUTION (71) FOR THE PRESSURE

(e_J = 0.8126700; e^* = 0.985226)

e	$a_1+S_2u_1^{(2)}$	$a_3+S_2u_3^{(2)}$	P_{13}	P_{11}	P_{33}
0.	0	0	0	0	0
0.20	−4.2916	−4.4424	1.7745	0.8592	0.9163
.35	−3.9722	−4.4338	1.7657	7979	0.9785
40.	−3.8251	−4.4276	1.7608	7700	1.0097
45.	−3.6567	−4.4185	1.7542	7384	1.0475
.50	−3 4666	−4.4054	1.7455	.7031	1.0933
.55	−3.2538	−4.3867	1.7339	6640	1.1489
.60	−3.0174	−4.3599	1.7183	.6212	1.2168
.65	−2.7563	−4.3217	1.6968	.5747	1.3007
70.	−2.4690	−4.2668	1.6665	.5246	1.4061
75.	−2.1538	−4.1865	1.6225	4707	1.5417
.80	−1.8082	−4.0660	1.5554	4134	1.7221
e_J	−1.7155	−4.0262	1.5328	.3983	1.7778
.82	−1.6609	−4.0010	1.5184	.3894	1.8123
.84	−1.5080	−3.9226	1.4727	.3650	1.9164
.85	−1.4294	−3.8771	1.4457	.3525	1.9749
.86	−1.3493	−3.8267	1.4153	.3400	2.0384
.87	−1.2676	−3.7706	1.3808	.3274	2.1076
.88	−1.1844	−3 7078	1.3414	.3147	2.1836
.89	−1.0996	−3 6372	1.2960	.3019	2.2674
.90	−1.0131	−3.5572	1.2431	.2891	2.3606
0.91	−0.9249	−3.4660	1.1805	0.2764	2.4651

e	$a_1+S_2u_1^{(2)}$	$a_3+S_2u_3^{(2)}$	P_{13}	P_{11}	P_{33}
0.92	−0.8350	−3.3610	+ 1.1055	+ 0.2639	+ 2.5834
e_{max}	−0.7438	−3.2392	+ 1.0136	+ 0.2520	+ 2.7187
.94	−0.6501	−3.0940	+ 0.8959	+ 0.2409	+ 2.8784
.95	−0.5555	−2.9197	+ 0.7391	+ 0.2320	+ 3.0699
.96	−0.4602	−2.7033	+ 0.5095	+ 0.2284	+ 3.3115
.97	−0.3675	−2.4202	+ 0.0990	+ 0.2420	+ 3.6503
.975	−0.3267	−2.2347	− 0.3141	+ 0.2717	+ 3.9088
.980	−0.3062	−1.9776	− 1.3536	+ 0.3770	+ 4.3949
.981	−0.3110	−1.9035	− 1.8272	+ 0.4308	+ 4.5836
.982	−0.3255	−1.8084	− 2.5799	+ 0.5194	+ 4.8663
.983	−0.3624	−1.6664	− 3.9863	+ 0.6895	+ 5.3682
.984	−0.4768	−1.3671	− 7 6444	+ 1.1409	+ 6.6220
.985	−1.7008	+1.1538	−43 49	+ 5.615	+18.61
e^*	± ∞	± ∞	± ∞	± ∞	± ∞
.986	+0.2637	−2.6999	+13 15	− 1.4710	− 0.2365
.987	+0.0261	−2.1464	+ 5.9150	− 0.5731	+ 2.2148
.988	−0.0324	−1.9478	+ 3.8692	− 0.3243	+ 2.9379
.989	−0.0542	−1.8189	+ 2.8866	− 0.2086	+ 3.3088
.990	−0.0622	−1.7133	+ 2.2976	− 0.1426	+ 3.5510
.995	−0.0440	−1.2094	+ 0.9865	− 0.0242	+ 4.2732
0.999	−0.0096	−0.5547	+ 0.3366	− 0.0015	+ 4.8907

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1967ApJ...147..334C

By virtue of the deformation, the gravitational potential, U_{Mc}, of the Maclaurin spheroid will change by the amount

$$\frac{\delta U}{\pi G\rho}=\frac{S_2}{a_1^2}\left[u_0^{(2)}+u_1^{(2)}\varpi^2+u_3^{(2)}z^2+u_{11}^{(2)}\varpi^4+u_{13}^{(2)}\varpi^2z^2+u_{33}^{(2)}z^4\right], \tag{111}$$

where $u_0^{(2)}$, $u_1^{(2)}$, etc., are the same quantities that are defined in equations (67). The angular velocity of rotation will also change; and its amount will be determined by

$$\delta\Omega^2=\delta\left(\frac{\mathfrak{W}_{33}-\mathfrak{W}_{11}}{I_{11}}\right)=-\frac{1}{I_{11}}\left[(\delta\mathfrak{W}_{11}-\delta\mathfrak{W}_{33})+\Omega_{Mc}^2\delta I_{11}\right]; \tag{112}$$

and by equations (51) and (54) (modified appropriately for the present circumstances) we can write

$$\frac{\delta\Omega^2}{\pi G\rho}=-4(a_1^2A_{11}-2a_3^2A_{13})\frac{V_{11}}{I_{11}}=-\tfrac{32}{7}S_2(a_1^2A_{11}-2a_3^2A_{13}). \tag{113}$$

Now if the deformed spheroid should continue to be in equilibrium, as we have assumed, then the pressure distribution in it will be given by

$$\frac{p}{\rho}=\pi G\rho a_3^2A_3\left(1-\frac{\varpi^2}{a_1^2}-\frac{z^2}{a_3^2}\right)+\tfrac{1}{2}\delta\Omega^2\varpi^2+\delta U+\text{constant}, \tag{114}$$

or, in view of equations (111) and (113),

$$\frac{p}{\pi G\rho^2}=a_3^2A_3\left(1-\frac{\varpi^2}{a_1^2}-\frac{z^2}{a_3^2}\right)+\frac{S_2}{a_1^2}\{[u_1^{(2)}-\tfrac{16}{7}a_1^2(a_1^2A_{11}-2a_3^2A_{13})]\varpi^2$$
$$+u_3^{(2)}z^2+u_{11}^{(2)}\varpi^4+u_{13}^{(2)}\varpi^2z^2+u_{33}^{(2)}z^4+\text{constant}\}, \tag{115}$$

where the additive constant is now an infinitesimal one. And we must require that the pressure given by equation (115) vanishes on the deformed surface, namely,

$$S(x)=\frac{\varpi^2}{a_1^2}+\frac{z^2}{a_3^2}-1-\frac{2S_2}{a_1^2}\left(\frac{\varpi^4}{a_1^2}-\frac{4\varpi^2z^2}{a_3^2}\right)=0. \tag{116}$$

On $S(x)$ the expression for p/ρ takes the form

$$\left(\frac{p}{\rho}\right)_S=\pi G\rho\,\frac{S_2}{a_1^2}(Q_1\varpi^2+Q_3z^2+Q_{11}\varpi^4+Q_{13}\varpi^2z^2+Q_{33}z^4+\text{constant}), \tag{117}$$

where

$$Q_1=u_1^{(2)}-\tfrac{16}{7}a_1^2(a_1^2A_{11}-2a_3^2A_{13}),\qquad Q_3=u_3^{(2)},$$
$$Q_{11}=u_{11}^{(2)}-2\frac{a_3^2}{a_1^2}A_3,\qquad Q_{13}=u_{13}^{(2)}+8A_3,\qquad\text{and}\qquad Q_{33}=u_{33}^{(2)}. \tag{118}$$

It is clearly sufficient that $(p/\rho)_S$ given by equation (117) vanishes on the original spheroid S_{Mc}; and the conditions for achieving it lead to the same equations (89), as formerly. With the present definitions of the Q's, the conditions are, explicitly,

$$a_1^4u_{11}^{(2)}+a_3^4u_{33}^{(2)}-a_1^2a_3^2u_{13}^{(2)}-10a_1^2a_3^2A_3=0 \tag{119}$$

and

$$a_1^4u_{11}^{(2)}-a_3^4u_{33}^{(2)}+a_1^2u_1^{(2)}-a_3^2u_3^{(2)}-2a_1^2a_3^2A_3$$
$$-\tfrac{16}{7}a_1^4(a_1^2A_{11}-2a_3^2A_{13})=0. \tag{120}$$

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1967ApJ...147..334C

We have already seen (cf. eq. [97]) that equation (120) is $-\frac{1}{7}$ of equation (119). It follows then that *the necessary and sufficient condition, that the Maclaurin spheroid allows an infinitesimal neutral deformation by a displacement of the form (110), is that equation (119) is satisfied.*

It is now apparent why the solution for the post-Newtonian configuration diverges at e^*: *the Newtonian instability of the Maclaurin spheroid at e^* is excited by the post-Newtonian effects of general relativity.*

X. CONCLUDING REMARKS

The completion of the solution of the post-Newtonian equations for the case of uniformly rotating homogeneous bodies with axial symmetry suggests the extension of the solution to include triaxial configurations which will arise from the relativistic deformation of the Jacobian ellipsoids. This extension of the present analysis will be considered in a separate paper.

I should like to record my indebtedness to Miss Donna Elbert for her assistance with the numerical work. I am also grateful to Dr. E. Krefetz for checking the analysis and to Dr. C. E. Rosenkilde for several clarifying discussions.

The research reported in this paper has in part been supported by the Office of Naval Research under contract Nonr-2121(24) with the University of Chicago.

APPENDIX I

In this Appendix we shall indicate how the relations (76) among the coefficients $u_i^{(n)}$ and $u_{ij}^{(n)}$ can be established The method is one of straightforward reduction in which systematic use is made of the following relations among the index symbols:

$$\begin{aligned} 8a_1^2A_{1111} &= 7A_{111} - a_3^2A_{1113}\,, & 6a_1^2A_{1113} &= 7A_{113} - 3a_3^2A_{1133}\,, \\ 7a_3^2A_{3333} &= 7A_{333} - 2a_1^2A_{1333}\,, & 4a_1^2A_{1133} &= 7A_{133} - 5a_3^2A_{1333}\,, \\ 6a_1^2A_{111} &= 5A_{11} - a_3^2A_{113}\,, & 4a_1^2A_{113} &= 5A_{13} - 3a_3^2A_{133} \\ 5a_3^2A_{333} &= 5A_{33} - 2a_1^2A_{133}\,, & 4a_1^2A_{11} &= 3A_1 - a_3^2A_{13}\,, \end{aligned} \tag{AI.1}$$

and

$$3a_3^2A_{33} = 3A_3 - 2a_1^2A_{13}\,.$$

The foregoing relations, appropriate for a spheroid ($a_1 = a_2$), are special cases of formulae valid for ellipsoids in general (cf. Chandrasekhar and Lebovitz 1962, eqs. [29], [40], [43], and [46]).

Substituting for the various u-coefficients their values listed in equations (66)–(68) and eliminating the B-symbols in favor of the A-symbols with the aid of the formula

$$B_{ijkl\ldots} = A_{jkl\ldots} - a_i^2A_{ijkl\ldots}\,, \tag{AI.2}$$

we find

$$a_3^2u_{11}^{(2)} + 3a_1^2u_{11}^{(3)} = \tfrac{1}{2}a_1^2a_3^2(24a_1^4A_{1111} - 3a_1^2a_3^2A_{1113} - 3a_3^4A_{1133} - 21a_1^2A_{111} + 7a_3^2A_{113})\,, \tag{AI.3}$$

$$a_3^2u_{13}^{(2)} + 3a_1^2u_{13}^{(3)} = a_1^2a_3^2(12a_1^4A_{1113} - 6a_1^2a_3^2A_{1133} - 15a_3^4A_{1333} + 21a_3^2A_{133} - 14a_1^2A_{113})\,, \tag{AI.4}$$

$$a_3^2u_{33}^{(2)} + 3a_1^2u_{33}^{(3)} = \tfrac{1}{2}a_1^2a_3^2(4a_1^4A_{1133} - 5a_1^2a_3^2A_{1333} - 35a_3^4A_{3333} + 35a_3^2A_{333} - 7a_1^2A_{133})\,, \tag{AI.5}$$

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1967ApJ...147..334C

$$a_3^2 u_1^{(2)} + 3a_1^2 u_1^{(3)} = a_1^2 a_3^2(-12a_1^4 A_{111} + 2a_1^2 a_3^2 A_{113} + 3a_3^4 A_{133} + 14a_1^2 A_{11} - 7a_3^2 A_{13}) , \tag{AI.6}$$

and

$$a_3^2 u_3^{(2)} + 3a_1^2 u_3^{(3)} = a_1^2 a_3^2(15a_3^4 A_{333} + 3a_1^2 a_3^2 A_{133} - 4a_1^4 A_{113} + 7a_1^2 A_{13} - 21a_3^2 A_{33}) . \tag{AI.7}$$

On reducing the right-hand sides of equations (AI.3), (AI.4), and (AI.5) systematically with the aid of the relations (AI 1), we find that they vanish identically. A similar reduction of the right-hand sides of equations (AI.6) and (AI.7) yields

$$2a_1^2 a_3^2(2a_1^2 A_{11} - a_3^2 A_{13}) = a_1^2 a_3^2 u_1^{(1)} , \tag{AI.8}$$

and

$$2a_1^2 a_3^2(a_1^2 A_{13} - 3a_3^2 A_{33}) = a_1^2 a_3^2 u_3^{(1)} , \tag{AI.9}$$

respectively. The relations to be established, therefore, follow.

APPENDIX II

In this Appendix we shall indicate how the relation (98) can be established.

By substituting for the various u-coefficients their values listed in equations (66) and (67) and eliminating the B-symbols in favor of the A-symbols, we find

$$8a_1^4 u_{11}^{(2)} - 6a_3^4 u_{33}^{(2)} - a_1^2 a_3^2 u_{13}^{(2)} = 96a_1^{10} A_{1111} - 60a_1^8 a_3^2 A_{1113} + 60a_1^2 a_3^8 A_{1333} + 12a_1^6 a_3^4 A_{1133} - 48a_1^8 A_{111} + 24a_1^6 a_3^2 A_{113} - 60a_1^2 a_3^6 A_{133} \tag{AII.1}$$

and

$$a_1^2 u_1^{(2)} - a_3^2 u_3^{(2)} = -12a_1^8 A_{111} + 12a_1^6 a_3^2 A_{113} - 12a_1^2 a_3^6 A_{133} + 8a_1^6 A_{11} - 8a_1^4 a_3^2 A_{13} + 12a_1^2 a_3^4 A_{13} . \tag{AII.2}$$

Making use of the foregoing reductions, we find

$$\begin{aligned} &8a_1^4 u_{11}^{(2)} - 6a_3^4 u_{33}^{(2)} - a_1^2 a_3^2 u_{13}^{(2)} + 7[a_1^2 u_1^{(2)} - a_3^2 u_3^{(2)}] \\ &\quad - 24a_1^2 a_3^2 A_3 - 16a_1^6 A_{11} + 32a_1^4 a_3^2 A_3 \\ &\quad = 96a_1^{10} A_{1111} - 60a_1^8 a_3^2 A_{1113} + 60a_1^2 a_3^8 A_{1333} + 12a_1^6 a_3^4 A_{1133} \\ &\quad - 132a_1^8 A_{111} + 108a_1^6 a_3^2 A_{113} - 144a_1^2 a_3^6 A_{133} \\ &\quad + 40a_1^6 A_{11} - 24a_1^4 a_3^2 A_{13} + 84a_1^2 a_3^4 A_{13} - 24a_1^2 a_3^2 A_3 . \end{aligned} \tag{AII.3}$$

On reducing the right-hand side of this last equation systematically with the aid of the relations listed in equation (AI.1), we find that it vanishes, as required.

REFERENCES

Chandrasekhar, S. 1964, *Lectures in Theoretical Physics* (Boulder: University of Colorado Press), **6**, 1.

———. 1965, *Ap. J.*, **142**, 1513 (Paper I).

Chandrasekhar, S., and Lebovitz, N. R. 1962, *Ap. J.*, **136**, 1037.

———. 1963, *ibid.*, **137**, 1172.

Krefetz, E. 1966, *Ap. J.*, **143**, 1004.

Routh, E. J. 1892, *A Treatise on Analytical Statics* (Cambridge: Cambridge University Press, 1922), **2**, 120–128.

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