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# ĀRYABHAṬA I'S ACHIEVEMENTS IN ANCIENT INDIAN MATHEMATICS & ASTRONOMY AND THEIR RELEVANCE IN THE TWENTY-FIRST CENTURY

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The period from fifth to twelfth century AD witnessed the rise of great scholars and is indeed a golden period in the history of Indian mathematics and astronomy. Āryabhaṭa I is pioneer in this group. He codified knowledge systematically in his *Āryabhaṭīya* for the first time into two major sections, *gaṇitapāda* (mathematics section including geometry) and *golapāda* (celestial sphere section dealing with astronomy). The elementary results are of course given by him in two other sections, *gitikāpāda* (elementary) data on astronomy and sine table) and *kālakriyāpāda* (section on reckoning time). The tradition of putting both *gaṇita* and *gola* sections in the same text continued more or less up to the time of Brahmagupta, and separate texts began to appear after that in each section and also on subsections. He wrote another work, *Āryabhaṭasiddhānta*, as envisaged from Bhāskara I's commentary. Both the works were popular up to twelfth century AD.

Āryabhaṭa I's rules are very short and cryptic. It could be understood only with Bhāskara I's commentary. My main objective in the paper is to give an outline of the major contributions of Āryabhaṭa I. Only one topic will be discussed in little more detail i.e. the solution of first degree indeterminate equation of the type: $by = ax + c$, and extensions made by other scholars in the field. This topic got a considerable importance, and has a far reaching impact. Lastly it will be emphasized that the mathematics looses its essence and vitality if it is bereft of context and is only restricted to internal logic and intellectual curiosity. The mathematics is getting unpopular day by day. To make it more meaningful, effort might be made to make it also an integral part of culture to avoid further damage from the domain of public perception in the twenty-first century.

## Introduction

The period between fifth to twelfth century is an extremely important period in the history of Indian mathematics and astronomy. The names of major authors and their works in the period will be of interest :

| *Authors* | *Works* |
|---|---|
| Āryabhaṭa I (b. 476 AD) | *Āryabhaṭīya*, and *Āryabhaṭasiddhānta* |
| Varāhamihira (d. 576 AD) | *Pañcasiddhāntikā*, *Bṛhatsaṃhitā* and others |
| Bhāskara I (c. 628 AD) | *Āryabhaṭīya-bhāṣya*, *Mahābhāskarīya* and *Laghubhāskarīya* |
| Brahmagupta (c. 628 AD) | *Brāhmasphuṭasiddhānta* and *Khaṇḍakhādyaka* |
| ........ | *Bākhshālī Ms.* |
| Lalla (748 AD) | *Śiṣyadhivṛddhidatantra* |
| Śrīdhara (c. 750 AD) | *Pāṭīgaṇita* |
| Mahāvīra (c. 850 AD) | *Gaṇitasārasaṃgraha* |
| Pṛthudakasvāmī (c. 860 AD) | *Brāhmasphuṭasiddhānta-bhāṣya* and *Khaṇḍakhādyaka-vivaraṇa* |
| ........ | *Sūryasiddhānta* (modern) |
| Vaṭeśvara (904 AD) | *Vatesvarasiddhanta* and *Gola* |
| Muñjāla (932 AD) | *Laghumānasa* |
| Āryabhaṭa II (950 AD) | *Mahāsiddhānta* |
| Śrīpati (1039 AD) | *Siddhāntaśekhara*, *Dhikoṭī*, *Gaṇitatilaka* and *Dhruvamānasa* |
| Bhāskara II (1150 AD) | *Siddhāntaśiromaṇi*, *Līlāvatī*, *Bījagaṇita*, *Karaṇakutūhala and others.* |

Āryabhaṭa I is pioneer in this group, and his work, *Āryabhaṭīya* is known as the ancient Svāyambhuva (which was revealed by Syāyambhu).[1] Bhāskara I, the earliest commentator of the *Āryabhaṭīya*, writes[2], "The learned people of Kusumpura (near Patna) held the *Svāyambhuva-siddhānta* in the highest esteem, even though the *Pauliśa*, the *Romaka*, the *Vaśiṣṭha* and the *Saura Siddhānta* were (known) there". These four texts along with *Paitāmahasiddhānta* have been redacted by Varāhamihira in his *Pañcasiddhāntikā*. The *Āryabhaṭīya* was a popular work and was studied throughout India. It was mentioned by Varāhamihira of Kapitthaka (near Ujjain) in the sixth century, by Bhāskara I of Valabhi (in Kathiawar) and Brahmagupta of Bhinmal (in Rajasthan) in the seventh century, and by Govindasvāmī of Kerala in the ninth century. Bhāskara II had also associations with the school of Ujjain as revealed from an inscription discovered by Bhaudaji. The *Āryabhaṭīya* measured day from one sunrise to the next where as his other work *Āryabhaṭa-*

*siddhānta* measured day from one midnight to the next. The astronomical parameters obviously differed because Āryabhaṭa I, as observed by a few scholars, wanted to improve them on the basis of his observations. The celebrated Brahmagupta who in his youth was a bitter critic of Āryabhaṭa I was so much impressed later by his popularity that he could not resist the temptation of bringing out an abridged edition based on Āryabhaṭa I's work under an attractive title, *Khaṇḍakhādyaka*, meaning 'food prepared with sugarcandy". The *Āryabhaṭīya* was even taken to Central Asia by the Abbasid Califs and translated it into Arabic by Abul Hasan Ahwazi under the title *Arajbahara* or *Arajbahaj*, as informed by Āl-Bīrūnī in the eleventh century.

Let us now look at Āryabhaṭa I's *Āryabhaṭīya*. Āryabhaṭa I's rules are very short and cryptic in style. The knowledge is codified systematically in this text for the first time into two major sections, *gaṇitapāda* (mathematics section including geometry) and *golapāda* (celestial sphere section dealing with astronomy). The elementary results of course are given by him in two other sections, *gītikāpāda* (elementary data on astronomy and sine table) and *kālakriyāpāda* (section on reckoning time). The tradition of putting both *gaṇita* and *gola* sections in the same text continued more or less up to the time of Brahmagupta, and separate texts began to appear after that in each section and also on subsections. Āryabhaṭa I introduced alphabets in his alphabetic numerical system to avoid confusion in regional variation of numerical signs, and in the process discovered the symbol of zero to fill up vacant places. Āryabhaṭa I's fundamental operations in arithmetic, like square, squaring, cube, cubing, square root and cube root, based on decimal place-value system, are unique in this direction. His formulae for the sum of $n$ natural numbers, squares of $n$ natural numbers, cubes of $n$ natural numbers and *saṅkalita* of different types are correct. His interest problem leads to the solution of a quadratic equation leading to a positive root. The rule of three, rule of inversion along with simplification of fraction, and even solution of indeterminate equations are great achievements at his time. Āryabhaṭa I gave correct formulae for area of right triangle, right pyramid, circle, trapezium, but failed to give a correct formula for volume of a right pyramid (correct one was only given by Brahmagupta), and sphere (which was only achieved by Bhāskara II after repeated failures by his predecessors). He of course gave the value of $\pi$ correct to four places of decimals, that to is approximate *(āsannamāna)* according to him. A number of hypothesis has already been made—how he achieved this result.

In astronomy, Āryabhaṭa I says, he does not believe in the theory of creation and annihilation of the world. For him, time is a continuous process without beginning and end (*anādi* and *ananta*). He might have borrowed the concept of Kalpa (1000 Yugas), and Yuga (4320000 years having four smaller subdivisions—*satya, tretā, dvāpara* and *kali* of 1080000 years each) and noted that they are not connected with any terrestrial affairs, but are purely based on astronomical phenomena depending on the position of the planets in the sky. Although the time divisions of Āryabhaṭa I and *Sūryasiddhānta* differ, they have been so adjusted that the beginning of current Kaliyuga falls on the same day, i.e. Friday, February 18, 3102 BC. He also accepted that all planets together with Moon's apogee together with Moon's node were in conjunction at the first point of the asterism Aśvinī ( $\zeta$ Piscium) [Gītikāpāda, vs. 4]. His concepts are based on geocentric circular orbits for planets, and gives the number of revolutions performed by the planets round the earth in a period of 4320000 years. He developed models for predicting true position of planets on the hypothesis that while the mean planet moves on the circular orbit round the earth, the true planets move in epicycles or in eccentrics. These types of epicyclic and eccentric models were also developed before by Greek astronomer Ptolemy. But the method used by Āryabhaṭa I for explaining the planetary motions are quite different, as reported by Bina Chatterjee. Further it may be pointed out that the epicycles of Ptolemy were of fixed dimension, where as those of Āryabhaṭa I vary in size from place to place. For correct prediction he developed for the first time a trigonometrical concept of *jyā* $\theta$ (=R Sine $\theta$), *kojyā* $\theta$ (=R Cosine $\theta$), and *utkramajyā* $\theta$ (=R – R Cosine $\theta$) for a circle of radius R = 3438 minutes. His *jyā* and *kojyā* tables drawn at an interval of $3°45'$ in a quarter circle were correct to four decimal places and his *jyā* table was based on R Sine differences. This table inspired later scholars to such an extent that Mādhava, a Kerala astronomer developed Sine and Co-sine series to fix better values for *jyā* table with better values of R. He even suggested four corrections for superior planets (Mars, Jupiter and Saturn) and a few others for inferior planets (Mercury and Venus). Āryabhaṭa I has given a detailed parameters for calculating lunar eclipse like angular diameters of Sun, Moon while in conjunction and opposition, horizontal parallax of Sun and Moon, duration of the eclipse which have given great fillip to later scholars.

So long I have given just an outline of the major contributions made by Āryabhaṭa I. Now I wish to discuss his rule for the solution of first degree indeterminate equation of the type by = ax + c, and the extension made by other scholars in little detail.

**Solution of Indeterminate Equations and their extensions by other scholars :**

**Āryabhaṭa I :** He posed a problem as how to get a number N which when divided by a number a leaves a remainder $r_1$, and when divided by a number b leaves a remainder $r_2$. The equation may obviously written as : $N = ax + r_1 = by + r_2$ .....(1). The technical terms for $r_1$, $r_2$ = remainders *(agra, śeṣa* etc), *a, b* = divisor *(bhāgahāra, bhājaka, cheda* etc). If $r_1 > r_2$, then a is known as, *adhikāgra bhāgahāra* (i.e. *bhāga* is taken as greater when remainder is greater), likewise *b* as *unāgrabhāgahāra;* the difference $r_1 - r_2 = c$, say, is known as *kṣepa.* The problem then reduces to the solution of :

i) $by = ax + c$, $r_1 > r_2$ and

ii) $ax = by + c$ where $r_2 > r_1$

When *x* and *y* are known, *N* can be determined from (1).

Āryabhaṭa I's rule runs as follows[3] :

*adhikāgra bhāgahāraṃ chindyāt ūnāgra bhāgahārena /*
*śeṣa paraspara bhaktaṃ matiguṇam agrāntare kṣiptam //*
*adha upari guṇitam antyā yug ūnagraccheda bhājite śeṣam /*
*adhikāgracchedaguṇaṃ dvicchedāgram adhikāgrayutaṃ //*

The other technical terms are : *matiguṇam* meaning multiplied by a desired quantity; the word *kṣiptam* has been associated with *prakṣipya* by Bhāskara I meaning *viśodhya*[4], i.e. the *kṣepa* quantity is to be subtracted when the number of partial quotients are even, and added when the number of partial quotients are odd, as followed by Bhāskara I and interpreted by Datta. Bhāskara I admits that there was an alternative view maintained by other groups *(sampradāyaviccheda)* that it has to be added when the number is even and subtracted when the number is odd *(sameṣu kṣiptam, viṣamesu śodhyaṃ).* It means the same thing if we take the number of partial remainders in the later case, since number of partial remainder is always one less than number of partial quotients. However, we have followed Bhāskara I's interpretation, and as followed by Datta[5], which may be explained as follows :

Eng. Translation :

"The divisor corresponding to the greater remainder *(adhikāgra bhāgaharaṃ)* is to be mutually divided by the divisor corresponding to the smaller remainder (as in H.C.F. division process), then the remainder (after some steps) is to be multiplied by a desired integer *(mati)* such that the product being increased (if the number of partial quotients in the division process is odd) and diminished (if the number of partial quotients in the division process is even) by the difference of remainders, be exactly divisible by the last remainder. The chosen number *(mati)* is then placed below the quotients in a column, and then the new quotient underneath. The *mati* is then multiplied by the number lying above it and the product is added to the number below it, and the process is repeated (until only two numbers remain). The (second) last number is then divided by the divisor of the smaller remainder, and the residue is multiplied by the divisor corresponding to the greater remainder in which the greater remainder is added. The result will be the number corresponding to the divisors."

Let us solve an example : $29y = 45x + 7$.

Here $a = 45$, $b = 29$ and here $a > b$, and $c = 7$.

As per rule, 45 is divided by 29 as is done in H.C.F. Two cases are to be considered when the no. of partial quotients in the division process are even or odd.

Case I : When the no. of partial quotions = even

```
29 ) 45 ( 1
     29
     16 ) 29 ( 1
          16
          13 ) 16 ( 1
               13
                3 ) 13 ( 4
                    12
                     1
```

In modern notation, $a/b = 45/29 = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{4+}\frac{1}{3}$

where convergents $p_1 / q_1 = 1$, $p_2 / q_2 = 2$, $p_3 / q_3 = 3/2$

$p_4 / q_4 = 14/9$, $p_5 / q_5 = 45/29$

Here the no. of partial quotients is even, the *kṣepa* quantity $c = 7$ will be negative. Obviously $[1 \times 10\ (mati) - 7] \div 3$ (previous quotient) = 1 (final quotient = $q$, say).

Then the chain *(valli)* is as follows :

```
1    1    1    1    143 (= u)
1    1    1    92    92 (= v)
1    1    51   51 (i.e. 41 × 1 + 10 and so on)
4    41   41 (i.e. 10 x 4 +1)
10 (mati) 10
1
```

then, $u = 143 = 45 \times 3 + 8 = 8 \pmod{45}$,

and $v = 92 = 29 \times 3 + 5 = 5 \pmod{29}$.

Here $y_1 = u = 8$, $x_1 = v = 5$

Then $y = y_1$, $x = x_1$ is the solution of $29y = 45x + 7$.
N obviously comes out to be $N = ax + c = 45 \times 5 + 7 = 232$, which could verified by the other value.

Case II : When the no. of partial quotients = odd

Let us consider the partial quotients in the above division be stopped after quotients, 1, 1, 1, and the remainder is 3, then $[3 \times 2\ (mati) + 7] \div 13$ (previous quotient) = 1.

When the chain is completed with 1, 1, 1, 2 *(mati)*, 1 and the operation is completed as before, then $y_1 = 8$, $x_1 = 5$; the same solution as before. The solution may be grouped in the following table as follows:

| *Valli* | *Kṣepa* | values of $y$ | values of $x$ |
|---|---|---|---|
| Even | negative | $y_1$ | $x_1$ |
| Odd | positive | $y_1$ | $x_1$ |

For Case I, if the rational of u and v are analyzed, it will be found that
$u/v = (q\,a + c\,p_4) / (q\,b + c\,q_4)$, where $q = 1$, $p_4 = 14$, $q_4 = 9$ and $c = 7$.
For Case II, $u/v = (q\,a - c\,p_3) / (q\,b - c\,q_3)$, where $q = 1$, $p_3 = 3$, $q_3 = 2$, and $c = 7$. The plus or minus sign is applicable according as the number of partial quotients is even or odd. So it is a short cut method for finding the $(n\text{-}1)$th convergent of a fraction.
Here the *valli* operation takes into account straightway of the remainder $c$ for solution of $by = ax + c$. Āryabhaṭa has also given the solution of the equation : $by = ax - c$ by transposing in the form : $ax = by + c$. In that case last two numbers will be x and y. Further, Āryabhaṭa I's method is unique in the sense that by selecting *mati* at any stage, one can shorten the steps of *valli* at his own will.

**Bhāskara I :** Bhāskara I for the first time pointed out[6] that a and b should be prime *(dṛḍa)* to each other, being reduced by their (non-zero) common factor, and has given exactly the same method as that of Āryabhaṭa I. He has used the technical words: *mati* (desired number), *valli* (chain operation, not used by Āryabhaṭa I), *yuktam* (addition), *śuddhaṃ* (subtraction), *guṇa* (for x) and labdhi (for y). He has also given an alternative rule for the solution of indeterminate equation which has wide astronomical applications.

In fact the mean longitude of planets(y) in astronomical siddhāntas is obtained when *ahargaṇa* (with reference to an epoch) is multiplied by revolution number of planets and the product is divided by the no. of civil days. The revolution of planets is always not complete and remains a residue for a particular position. The equation in this case may be written as

$Y = (aX - c) / b$ ....(1)

where $Y$ = mean motion of planets, $X$ = *ahargaṇa* (no. of civil days), $a$ = revolution of planets, $b$ = no. of civil days, and $c$ = residue.

The equation could be written as

$y = (ax - 1) / b$ ....(2),

where $X = cx$, and $Y = cy$. If $x = \alpha$, $y = \beta$ is a solution of (2), then $x = c\alpha$, $y = c\beta$ will be solution of (1). Bhaskara I has given as many as 24 examples[7] to solve astronomical problem where he has used the equation(2). The solution is made as usual dividing $a$ by $b$ where the numbers a = revolution no. of sun = 4320000, and $b$ = no. of civil days = 1577917500, after reduction by the common factor 75, it becomes, $a = 576$, $b = 210389$. The fraction $a/b$ even then is so big, the solution of $ax - 1 = by$ is not easy. To avoid this, Bhāskara I has computed table for planets from which the values of x and y could be found easily from the table given below :

| | $a$ | $b$ | $x$ | $y$ |
|---|---|---|---|---|
| Sun | 576 | 210389 | 94602 | 259 |
| Moon | 78898 | 2155625 | 776837 | 28433 |
| Moon's Apogee | 488219 | 1577917500 | 718667879 | 222361 |
| Moon's As. Node | 116113 | 788958750 | 625606177 | 92072 |
| Mars | 191402 | 131493125 | 16101213 | 23437 |
| Mercury | 896851 | 78895875 | 23587276 | 268129 |
| Jupiter | 30352 | 131493125 | 76053038 | 17555 |
| Venus | 585199 | 131493125 | 70046049 | 311734 |

There are other tables for all planets where the values of a and b are given in further subdivisions in minutes, and their sixtieth parts from which the values of *ahargaṇa* and mean longitudes are calculated. The table gives instant solution for $x$ and $y$, from which $X = cx$ and $Y = cy$ may be calculated. The method became popular in the 7th century since it avoids tedious calculation. By the by it may be noted that the Chinese indeterminate analysis, called *t'ai-yen-shu* or *t'ai-yen-ch'iu-I-shu* (great extension method of finding unity) was materially developed by the Buddhist priest I-tsing in 727 AD and later on by Ch'in Chiu Shao in 1247 AD. I-tsing, a Sanskrit scholar came to India in 673 AD and have learnt the technique along with various other things. Mikami has pointed out that the Chinese interest in indeterminate analysis grew after their contact with Hindu culture.[8]

As regards solution of $by = ax + 1$, Bhāskara I says that if $x = \alpha$, $y = \beta$ be a solution of $by = ax - 1$, then $x = b - \alpha$, $y = a - \beta$ is a solution of $by = ax + 1$.

**Brahmagupta, Mahāvira :** Brahmagupta[9] has used the same method of Āryabhaṭa I. He has used clearly that the *kṣepa* quantity has to be added *(saṃyuktam)* when the no. of quotients are odd, and subtracted *(śuddhaṃ)* when the no. is even. He has also used three other technical words, *iṣṭaguṇa* for *matiguṇa*, *guṇa* (for *x*) and *labdhi* (for *y*). Mahāvira[10] likewise followed Āryabhaṭa I and has retained the word *mati*, but has used *kṣepa* as *dhanam* (positive) for odd *(ayuja)* no. of quotients, and *ṛṇaṃ* (negative) for even *(yuja)* no. of quotients. He has divided the last number by the divisor of greater remainder *(adhikāgrasya hāraus)* unlike Āryabhaṭa I & Brahmagupta, and the remainder is added with their corresponding divisor to get the value of N. So Mahāvira is more explicit.

**Āryabhaṭa II, Śripati and Bhāskara II :** Āryabhaṭa II[11] gave serial number to the steps for the solution of $by = ax \pm 1$. The novelty in his method is that he made a and b prime and then continued the mutual division till the remainder becomes 1, then he prepared the valli with quotients, then 1, then o one below another, and the operation was repeated as usual. Śripati[12] gave the same method as of Āryabhaṭa II.

Bhāskara II[13] followed Āryabhaṭa II for his solution of the problem $by = ax \pm c$ (where $\pm$ c = *kṣepa*), and continued the mutual division till the remainder becomes 1, and then prepared the *valli* with quotients along with *kṣepa* c and 0.. The values of *y* and *x* are obtained by him as per the following table :

| No. of quotients in the *valli* | *kṣepa* | *labdhi(y)* | *guṇa(x)* |
|---|---|---|---|
| even | positive | $y_1$ | $x_1$ |
| odd | positive | $a - y_1$ | $b - x_1$ |
| even | negative | $a - y_1$ | $b - x_1$ |
| odd | negative | $y_1$ | $x_1$ |

This gives the method of solution obtained by Siddhantic workers between fifth to twelfth century AD. It will not be out of place to mention a few words about successive convergents.

### Successive Convergents

Āryabhaṭa I gave a value of $\pi$ = circumference / diameter = 62832/20000 (=3927/1250) = 3.1416 which is correct to 4 places of decimals, he calls it as *āsannamāna* (approximate value).[14] How he obtained is not clear. But this fraction gives four convergents, 3,22/7, 355/113, 3927/1250 which have been used by later scholars.

Bhāskara I gives the formula for declination as under[15]

$$\text{R Sine } \delta = \frac{(1397 \times \text{R Sine } \lambda)}{3438}, \text{ where } \delta = \text{declination, } \lambda = \text{longitude.}$$

Now the convergents of the fraction

$$\frac{1397}{3438} = \frac{1}{2}+\frac{1}{2}+\frac{1}{5}+\frac{1}{1}+\frac{1}{9}+\frac{1}{1}+\frac{1}{9}$$
$$= 1/2,\ 2/5,\ 11/27,\ 13/32,\ 128/315...$$

In another place[16] Bhāskara I gives the same formula as,

$$\text{R Sine } \delta = \frac{(13 \times \text{R Sine } \lambda)}{32}$$

This shows that Bhāskara I knew that convergents are approximate values of the fraction and used the fourth convergent of 1397/3438 for finding value of the declination. Application of convergents are found to have been used by scholars from Bhāskara I onwards.

### Relevance in the twenty-first century

As regards relevance of mathematics it has to be admitted that mathematics has become a major cultural force in western civilization. Every one now admits that mathematics has become a practical tool of engineering design, carries main burden of scientific reasoning and is the core of the major theories of physical science. It has determined the direction and content of much philosophic thought, destroyed and rebuilt religious doctrines, and supplied basis to economic and political theories. It has even established mathematical logic to answer fundamental questions about nature of man and his universe. The achievements of mathematics in a way gives satisfaction and aesthetic values to mathematicians themselves at least equal to those offered by any other branch of our culture. Despite these achievements and their contributions to our life and thought, why the people is loosing interest or reject mathematics as a subject not of intellectual interest ? Layman likewise makes little use of technical mathematics and obviously has objected to the naked and dry material usually presented in the text books. Consequently, a subject that is basic, vital and elevating is neglected and even scorned otherwise by highly educated people.

It appears to me that just as a phrase either looses meaning or acquires an unintended meaning when removed from its context, so is mathematics detached from its rich intellectual setting in the culture of our civilization and is reduced to a series of techniques. After an unbroken tradition of many centuries, mathematics has ceased to be considered as an integral part of culture in our era of mass education. The isolation of research scientists, the pitiful scarcity of inspiring teachers, the host of dull and commercial text books and general educational trend away from intellectual discipline have contributed more to this problem. I personally feel that mathematics will loose its essence and vitality more and more if it is bereft of context and is only restricted to internal logic and intellectual curiosity. The mathematics is getting unpopular day by day. To make it more meaningful, effort might be made to make it also an integral part of culture to avoid further damage from the domain of public perception.

## Notes and References


1. *Āryabhaṭīya* of Āryabhaṭa, critically ed. with introduction, Eng. Tr., notes, comments and indexes by Kripa Shankar Shukla, in collaboration with K.V. Sarma, Indian National Science Academy, New Delhi, 1976, Vide Gola section, vs 50.
2. Vide *Āryabhaṭīya*, Ibid., Introduction by Kripa Shankar Shukla, p. xxvii.
3. *Āryabhaṭīya*, Gaṇita, vs. 32-33.
4. *Āryabhaṭīya* of Āryabhaṭa with commentary of Bhāskara I and Someśvara, ed. by Kripa Shankar Shukla with introduction and appendices, Indian National Science Academy, New Delhi, 1976, See commentary on *Gaṇita*, vs. 32-33.
5. Datta, B. "Elder Āryabhaṭa's rule for the Solution of Indeterminate Equations of the First Degree", *Bulletin of the Calcutta Mathematical Society*, 24, 19-36, 1932; See also Bag, A.K., *Mathematics in Ancient*

*and Medieval India*, pp. 194-209, Chaukhamba Orientalia, Varanasi, 1979.

6. *Mahābhāskarīya*, i. 41-44.
*hārabhājyau dṛdau syātām kuṭṭākāraṃ tayorviduḥ //*
*bhājyam nyasedupari hāramadhaśca*
*tasya khaṇḍyātparasparamadho vinidhāya labdham /*
*kenāa 'hato 'yamapanīya yathā 'sya śeṣam*
*bhāgam dadāti pariśuddhamiti pracintyam //*
*āptāṃ matiṃ tāṃ binidhāya vallyāṃ*
*nityaṃ hi adhodhaḥ kramaśaśca labdhaṃ //*
*matyāṃ hataṃ syāduparisthitaṃ yat*
*labdhena yuktaṃ parataśca tadvat //*
*hāreṇa bhājyovidhinoparistho*
*bhājyena nityaṃ tadadhaḥsthitaśca /*
*ahṇāmgaṇo 'smin bhāgaṇādayaśca*
*tadvā bhavedyasya samihitam yat //*
7. Reference 4, Appendix II, pp. 311-339.
8. Yoshio Mikami, *The Development of Mathematics in China and Japan*, Leipzig, 1913, p. 58; See also N.K. Majumder, "On Chinese indeterminate analysis, *Bulletin of the Calcutta Mathematical Society*, 5, 9-11.
9. *Brāhmasphuṭasiddhānta*, *Kuṭṭakādhyāya*, vs. 3-5.
*adhikāgrabhāgahārād ūnagracchedabhājitāt śeṣaṃ /*
*yat tat parasparahṛtam labdham adho 'dhah pṛthak sthāpyaṃ //*
*śeṣam tathā iṣṭaguṇitam yathā 'grayorantarena samyuktaṃ /*
*śudhyati guṇakaḥ sthāpyo labdhaṃ ca antyadupāntyaguṇaḥ //*
*svordhvo 'ntyayuto 'grānto hināgracched bhājitaḥ śesaṃ /*
*adhikāgracchedahatamadhikāgrayutaṃ bhavati agram //*
10. *Gaṇitasārasaṃgraha*, vi. 114 1/2.
*cchittvā chedena rāśim prathamaphalamapohyāptamanyonyabhaktaṃ*
*sthāpyordhvādharyato 'dha matiguṇamayujalpe 'vaśiṣte dhanaṇṇam /*
*cchittvādhaḥ svoparighnopariyutaharabhāgo 'dhikāgrasya hāraṃ*
*cchittvā chedena sāgrāntaraphalamadhikāgrānvitam hāraghātam //*
11. *Mahāsiddhānta*, xviii, 1-16.
12. *Siddhāntaśekhara*, xiv. 22-25.
*vibhājyahāraṃ ca yutiṃ nijacchidā*
*samena va 'adāvapavartya sambhave /*
*vibhājyahārau vibhajet parasparam*
*tathā yathā śeṣakameva rūpakam //*
*phalanyadho 'dhaḥ kramaśo niveśayen*
*matiṃ tathā 'dhastadadhaśca tatphalam /*
*idaṃ hataṃ kena yutaṃ vivarjitaṃ*
*harena bhaktaṃ sadaho niragrakam //*
*sameṣu labdhesu asameṣu ṛṇam dhanaṃ*
*dhanaṃ tvṛṇaṃ kṣepamuśanti tadvidaḥ /*
*matiṃ vicintyeti tadurdhagaṃ tayā*
*nihatya labdhaṃ ca tathā niyojayet //*

*punaḥ punaḥ karma yathotkramādidam*
*yadā tu rāśidvayameva jāyate /*
*harena bhataḥ prathamo guṇo bhavet*
*phalaṃ dvitīyaṃ tu vibhajyarāśinā //*

13. *Bijagaṇita.* vi. Kuṭṭakavivaraṇa.
14. *Āryabhaṭiya,* Gaṇita, vs. 10.
15. *Mahābhāskariya,* Vi. 6-7.
16. *Mahābhāskariya,* iv. 25.