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ASYMPTOTIC BEHAVIOUR UNDER ITERATED RANDOM LINEAR TRANSFORMATIONS

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Abstract. Let V be a finite-dimensional real vector space. We describe conditions for a sequence of the form $\{\mu^i * \nu\}$, where μ is a probability measure on $\mathrm{GL}(V)$ (μ^i denotes the i-th convolution power of μ) and ν is a finite positive measure on V, to converge in distribution (in the vague topology) to the zero measure on V. The conditions depend on μ only via the closed subgroup of $\mathrm{GL}(V)$ generated by the support of μ.

1. Introduction

Let V be a finite-dimensional real vector space (namely $\mathbb{R}^d$ for some d). It is well-known and easy to see that given a linear transformation T of V onto itself and a vector v in V, the trajectory $\{v, Tv, T^2v, \dots\}$ goes off to infinity unless v is contained in a T-invariant subspace W of V on which the restrictions of $\{T^i \mid i = 1, 2, \dots\}$ form a bounded set of linear endomorphisms. Similarly, if we start with an initial distribution ν on V, the sequence $\{\nu, T\nu, T^2\nu, \dots\}$ 'dissipates to infinity' unless $\nu(W) > 0$ for a subspace W as above. Here we ask the question as to what happens if we start with an initial distribution ν and apply iteratedly random linear transformations, rather than a fixed transformation T. In other words, we consider the behaviour of the sequence $\{X_i X_{i-1} \cdots X_1 \nu\}$, as $i \to \infty$, where $\{X_i\}$ is a sequence of independent $\mathrm{GL}(V)$-valued random variables and ν is an initial distribution (we shall mainly consider the case where $\{X_i\}$ are identically distributed, but some results are proved in somewhat greater generality). We show that under certain quite general conditions a behaviour similar to the deterministic case holds (see Corollary 1.4).

Let $M^1(\mathrm{GL}(V))$ denote the space of probability measures on $\mathrm{GL}(V)$. Let $M(\mathrm{End}(V))$ and $M(V)$ denote the spaces of finite (nonnegative real valued) measures on $\mathrm{End}(V)$ and V respectively, equipped with the usual vague topology. We note that dissipation of a sequence of measures on V corresponds to convergence to the zero measure in $M(V)$. We shall denote the zero measures in $M(V)$ and $M(\mathrm{End}(V))$ by 0. For $\mu \in M^1(\mathrm{GL}(V))$ we denote by $G(\mu)$ the smallest closed subgroup of $\mathrm{GL}(V)$ containing $\operatorname{supp}\mu$, the support of μ. It turns

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out that the conclusions depend on the subgroup $G(\mu)$, rather than the specific measure μ. We prove the following.

Theorem 1.1. *Let $\{\mu_i\}$ be a sequence in $M^1(\mathrm{GL}(V))$ such that $\mu_i \to 0$ in $M(\mathrm{End}(V))$. Let $\mu \in M^1(\mathrm{GL}(V))$ be such that $\mu_i * \mu = \mu * \mu_i$ for all i. Then for any $\nu \in M(V)$ at least one of the following conditions holds:*

i) $\mu_i * \nu \to 0$ in $M(V)$, as $i \to \infty$, or

ii) there exists a proper subspace W of V such that $\nu(W) > 0$ and W is invariant under the action of a subgroup of finite index in $G(\mu)$.

The theorem can be applied, in particular, to sequences of the form $\{\mu^{n_i}\}$, where $\{n_i\}$ is a sequence of natural numbers, and $\mu \in M^1(\mathrm{GL}(V))$ is such that $\mu^{n_i} \to 0$ in $M(\mathrm{End}(V))$. For measures supported on $\mathrm{SL}(V)$, the special linear group, it is known that $\mu^i \to 0$ in $M(\mathrm{End}(V))$ whenever $G(\mu)$ is noncompact (see Proposition 5.4). Together with Theorem 1.1 this implies the following.

Corollary 1.2. *Let $\mu \in M^1(\mathrm{SL}(V))$ be such that $G(\mu)$ is noncompact. Then $\mu^i * \nu \to 0$ in $M(V)$, for any $\nu \in M(V)$ such that $\nu(W) = 0$ for every proper subspace W which is invariant under a subgroup of finite index in $G(\mu)$. In particular, if $G(\mu)$ acts strongly irreducibly on V, then $\mu^i * \nu \to \nu(\{0\})\delta_0$, as $i \to \infty$, for all $\nu \in M(V)$.*

The second assertion in the Corollary can also be deduced from the classical theorem of Furstenberg ([3], Theorem 8.6) asserting that if $\{X_i\}$ is an i.i.d. sequence of $\mathrm{SL}(V)$-valued random variables with distribution $\mu \in M^1(\mathrm{SL}(V))$ such that $G(\mu)$ is noncompact and acts strongly irreducibly on V, then, in fact, for any $v \in V$, $v \neq 0$, $\|X_i X_{i-1} \cdots X_1 v\|$ grows exponentially to ∞. Though there has been considerable subsequent development along the lines initiated by Furstenberg (see, for instance, [5], [6] and [8], and other works cited there), it does not seem to apply to the general case as above. We note also that in the general case exponential growth is not to be expected for the norms of corresponding random sequences of vectors, as can be seen even from the deterministic case (e.g. μ a point measure supported on a unipotent transformation). It may also be noted that, in general, irreducibility of the $G(\mu)$-action is not adequate, in the place of strong irreducibility, to ensure the conclusion as in the second statement in the corollary (see Remark 5.5).

For measures μ not supported on $\mathrm{SL}(V)$, conditions on $G(\mu)$ do not determine whether $\mu^{n_i} \to 0$ in $M(\mathrm{End}(V))$ for some sequence $\{n_i\}$ in $\mathbb{N}$. Some of the theory as in [6] and [8] provides conditions on $\mu \in M^1(\mathrm{GL}(V))$ which can ensure that $\mu^i \to 0$ in $M(\mathrm{End}(V))$, and these may be used in applying Theorem 1.1 for measures μ not supported on $\mathrm{SL}(V)$. We shall however not go into the details of this here, especially as the conditions involved are relatively technical.

The first part of Corollary 1.2 reduces the task of studying the limit points of $\{\mu^i * \nu\}$ in $M(V)$, to ν such that $\nu(W) > 0$ for some proper subspace invariant under a subgroup of finite index in $G(\mu)$. For a class of measures μ we get a characterisation in terms of only invariant subspaces on which the $G(\mu)$-orbits are bounded.

Definition 1.3. *A subgroup H of* $\mathrm{GL}(V)$ *is said to be of type* $\mathcal{S}$ *if there exists a closed subgroup* $\tilde{H}$ *of* $\mathrm{GL}(V)$ *containing* H*, such that the following holds:*

i) if E is a subset of V which is H-invariant and can be expressed as a finite union of subspaces of V, then E is also $\tilde{H}$-invariant, and

ii) if S is the subset of $\tilde{H}$ consisting of all the elements x such that x is either unipotent or contained in a compact connected subgroup of $\tilde{H}$, then the subgroup generated by S is dense in $\tilde{H}$.

Clearly if H is a subgroup which is generated by the unipotent elements and compact connected subgroups contained in it, then it is of type $\mathcal{S}$. However, a subgroup H can be of type $\mathcal{S}$ even without this being satisfied. We note, in particular, that if H is such that the Zariski closure of H in $\mathrm{GL}(V)$ is a connected semisimple Lie group or, more generally, a connected Lie group whose solvable radical is nilpotent, then H is of type $\mathcal{S}$. We note also that a subgroup of type $\mathcal{S}$ is necessarily contained in $\mathrm{SL}(V)$.

Corollary 1.4. *Let $\mu \in M^1(\mathrm{SL}(V))$ be such that $G(\mu)$ is of type $\mathcal{S}$. Let B be the subspace of V consisting of all the elements v such that the $G(\mu)$-orbit of v is bounded in V. Let $\nu \in M(V)$. Then $\mu^i * \nu \to 0$ in $M(V)$ if and only if $\nu(B) = 0$.*

In the case of measures $\mu \in M^1(\mathrm{GL}(V))$ whose support contains the identity element our proof of Theorem 1.1 is essentially self-contained. For the general case however we need a result (Proposition 5.1) whose validity we are at present able to establish in full generality only via the Szemeredi theorem (see also Remark 5.2 for an intermediate case).

Finally we may also mention that while for convenience of exposition we have restricted to vector spaces over the field of real numbers, the proofs show that analogous results hold also for vector spaces over p-adic fields.

2. Preliminaries

For a locally compact space X we denote by $M(X)$ the space of all finite non-negative real-valued measures on X, equipped with the usual vague topology. We denote by $M^1(X)$ the subset of $M(X)$ consisting of all probability measures. For $x \in X$ we denote by δ_x the unit point mass on X supported at the point x.

Let G be a locally compact second countable topological group and $\mu \in M^1(G)$. Recall that $G(\mu)$ denotes the smallest closed subgroup of G containing $\operatorname{supp}\mu$. There exists a unique smallest closed normal subgroup H of $G(\mu)$ such that $\operatorname{supp}\mu$ is contained in a single coset of H; this subgroup will be denoted by $N(\mu)$. For $\mu \in M^1(G)$ we denote by $\tilde{\mu}$ the measure defined by $\tilde{\mu}(E) = \mu(\{g^{-1} \mid g \in E\})$, for all Borel subsets E of G. It can be seen that $G(\mu * \tilde{\mu})$ and $G(\tilde{\mu} * \mu)$ are contained in $N(\mu)$, and that they generate a dense subgroup of $N(\mu)$.

We denote by V a finite-dimensional (real) vector space and by $\mathrm{End}(V)$ the algebra of linear endomorphisms of V, equipped with the usual topology. We recall that for $\mu \in M^1(\mathrm{End}(V))$ and $\nu \in M(V)$, the convolution product $\mu * \nu$

is the measure on V defined by the condition $\int f \mathrm{d}(\mu * \nu) = \int f(gx)\mathrm{d}\mu(g)\mathrm{d}\nu(x)$, for all continuous functions f on V, with compact support.

Definition: A measure $\nu \in M(V)$ is said to be *r-pure* if there exists a sequence $\{V_i\}$ of r-dimensional subspaces of V such that $\nu(V - \cup_1^\infty V_i) = 0$ and $\nu(W) = 0$ for all subspaces W of dimensional less than r; ν is said to be *pure* if it is r-pure for some r.

Any measure $\nu \in M(V)$ can be decomposed uniquely as $\nu = \Sigma_{r=0}^d \nu_r$, where d is the dimension of V and ν_r is a r-pure measure for each r.

Lemma 2.1. *Let $\nu \in M(V)$. Let $a > 0$ and $\{W_i\}$ be a sequence of subspaces of V such that $\nu(W_i) \leq a$ and $\nu(W_i) \to a$. Then $\nu(W_i) = a$ for all large i. In particular, there exists a proper subspace W_0 such that $\nu(W) \leq \nu(W_0)$ for every proper subspace W of V.*

Proof: Suppose that the first assertion of the lemma does not hold. Then there exists an increasing sequence $\{k_i\}$ of natural numbers such that $\nu(W_{k_i}) < a$ and $\nu(W_{k_i})$ increases monotonically to a. Passing to such a subsequence and modifying notation we may assume that $\{\nu(W_i)\}$ increases monotonically to a.

Now, let $\nu = \Sigma_{r=0}^d \nu_r$ be the decomposition of ν into pure measures, ν_r being r-pure for each r. Passing to a subsequence we may assume that for each r, $\{\nu_r(W_i)\}$ is convergent, say $\nu_r(W_i) \to a_r$. Then $a = \Sigma_{r=0}^d a_r$. Since $\nu(W_i) < a$ for all i there exists $r \geq 1$ such that $\nu_r(W_i) < a_r$ for infinitely many i. This shows that without loss of generality we may assume that ν is an r-pure measure for some $r = 1, \ldots, d$.

Let $\{V_j\}$ be the sequence of r-dimensional subspaces such that $\nu(V_j) > 0$ for all j, and $\nu(V - \cup V_j) = 0$. We now define inductively a sequence $\{N_j\}$ of subsets of $\mathbb{N}$ as follows. Let $N_1 = \mathbb{N}$. For $j \geq 2$, having chosen $N_1, \ldots N_{j-1}$, we choose N_j to be an infinite subset of N_{j-1} as follows: if V_j is contained in W_n for infinitely many n in N_{j-1} we choose N_j to be the subset consisting of n for which this holds; if V_j is contained in W_n only for finitely many n in N_{j-1} then we choose N_j to be the subset consisting n in N_{j-1} for which V_j is not contained in W_n.

Now for each i, let n_i be the least element of N_i, and $W_i' = W_{n_i}$. Then we see that for every j either V_j is contained in W_i' for all large i, or $\nu(V_j \cap W_i') = 0$ for all large i. Let J be the subset of $\mathbb{N}$ consisting of all j such that V_j is contained W_i' for all large i. Let V' be the subspace of V spanned by $\cup_{j\in J} V_j$. Then there exists a finite subset F of J such that V' is also spanned by $\cup_{j \in F} V_j$. It follows therefore that V' is contained in W_i', and in particular $\nu(W_i') \geq \nu(V')$, for all large i, say $i \geq i_0$. Now suppose, if possible, that there exists a $p \geq i_0$ such that $\nu(W_p') > \nu(V')$. Then there exists $q \in \mathbb{N}$, $q \notin J$, such that V_q is contained in W_p', and $\nu(W_p') \geq \nu(V') + \nu(V_q)$. There exists a finite subset, say F' of $\mathbb{N}$, complementary to J, such that $\nu(\cup_{j \notin J \cup F'} V_j) < \nu(V_q)$. The choice of J shows that $\nu(W_i' \cap V_j) = 0$ for all large i, say $i \geq i_1$, and all $j \in F'$. Hence, for $i \geq i_1$,

$$\nu(W_i') \leq \nu(V') + \nu(\cup_{j \notin J \cup F'} V_j) < \nu(V') + \nu(V_q) \leq \nu(W_p'),$$

which is a contradiction since $\{\nu(W_i')\}$ is nondecreasing. This shows that $\nu(W_i') = \nu(V')$, for all large i. But then $a = \nu(V')$ and we have $\nu(W_{n_i}) = \nu(W_i') = a$ for all large i. This proves the lemma.

Proposition 2.2. *Let $\nu \in M(V)$. Let $\{g_i\}$ be a divergent sequence in* $\mathrm{End}(V)$ *(viz. $\{g_i\}$ has no convergent subsequence), and let W be the subspace of V consisting of all points v in V such that $\{g_i v\}$ is bounded in V. If $\lambda \in M(V)$ is a limit point of $\{g_i\nu\}$ then $\nu(W) \geq \lambda(V)$. In particular, if there exist a compact subset K of V and $a > 0$ such that $(g_i\nu)(K) \geq a$ for all i, then $\nu(W) \geq a$.*

Proof: Let $\tilde{V}$ be the one-point compactification of V, and let ∞ denote the point at infinity. We view $g_i\nu$, $i \in \mathbb{N}$, as measures on $\tilde{V}$. Let $\tilde{\lambda}$ be the measure on $\tilde{V}$ such that $\tilde{\lambda}(V) = \lambda(V)$ and $\tilde{\lambda}(\{\infty\}) = \nu(V) - \lambda(V)$. Then the condition in the hypothesis implies that $\tilde{\lambda}$ is a limit point of the sequence $\{g_i\nu\}$ in $M(\tilde{V})$. Passing to a subsequence we may assume that $\{g_i\}$ converges pointwise on $\tilde{V}$, say $g_i \to \gamma$, where $\gamma : \tilde{V} \to \tilde{V}$ is a Borel map. Then we have $\gamma(\nu) = \tilde{\lambda}$. Since $\gamma(v) = \infty$ for all $v \notin W$, it follows that $\lambda(V) = \tilde{\lambda}(V) = \nu(\gamma^{-1}(V)) \leq \nu(W)$. This proves the first statement. The second may be proved by considering a limit point of $\{g_i\nu\}$ viewed as a sequence of measures on $\tilde{V}$.

Proposition 2.3. *Let $\{\mu_k\}$ be a sequence of probability measures on* $\mathrm{End}(V)$ *such that $\mu_k \to 0$ in $M^1(\mathrm{End}(V))$. Let $\nu \in M^1(V)$ be such that $\mu_k * \nu \to \lambda$ in $M(V)$. Then there exists a proper subspace W of V such that $\nu(W) \geq \lambda(V)$.*

Proof: Clearly we may suppose that $\lambda(V) > 0$. Let $s \in (0, \lambda(V))$ and $t \in (s, \lambda(V))$ be arbitrary. Let B be a ball in V centered at 0, such that $\lambda(B) > t$. Then $(\mu_k * \nu)(B) > t$ for all large k, say all $k \geq k_0$. Let $A = \{x \in \mathrm{End}(V) \mid \nu(x^{-1}(B)) > s\}$. Then for $k \geq k_0$ we have $t < (\mu_k * \nu)(B) = \int \nu(x^{-1}B)\mathrm{d}\mu_k(x) \leq \mu_k(A) + s\mu_k(A^c) = (1-s)\mu_k(A) + s$, and hence $\mu_k(A) > (t-s)/(1-s) > 0$.

Let $\{K_r\}$ be a sequence of compact subsets of $\mathrm{End}(V)$ such that every compact subset of $\mathrm{End}(V)$ is contained in K_r for some r. For each r, we have $\mu_k(K_r) \to 0$ as $k \to \infty$. Therefore for all large k, $\mu_k(K_r^c \cap A) \neq 0$ and hence $K_r^c \cap A$ is nonempty. Choose $x_r \in K_r^c \cap A$ for each r. Then $\{x_r\}$ is a divergent sequence in $\mathrm{End}(V)$ and $x_r\nu(B) > s$. By Proposition 2.2 this implies that there exists a proper subspace W_s of V such that $\nu(W_s) \geq s$. Since this holds for any $s \in (0, \lambda(V))$, by Lemma 2.1 there exists a proper subspace W such that $\nu(W) \geq \lambda(V)$. This proves the proposition.

3. Measures on linear groups and invariance of subspaces

In this section we prove the following.

Theorem 3.1. *Let G be a Lie group and $\rho : G \to \mathrm{GL}(V)$ be a representation over a finite-dimensional $\mathbb{R}$-vector space V. Let $\mu \in M^1(G)$, and $\{H_i\}$ be a sequence of closed subgroups of G such that $\rho(H_i)$ has only finitely many connected components for each i, and $\mu(H_i) \to 1$ as $i \to \infty$. Let $l \geq 1$, and for each i let E_i be a $\rho(H_i)$-invariant subset of V which is a union of at most l proper (vector) subspaces of V. Then E_i is $\rho(G(\mu))$-invariant for all large i.*

Proof. To begin with we note that it suffices to show that E_i is $\rho(G(\mu))$-invariant for infinitely many i (since such a result can then be applied to subsequences of the given sequence). Therefore there is no loss of generality in passing to subsequences of $\{E_i\}$, which we shall do in the sequel wherever convenient.

Clearly, for each i and any r, the subset $E_i^{(r)}$ of E_i defined to be the union of all maximal subspaces of E_i with dimension r is $\rho(H_i)$-invariant. It suffices to prove the theorem for the sequences of sets $\{E_i^{(r)}\}$ for each r, in the place of $\{E_i\}$, and hence, passing to a subsequence, we may assume that all maximal subspaces of E_i are of the same dimension $r \geq 1$.

Now let $\wedge^r V$ denote the r-th exterior power of V and let $\wedge^r \rho : G \to \mathrm{GL}(\wedge^r V)$ be the rth exterior power of ρ. For each i let E_i' be the subset of $\wedge^r V$ consisting of (at most l) lines L, of the form $\wedge^r U$ where U is an r-dimensional subspace of E_i. Then each E_i' is $\wedge^r \rho(H_i)$-invariant, and to prove the theorem it suffices to show that E_i' is $\wedge^r \rho(G(\mu))$-invariant for all large i. This shows that it suffices to prove the theorem in the case when $r = 1$. Furthermore, by an argument as above we may also assume that E_i does not contain any proper $\rho(H_i)$-invariant subset which is a union of lines, for any i.

Now suppose first that all H_i are finite subgroups. In this case μ is an atomic measure. Also, in view of the above reduction, for each i, E_i consists of points whose $\rho(H_i)$-orbit has at most $2l$ points (note that in each of the lines in E_i the orbit can have at most 2 points, namely a vector and its negative). For each $k \geq 1$ let B_k be the subgroup of G generated by $\{g \in G \mid \mu(\{g\}) \geq \frac{1}{k}\}$. As $\mu(H_i) \to 1$ it follows that each B_k is contained in H_i for all large i; specifically this holds for all i such that $\mu(H_i) > 1 - \frac{1}{k}$. In particular each B_k is a finite subgroup. We note also that $\{B_k\}$ is an increasing sequence of subgroups and $\cup_k B_k$ is dense in $G(\mu)$. For each $k \geq 1$ let P_k be the subset of V consisting of all points whose $\rho(B_k)$-orbit has at most $2l$ points. Then each P_k is a finite union of subspaces, and since $\{B_k\}$ is an increasing sequence of subgroups, $\{P_k\}$ is a decreasing sequence of sets. By considerations of dimension and the number of maximal subspaces we see that there exists $m \geq 1$ such that $P_m = \cap_k P_k$. Then for $v \in P_m$, for each k the $\rho(B_k)$-orbit of v has at most $2l$ points, and as $\{B_k\}$ is an increasing sequence of subgroups whose union is dense in $G(\mu)$ this implies that the $\rho(G(\mu))$-orbit of v has at most $2l$ points. Since this holds for every v in P_m it follows that there is a closed subgroup of finite index in $G(\mu)$, say G', such that $\rho(G')$ fixes every point of P_m. Then there exists $n \geq m$ such that $G(\mu) = B_n G'$. In turn this implies that every $\rho(G(\mu))$-orbit in $P_n = P_m$ is an orbit of $\rho(B_n)$. Recall that B_n is contained in H_i for all large i, say $i \geq i_0$. As $\rho(H_i)$-orbits of the points in E_i have at most $2l$ points, this implies that for all $i \geq i_0$ the set E_i is contained in P_n. Thus the $\rho(G(\mu))$-orbit of any point in E_i, $i \geq i_0$, is an orbit of $\rho(B_n)$. Since $\rho(B_n)$ is contained in $\rho(H_i)$ and E_i is $\rho(H_i)$-invariant, this shows that for $i \geq i_0$, E_i is $\rho(G(\mu))$-invariant. This proves the theorem in the case at hand.

Now consider the general case. For each i let $\tilde{H}_i$ be the Zariski closure of $\rho(H_i)$ in $\mathrm{GL}(V)$. Then each $\tilde{H}_i$ is a real algebraic subgroup, in particular a closed subgroup with finitely many connected components, leaving invariant the subset E_i. This shows that to prove the theorem it now suffices to consider the case when $G = \mathrm{GL}(V)$, ρ is the identity representation, and each H_i is a real algebraic subgroup of $\mathrm{GL}(V)$; (also the maximal subspaces of E_i may be assumed to be one-dimensional as before).

Passing to a subsequence we may assume that H_i are all of the same dimension. Now, unless H_i^0 is the same for all large i, we can get a sequence $\{H_i'\}$ of real algebraic subgroups such that H_i' is a lower dimensional subgroup of H_i for all i, and $\mu(H_i') \to 1$ as $i \to \infty$ (consider $H_i' = H_i \cap H_{k_i}$, for suitable k_i). Therefore without loss of generality we may further assume that H_i^0 are all same, say $H_i^0 = H$ for all i.

Since H is connected it follows that each of the lines contained in E_i is H-invariant. For a given i and a given character $\chi : H \to I\!R^+$ the subset $\{v \in E_i \mid h(v) = \chi(h)v \text{ for all } h \in H\}$ is invariant under the action of H_i. Hence from our choice, made earlier, it follows that for each i there exists a character χ_i such that $h(v) = \chi_i(h)v$ for all $h \in H$ and $v \in E_i$. There are only finitely many characters that can occur for the given action of H on V. Therefore passing to a subsequence we may assume that χ_i is the same for all i; thus there exists $\chi : H \to I\!R^+$ such that $h(v) = \chi(h)v$ for all $h \in H$ and $v \in E_i$ for any i.

Let $W = \{v \in V \mid h(v) = \chi(h)v \text{ for all } h \in H\}$ and let $G' = \{g \in G \mid g(W) = W\}$. Let $\sigma : G' \to \mathrm{GL}(W)$ be the representation obtained by restriction. Clearly, for each i, E_i is contained in W and H_i is contained in G', and in turn $G(\mu)$ is contained in G'. Let $H' = \sigma^{-1}(\sigma(H))$. Since $\sigma(H)$ acts as scalars it follows that H' is normal in G'.

Now let $G_1 = G'/H'$. Let S be the subgroup of $\mathrm{GL}(W)$ consisting of positive scalar transformations. We have a canonical isomorphism, say φ, of $\mathrm{GL}(W)/S$ onto $\mathrm{GL}_1(W)$, the latter being the subgroup of $\mathrm{GL}(W)$ consisting of the elements with determinant ± 1. We now define a representation $\sigma' : G_1 \to \mathrm{GL}(W)$, by setting $\sigma'(gH') = \varphi(\sigma(gH')S/S)$; it is well-defined since $\sigma(H')$ is contained in S. Let μ_1 be the measure on G_1, which is the image of μ under the quotient map $g \mapsto gH'$ for all $g \in G'$ (recall that $G(\mu)$ is contained in G'). Now H_iH'/H' are finite subgroups of G_1 such that $\mu_1(H_iH'/H') \to 1$. Also E_i is invariant under $\sigma'(H_iH'/H')$ for all i. Hence, by the special case considered earlier, E_i is $\sigma'(G(\mu_1))$-invariant for all large i. This implies in turn that E_i is $\rho(G(\mu))$-invariant for all large i, thus proving the theorem.

4. Invariant subspaces of positive measure

In this section we prove the following theorem.

Theorem 4.1. *Let $\mu \in M^1(\mathrm{GL}(V))$. Let $\nu \in M^1(V)$ and suppose that there exist a $\delta > 0$ and proper subspaces W_j, $j \in I\!N$, of V such that $(\mu^j * \nu)(W_j) > \delta$ for all $j \in I\!N$. Then there exists a proper subspace W of V, such that the following conditions are satisfied:*

i) $\nu(W) > 0$, *and* W *is invariant under the action of a subgroup of finite index in* $N(\mu)$; *and*

ii) for any $\epsilon > 0$ *there exists a* $j_0 \in \mathbb{N}$ *such that for all* $j \geq j_0$ *we have* $(\mu^j * \nu)(U) < \epsilon$ *for all subspaces* U *of dimension less than that of* W, *and* $|(\mu^j * \nu)(g^j xW) - (\mu^j * \nu)(g^j W)| < \epsilon$ *for all* $x \in N(\mu)$ *and* $g \in \operatorname{supp} \mu$.

Proof: Let d be the dimension of V. For all $j \in \mathbb{N}$ and $0 < m \leq d$, let c_{jm} denote the maximum of $(\mu^j * \nu)(W)$ over all m-dimensional subspaces W of V; we recall that the maximum is attained - see Lemma 2.1. Then for each m, $\{c_{jm}\}$ is a nonincreasing sequence, and hence converges to a limit, say c_m. Clearly $0 \leq c_1 \leq c_2 \leq \cdots \leq c_d = 1$. Let r be the minimum integer such that $c_r > 0$; we note that $r \leq d-1$ since by the condition in the hypothesis $c_{d-1} > \delta > 0$.

For any $k \in \mathbb{N}$ we shall denote by $\mathcal{F}_k$ the class of subsets of V which can be expressed as a union of k subspaces of dimension r.

Now, for any $j, k \in \mathbb{N}$ let α_{jk} denote the supremum of $(\mu^{j-1} * \nu)(S)$, over all $S \in \mathcal{F}_k$, where $\mu^0 = \delta_I$, I being the identity element in $\mathrm{GL}(V)$. Then clearly, for any fixed k, $\{\alpha_{jk}\}$ is a nonincreasing sequence in j; we denote by β_k the limit of $\{\alpha_{jk}\}$. Then, by the choice of r as above, $\beta_1 > 0$. Also, clearly $\beta_1 \leq \beta_k \leq k\beta_1$ and $\beta_k \leq 1$ for all k. Therefore there exists $l \in \mathbb{N}$ such that $\beta_k = k\beta_1$ for $k = 1, \ldots, l$ and $\beta_{l+1} < (l+1)\beta_1$.

We show that for all large j there exists a unique subset $A_j \in \mathcal{F}_l$ such that $(\mu^{j-1} * \nu)(A_j) = \alpha_{jl}$. Let $0 < \epsilon \leq (\beta_1 - (l+1)^{-1}\beta_{l+1})/2$. There exists $j_1 \in \mathbb{N}$ such that for $j \geq j_1$ we have $\alpha_{jk} \in (\beta_k - \epsilon, \beta_k + \epsilon)$ for all $1 \leq k \leq l+1$, and $(\mu^{j-1} * \nu)(W) < \epsilon$ for all subspaces W of dimension at most $r-1$. Let $j \geq j_1$ and $A_j, A_j' \in \mathcal{F}_l$ be such that $(\mu^{j-1} * \nu)(A_j) > \alpha_{jl} - \epsilon$ and $(\mu^{j-1} * \nu)(A_j') > \alpha_{jl} - \epsilon$. Suppose that $A_j' \neq A_j$ and let W be an r-dimensional subspace of A_j' not contained in A_j. Then $(\mu^{j-1} * \nu)(W) \geq (\mu^{j-1} * \nu)(A_j') - (l-1)(\beta_1 + \epsilon) \geq (\beta_l - \epsilon) - (l-1)(\beta_1 + \epsilon) = \beta_1 - l\epsilon$. Therefore

$$\begin{aligned}
(\mu^{j-1} * \nu)(A_j \cup W) &= (\mu^{j-1} * \nu)(A_j) + (\mu^{j-1} * \nu)(W) - (\mu^{j-1} * \nu)(A_j \cap W) \\
&\geq (\beta_l - \epsilon) + (\beta_1 - l\epsilon) - l\epsilon = (l+1)\beta_1 - (2l+1)\epsilon \\
&\geq \beta_{l+1} + \epsilon > \alpha_{j(l+1)},
\end{aligned}$$

which contradicts the definition of $\alpha_{j(l+1)}$. Therefore A_j and A_j' as above are same, which shows that there exists a unique $A_j \in \mathcal{F}_l$ such that $(\mu^{j-1} * \nu)(A_j) = \alpha_{jl}$ for all large j; let $j' \in \mathbb{N}$ be such that this holds for all $j \geq j'$.

Now, for all $j \geq j'$ let $T_j = \{x \in \mathrm{GL}(V) \mid xA_j \subset A_{j+1}\}$ and $\theta_j = \mu(T_j)$. Let

$$\alpha_{jl}' = \sup\{(\mu^{j-1} * \nu)(x^{-1}A_{j+1}) \mid x \notin T_j\}.$$

Then

$$\alpha_{(j+1)l} = (\mu^j * \nu)(A_{j+1}) \leq \theta_j \alpha_{jl} + (1 - \theta_j)\alpha_{jl}' = \theta_j(\alpha_{jl} - \alpha_{jl}') + \alpha_{jl}'.$$

Therefore $\theta_j \geq (\alpha_{jl} - \alpha_{jl}')^{-1}(\alpha_{(j+1)l} - \alpha_{jl}')$, provided $\alpha_{jl} \neq \alpha_{jl}'$. If W is an r-dimensional subspace of V not contained in A_j, then

$$\alpha_{j(l+1)} \geq (\mu^{j-1} * \nu)(A_j \cup W) = \alpha_{jl} + (\mu^{j-1} * \nu)(W) - (\mu^{j-1} * \nu)(A_j \cap W),$$

and hence $(\mu^{j-1} * \nu)(W) \leq \alpha_{j(l+1)} - \alpha_{jl} + lc_{(j-1)(r-1)}$, where $c_{..}$ is as defined earlier. Therefore for any set $B \in \mathcal{F}_l$ we have

$$(\mu^{j-1} * \nu)(B) \leq \alpha_{j(l-1)} + (\alpha_{j(l+1)} - \alpha_{jl}) + lc_{(j-1)(r-1)}.$$

In particular, this applies to $x^{-1}A_{j+1}$ for $x \notin T_j$, and hence we get that

$$\alpha'_{jl} \leq \alpha_{j(l-1)} + (\alpha_{j(l+1)} - \alpha_{jl}) + lc_{(j-1)(r-1)}.$$

As $j \to \infty$ we have that $c_{(j-1)(r-1)} \to 0$ and therefore $\alpha'_{jl} \to \beta_{l-1} + \beta_{l+1} - \beta_l = \beta_{l+1} - \beta_1$. Thus $\alpha_{jl} - \alpha'_{jl} \to \beta_l - (\beta_{l+1} - \beta_1) = (l+1)\beta_1 - \beta_{l+1}$, and since $\beta_{l+1} < (l+1)\beta_1$ it now follows that $(\alpha_{jl} - \alpha'_{jl})^{-1}(\alpha_{(j+1)l} - \alpha'_{jl}) \to 1$. This shows that $\theta_j \to 1$, i.e. $\mu(T_j) = \mu(\{x \in \mathrm{GL}(V) \mid xA_j \subset A_{j+1}\}) \to 1$.

Recall that for all $x \in T_j$, $j \geq j'$, we have $xA_j \subset A_{j+1}$, and since each A_j is a union of l subspaces of dimension r, in fact $xA_j = A_{j+1}$. For $j \geq j'$ let H_j denote the subgroup $\{g \in \mathrm{GL}(V) \mid gA_j = A_j\}$. Then we have, for all $x, y \in T_j$, $x^{-1}y \in H_j$ and $xy^{-1} \in H_{j+1}$ for all $j \geq j'$. Hence, denoting by χ_j the characteristic function of H_j, for $j \geq j'+1$, we see that

$$(\mu * \tilde{\mu})(H_j) = \int \chi_j(xy^{-1})\mathrm{d}\mu(x)\mathrm{d}\mu(y) \geq \mu(T_{j-1})^2 \to 1, \text{ as } j \to \infty,$$

and, similarly, $(\tilde{\mu} * \mu)(H_j) \geq \mu(T_j)^2 \to 1$ as $j \to \infty$. We note that each H_j, $j \geq j'$, is a closed subgroup of $\mathrm{GL}(V)$, with finitely many connected components. Applying Theorem 3.1, with $\mu * \tilde{\mu}$ and $\tilde{\mu} * \mu$ in the place of the μ there, we get that for all large j the set A_j is invariant under the action of the subgroups $G(\mu * \tilde{\mu})$ and $G(\tilde{\mu} * \mu)$. As noted before $G(\mu * \tilde{\mu})$ and $G(\tilde{\mu} * \mu)$ generate a dense subgroup of $N(\mu)$, and hence for all large j the set A_j is $N(\mu)$-invariant.

Let $j_2 \in \mathbb{N}$ be such that A_j is $N(\mu)$-invariant for all $j \geq j_2$. Since $\operatorname{supp} \mu$ is contained in a single coset of $N(\mu)$ it follows that for any $g \in \operatorname{supp} \mu$ and $j \geq j_2$, $g^{-j}A_j$ is the same set, say A. Now, A is a union of l subspaces of dimension r, and it is invariant under the action of $N(\mu)$. Also, $\nu(A) = (\mu^j * \nu)(A_j) = \alpha_{jl}$ for all large j, and hence $\nu(A) = \beta_l = l\beta_1 > 0$. Therefore A contains an r-dimensional subspace W such that $\nu(W) > 0$. As A is $N(\mu)$-invariant, W is invariant under a subgroup, say N, of finite index (at most l) in $N(\mu)$. This proves assertion (i) in the theorem.

Now let $\epsilon > 0$ be given. Let $x_1, \dots, x_q$, where $q \leq l$ be such that $N(\mu)$ is a disjoint union of x_pN, $1 \leq p \leq q$. Clearly, for all $g \in \operatorname{supp} \mu$ and $1 \leq p \leq q$, g^jx_pW are (maximal) r-dimensional subspaces of $g^jA = A_j$. From the definition of r and α_{jl} it follows that there exists a $j_0 \geq j_2$ such that for all $j \geq j_0$ and any subspace U we have $(\mu^j * \nu)(U) \leq \beta_1 + (2l)^{-1}\epsilon$ if U is r-dimensional, and $(\mu^j * \nu)(U) < \epsilon$ if U is of dimension less than r. Since $(\mu^j * \nu)(A_j) = \beta_l = l\beta_1$ for all $j \geq j_2$, the first part also yields that, for any $g \in \operatorname{supp} \mu$ and $1 \leq p \leq q$, $(\mu^j * \nu)(g^jx_pW) \geq l\beta_1 - (l-1)(\beta_1 + (2l)^{-1}\epsilon) > \beta_1 - \frac{1}{2}\epsilon$. Since for every $x \in N(\mu)$ there exists x_p such that $xW = x_pW$, this shows that $|(\mu^j * \nu)(g^jxW) - (\mu^j * \nu)(g^jW)| < \epsilon$ for all $j \geq j_0$. This proves the theorem.

5. Proofs of Theorem 1.1 and the Corollaries

In this section we complete the proof of Theorems 1.1, and the Corollaries stated in the Introduction. In the case of measures μ whose support contains the identity element Theorem 1.1 follows easily from Theorem 4.1 (see the first part of the proof of the theorem below). For the general case we need the following proposition; though one may expect that it should be possible to give an elementary proof of the statement, and this is indeed possible under an additional condition (see Remark 5.2 below), at this time we are able to uphold the general assertion only using a version of the Szemeredi theorem!

Proposition 5.1. *Let V be a finite-dimensional real vector space and $T : V \to V$ be a linear transformation. Then for every $\delta > 0$ there exists $k \in \mathbb{N}$ such that the following holds: if W is a proper subspace of V and there exists a proper subspace W' of V such that*

$$|\{1 \le j \le k \mid T^j W \subset W'\}| > \delta k,$$

(where $|\cdot|$ denotes the cardinality of the set), then there exists $m \in \mathbb{N}$ such that W is contained in a proper T^m-invariant subspace of V.

Proof: Let d be the dimension of V. By the Szemeredi theorem (cf. [4], Theorem 1.2), for every $\delta > 0$ there exists a $k \in \mathbb{N}$ such that every subset of $\{1, 2, \dots, k\}$ of cardinality exceeding δk contains an arithmetic progression of length 2^d. For this k the condition in the hypothesis implies that the set $\{1 \le j \le k \mid T^j W \subset W'\}$ contains an arithmetic progression of length 2^d, say $\{a, a+m, \dots, a+(2^d-1)m\}$, where $m \in \mathbb{N}$. Then the subspace spanned by $\{W, T^m W, \dots, T^{m(2^d-1)} W\}$ is contained in $T^{-a}W'$, which is a proper subspace. Since d is the dimension of V, the subspace spanned by $\{W, T^m W, \dots, T^{m(2^d-1)} W\}$ is T^m-invariant. (The last assertion may be seen as follows: Let W_1 and W_2 be the subspaces spanned by $\{W, T^m W, \dots, T^{m(2^d-1)} W\}$ and $\cup_{j\in \mathbf{Z}} T^{mj} W$ respectively. Let r be the dimension of W, which may be assumed to be positive, and consider the r-th exterior power $\wedge^r V$ of V and the exterior linear transformation $\wedge^r T : \wedge^r V \to \wedge^r V$ corresponding to T. Let θ be a nonzero element of the one-dimensional subspace $\wedge^r W$. Then $\wedge^r W_1$ and $\wedge^r W_2$ are spanned, as subspaces of $\wedge^r V$ by $\{\theta, \wedge^r T^m(\theta), \dots, \wedge^r T^{m(2^d-1)}(\theta)\}$ and $\{\wedge^r T^{mj}(\theta) \mid j \in \mathbf{Z}\}$ respectively. Since the dimension of $\wedge^r V$ is less than 2^d, $\{\theta, \wedge^r T^m(\theta), \dots, \wedge^r T^{m(2^d-1)}(\theta)\}$ spans a $\wedge^r T^m$-invariant subspace, and hence coincides with the latter, namely $\wedge^r W_2$. Therefore $\wedge^r W_1 = \wedge^r W_2$ and hence $W_1 = W_2$, which shows that W_1 is T^m-invariant.). This proves the proposition.

Remark 5.2. In the case when all eigenvalues of T are real, it is possible to give a proof of Proposition 5.1 using only standard techniques. In fact in this case one can show that there exists $p \in \mathbb{N}$, depending only on V and T, such that for any two proper subspaces W and W' the set $\{j \in \mathbf{Z} \mid T^j W \subset W'\}$ contains at most p elements, unless it contains an infinite arithmetic progression. The essential point involved is that, for each choice for the dimension of W, the points in the set are zeros of a function of the form $\Sigma_{i=1}^r e^{\lambda_i t} P_i(t)$, on $I\!R$, where λ_i are eigenvalues of T and P_i are certain polynomials whose degrees (though not the coefficients) are independent of W and W'.

Before going over to the proof of the theorem we note also the following.

Remark 5.3. Let $\{\mu_i\}$ be a sequence in $M^1(\mathrm{GL}(V))$ and $\nu \in M(V)$ be such that $\{\mu_i * \nu\}$ does not converge to 0 in $M(V)$. Then there exists a subspace V' of V such that $\nu(V') > 0$, $\nu(U) = 0$ for all subspaces properly contained in V', and if ν' is the restriction of ν to V' (namely the measure defined by $\nu'(E) = \nu(E \cap V')$ for all Borel subsets E of V), then $\{\mu_i * \nu'\}$ does not converge to 0 in $M(V)$. To see this, consider first the decomposition of ν as $\Sigma\, \nu_r$ into pure components (see § 2). Since $\{\mu_i * \nu\}$ does not converge to 0 in $M(V)$ there exists r such that ν_r is a positive measure, and $\{\mu_i * \nu_r\}$ does not converge to 0 in $M(V)$. Now if $\{V_j\}$ is the sequence consisting of all r-dimensional subspaces of positive ν_r-measure, then it can be seen that there exists j such that the desired condition holds for $V' = V_j$.

Proof of Theorem 1.1 Suppose that $\{\mu_i * \nu\}$ does not converge to the zero measure in $M(V)$. Then by passing to a subsequence we may assume that $\{\mu_i * \nu\}$ converges to a $\lambda \in M(V)$, with $\lambda(V) > 0$. Then for all $j \in I\!N$ we have $\mu_i * (\mu^j * \nu) = \mu^j * (\mu_i * \nu) \to \mu^j * \lambda$, as $i \to \infty$. Therefore by Proposition 2.3, it follows that for every j there exists a proper subspace W_j of V such that $(\mu^j * \nu)(W_j) \geq \lambda(V)$. Hence the hypothesis of Theorem 4.1 is satisfied, for any $\delta < \lambda(V)$. Let W be a subspace of V such that the assertions (i) and (ii) as in the theorem hold. If the support of μ contains the identity element then $G(\mu) = N(\mu)$ and therefore assertion (i) shows that the theorem holds in this case.

We now continue with the proof in the general case. For this we draw upon assertion (ii) in Theorem 4.1. In view of Remark 5.3, replacing ν by its restriction to a subspace, we may assume that there exists a subspace V' of V such that ν is supported on V' and $\nu(U) = 0$ for every subspace properly contained in V'; then a subspace of V has positive measure if and only if it contains V'. Now, as above, passing to a subsequence we may further assume that $\{\mu_i * \nu\}$ converges to a $\lambda \in M(V)$, with $\lambda(V) > 0$.

Let N be the subgroup of $N(\mu)$ consisting of all elements leaving W invariant. Let q be the index of N in $N(\mu)$ and $x_1, \ldots, x_q \in N(\mu)$ be such that $x_1 N, \ldots, x_q N$ are the distinct cosets of N in $N(\mu)$. Let $A = N(\mu)W = \cup_{p=1}^q x_p W$. Since $\nu(W) > 0$, W contains V', and therefore, in particular $\operatorname{supp} \nu$ is contained in A. Since A is $N(\mu)$-invariant, it follows that, for all j, $\mu^j * \nu$ is supported on $g^j A$, for any element $g \in \operatorname{supp} \mu$; we shall consider a fixed $g \in \operatorname{supp} \mu$. (If g can be chosen to have only real eigenvalues, the proof can be completed using only the special case of Proposition 5.1 as in Remark 5.2.)

Let $\epsilon = \lambda(V)/2q > 0$. Then by assertion (ii) there exists $j_0 \in I\!N$ such that $(\mu^j * \nu)(U) < \epsilon$ for all $j \geq j_0$, and all subspaces U of dimension less than that of W. Now let $k \in I\!N$ be such that the conclusion of Proposition 5.1 holds for W as above, with $\delta = \lambda(V)/2q\nu(V)$.

Since $\{\mu_i * \nu\}$ converges to $\lambda \in M(V)$ and $\mu * \mu_i = \mu_i * \mu$ for all i, it follows that the sequence $\{\mu_i * (\frac{1}{k}\Sigma_{j=1}^k \mu^{j_0+j} * \nu)\}$ converges to the measure $\lambda_k = \frac{1}{k}\Sigma_{j=1}^k \mu^{j_0+j} * \lambda$ in $M(V)$. Therefore, by Proposition 2.3, there exists a proper subspace W' of V such that $\frac{1}{k}\Sigma_{j=1}^k (\mu^{j_0+j} * \nu)(W') \geq \lambda_k(V) = \lambda(V)$.

Since $\mu^j * \nu$ is supported on $g^j A$, for all j, we have

$$(\mu^j * \nu)(W') = (\mu^j * \nu)(W' \cap g^j A) \leq \Sigma_{p=1}^q (\mu^j * \nu)(g^j x_p W \cap W').$$

Let E_k be the subset of $\{1, \dots, k\}$ consisting of j such that $g^{j_0+j} x_p W \subset W'$ for some p. For $j \in \{1, \dots, k\} - E_k$, each $g^{j_0+j} x_p W \cap W'$ is a subspace of dimension less than r, and hence

$$(\mu^{j_0+j} * \nu)(W') \leq \Sigma_{p=1}^q (\mu^{j_0+j} * \nu)(g^{j_0+j} x_p W \cap W') < q\epsilon < \lambda(V)/2.$$

Since $\Sigma_{j=1}^k (\mu^{j_0+j} * \nu)(W') \geq k\lambda(V)$, it follows that $\Sigma_{j \in E_k} (\mu^{j_0+j} * \nu)(W') \geq k\lambda(V)/2$. In particular, it follows that $|E_k| \geq k\lambda(V)/2\nu(V)$. Therefore there exists p, $1 \leq p \leq q$, such that

$$|\{1 \leq j \leq k \mid g^{j_0+j} x_p W \subset W'\}| \geq k\lambda(V)/2q\nu(V) = k\delta.$$

Therefore, by Proposition 5.1, there exists $m \in I\!N$ such that the subspace spanned by $\{g^{mj} x_p W \mid j \in \mathbf{Z}\}$ is a proper subspace of V. Hence the subspace spanned by $\{(x_p^{-1} g^{mj} x_p) W \mid j \in \mathbf{Z}\}$ is also proper subspace of V. The latter contains W and hence its ν-measure is positive. Also, it is invariant under the subgroup generated by N and $x_p^{-1} g^m x_p$, which is easily seen to be a subgroup of finite index in $G(\mu)$. This completes the proof of the theorem.

For the proof of Corollary 1.2 we need the following proposition, deduced from the work on concentration functions of measures; the reader is referred to [7] for general results on this topic, and to [2] and [1] for another approach that seems more convenient for matrix groups.

Proposition 5.4. *Let $\mu \in M^1(\mathrm{SL}(V)) \subset M^1(\mathrm{End}(V))$. Suppose that $G(\mu)$ is noncompact. Then $\mu^i \to 0$ in $M^1(\mathrm{End}(V))$.*

Proof: Since $\mathrm{SL}(V)$ is a closed subset of $\mathrm{End}(V)$, one only needs to know that $\mu^i(K) \to 0$ for all compact subsets of $\mathrm{SL}(V)$. This is true by Corollary 3.6 of [1]; the desired statement can also be deduced from Corollary 2 in [2].

Proof of Corollary 1.2 The first statement in the corollary follows immediately from Theorem 1.1 and Proposition 5.4. We recall that an action of a subgroup H of $\mathrm{GL}(V)$ on V is said to be strongly irreducible if there is no proper nonzero subspace which is invariant under a subgroup of finite index in H. Therefore the first statement implies the second. This proves the corollary.

Proof of Corollary 1.4 Clearly we need to prove only the 'if' part, which we carry out by induction on the dimension of V. The statement is clear for dimension 1. We now proceed assuming (as we may) that it holds in lower-dimensional cases than the one under consideration. Suppose that $\{\mu^i * \nu\}$ does not converge to 0 in $M(V)$. In view of Remark 5.3 we may assume that there exists a subspace V' of V such that a subspace has positive ν-measure if and only if it contains V'. By Corollary 1.2 there exists a proper subspace W such that $\nu(W) > 0$ and W is invariant under a subgroup of finite index in $G(\mu)$. As $G(\mu)$ is of type $\mathcal{S}$ there exists a subgroup $\tilde{H}$ of $\mathrm{GL}(V)$ such that W is invariant under a subgroup of finite index in $\tilde{H}$, and the subset S of $\tilde{H}$ consisting of all elements which are either unipotent or contained in compact connected subgroups of $\tilde{H}$, generates a dense subgroup of $\tilde{H}$. From the first part it follows that there exists a $n \in \mathbb{N}$ such that for any $g \in \tilde{G}(\mu)$, W is g^n-invariant. For a unipotent element g, a subspace invariant under g^n is also g-invariant. Also the elements of $\tilde{H}$ contained in compact connected subgroups leave W invariant. Thus W is invariant under all elements of S and hence under $\tilde{H}$. Let $\eta : \tilde{H} \to \mathrm{GL}(W)$ be the homomorphism given by restriction, and let $\overline{\mu} = \eta(\mu)$. Since $G(\mu)$ is of type $\mathcal{S}$ (in $\mathrm{GL}(V)$) it follows $G(\overline{\mu})$ is of type $\mathcal{S}$ (in $\mathrm{GL}(W)$). Since $\nu(W) > 0$, W contains V', and hence ν is supported on W. Thus ν may be viewed as a measure on W. Clearly $\mu^{n_i} * \nu = \overline{\mu}^{n_i} * \nu$ for all i, and hence $\{\overline{\mu}^{n_i} * \nu\}$ does not converge to 0 in $M(W)$. Since the dimension of W is less than that of V, by the induction hypothesis it follows that $\nu(B \cap W) > 0$, and hence $\nu(B) > 0$. This proves the corollary.

Remark 5.5. Statement (ii) as in the conclusion of Theorem 1.1 can not be strengthened to requiring that the subspace W be invariant under (the whole of) $G(\mu)$, and similarly in Corollary 1.2 it does not suffice to assume that $\nu(W) = 0$ for all $G(\mu)$-invariant proper subspaces. This can be seen from the following example. Let $V = \mathbb{R}^2$ and $\{e_1, e_2\}$ be the standard basis of V. Let $\lambda > 1$ be fixed. Let $g_1, g_2 \in \mathrm{GL}(V)$ be the elements defined by $g_1(x_1e_1 + x_2e_2) = \lambda x_1e_1 + \lambda^{-1}x_2e_2$, and $g_2(x_1e_1 + x_2e_2) = \lambda x_2e_1 + \lambda^{-1}x_1e_2$, for all $x_1, x_2 \in \mathbb{R}$. Let μ be the probability measure on $\mathrm{GL}(V)$ defined by $\mu(\{g_1\}) = \mu(\{g_2\}) = \frac{1}{2}$. Since $|\det(g_1)| = |\det(g_2)| = 1$, by Proposition 5.4 $\mu^i \to 0$ in $M(\mathrm{End}(V))$. On the other hand for $\nu = \delta_{e_1}$ it can be seen that $\{\mu^i * \nu\}$ converges to a measure λ with $\lambda(V) = \frac{1}{2}$. It can be seen that $N(\mu) = G(\mu)$ and that it acts irreducibly (though not strongly irreducibly). Since e_1 is not contained in any $G(\mu)$-invariant subspace this proves the claim as above.

We conclude with the following comment. It seems to us that in Theorem 1.1 if (i) does not hold, and $\mu_i * \nu \to \lambda$ in $M(V)$ with $\lambda(V) > 0$, then the subspace W as in (ii) can be chosen satisfying the stronger condition that $\nu(W) \geq \lambda(V)$. This indeed holds for the example in Remark 5.5 and also in other examples that we considered. We have however not been able to ascertain it in general.

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