# From Natural Numbers to Numbers and Curves in Nature – II

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**In this second part of the article we discuss how simple growth models based on Fibbonachi numbers, golden section, logarithmic spirals, etc. can explain frequently occuring numbers and curves in living objects. Such mathematical modelling techniques are becoming quite popular in the study of pattern formation in nature.**

## Fibonacci Sequence and Logarithmic Spiral as Dynamic (Growth) Model

Fibonacci invented his sequence with relation to a problem about the growth of rabbit population. It was not, of course, a very realistic model for this particular problem. However, this model may be applied to the cell growth of living organisms. Let us think of the point C, (see Part I *Figure* 1), as the starting point. Then a line grows from C towards both left and right. Suppose at every step (after a fixed time interval), the length increases so as to make the total length follow the Fibonacci sequence. Further, assume that due to some imbalance, the growth towards the right starts one step later than that towards the left.

[1] Part 1. *Resonance*, Vol.9, No.9, pp.29-37, 2004.

If the growth stops after a large number of steps to generate the line AB, then AC/AB = *j* (the Golden Section). Even if we model the growth rate such that instead of the total length, the increase in length at every step follows the Fibonacci sequence, we again get (under the same starting imbalance) AC/AB = *j*.

Let there be *n* steps of growth to the left, and *n*–1 to the right. Then

$$\frac{AC}{BC} = \frac{\sum_{1}^{n} F_i}{\sum_{1}^{n-1} F_i} = \frac{F_{n+2}-1}{F_{n+1}-1} \text{ (using the first result of Part I, } \textit{Box}\ 1\text{)}$$

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$= 1/j$ for large $n$,

or, $AC/AB = j$ (for large $n$).

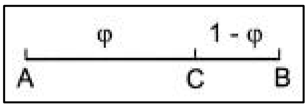



***Figure 1a (from Part 1). Geometric interpretation of Golden Section.***

If the growth of an organism generates a curve, then again a very simple model can generate a logarithmic spiral. Referring to *Figure* 1, let a unidirectional growth be taking place by deposition at the current point P, starting from the neighbourhood of the origin O, according to the following rule:

The rate of growth of the distance of the point P is proportional to its current value, and the angular velocity of the radius vector OP is constant. Mathematically,

$$\dot{r} = a\, r$$
$$\dot{q} = w \ , \qquad (1)$$

where a dot denotes differentiation with respect to time, and both $a$ and $w$ are constants. From equation (1), one gets

$$\frac{dr}{dq} = \frac{a}{w} r$$

or, $r = e^{(a/w)q}$

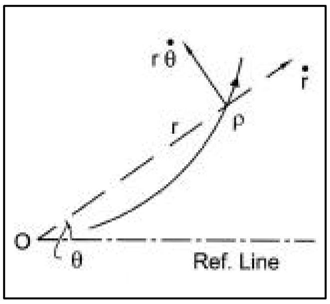



***Figure 1. Growth model for a logarithmic spiral.***

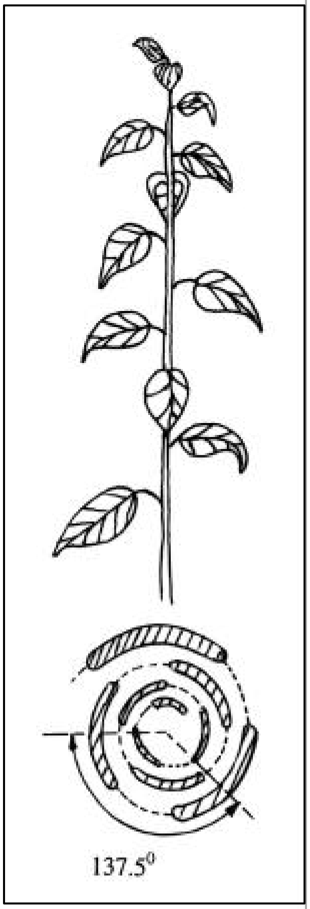
after arbitrarily defining the scale of the figure by putting $r$=1 at $q$ =0.

Thus, the form of the curve produced by this growth model is a logarithmic spiral with the growth rate $g = a/w$.

## Numbers and Curves in Nature

The Golden Rectangle was considered by Greeks to be most pleasing to the eye and they based the dimensions of many temples, vases and other artifacts based on this Golden Section. There is a long standing idea (sometimes ridiculed) that for a perfectly proportioned human body, the ratio of the height of the navel to the total height (and also of some other bodily proportions) is equal to the Golden Section. The total growth (or even the growth rate) following the Fibonacci sequence and starting from the navel (umbilical chord providing the source of initial nutrition) with a slight bias towards the lower portion during initiation might provide some clue to a possible explanation of this observation. The Golden Section is abundantly observed in nature, for example in various proportions of insect-body and flower-body dimensions, leafing patterns in trees, arrangement and width of human teeth, etc. In fact a London based dentist Eddy Levin has invented an ingenious three-legged (say A, B and C) callipers where the distances AB and AC (when changed) always maintain a fixed ratio equalling the Golden Section. For fascinating animated use of this instrument to measure Golden Sections appearing in nature, see [2].

***Figure 2. Spiral leaf pattern exhibiting Golden Angle (Adapted from [5]).***




The numbers of petals in a large number of flowers are seen to be Fibonacci numbers. A large number of common flowers have 1, 3, 5, 8, 13, 21 petals (see [3] for details on flower petals). Some giant sunflowers have 34, 55 or 89 petals. If one carefully observes the row of scales on the surface of a pineapple, then these diamond shaped marks are seen to follow two spirals sloping in opposite directions. It is found that the helix angles of these two spirals are different. If one counts the numbers of these marks along these two spirals, more often than not, these two numbers turn out to be consecutive Fibonacci numbers like (8, 13) with 8 spiralling along to the right and 13 to the left [4].

In 80% of the plant species, leaves execute a spiral up the stem, with each leaf displaced above the one below by an almost constant angle. The potato plant, for instance, has the arrangement shown in *Figure* 2. In this figure, the top view is a schematic representation where the successive leaves are depicted smaller the farther they are up the stem. In other words, newer leaves are shown smaller. The offset angle is close to 137.5 degrees (i.e., the Golden Angle), an observation that begs for an explanation [5].

Let us now look at the growth process of a plant. Plants grow from the tip of the stem, where one finds a bud of multicellular tissue called the meristem. Here cells multiply rapidly, and just behind the advancing tip, side buds called primordia begin to protrude

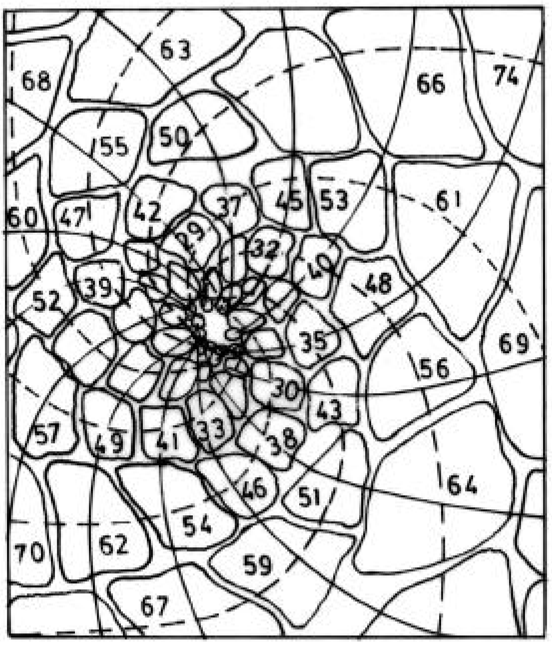



*Figure 3. Double spirals in leaf pattern exhibiting Fibonacci numbers and Golden Angle (Adapted from [5])*

one by one. These subsequently develop into leaves or petals or florets (tiny flowers that develop into seeds). Successive primordia form roughly at a constant time interval (of the order of 1 day). The pattern of leafing is determined by where the primordia appear on the boundary of the apex. By projecting the leaf pattern (of some species of plants) on a plane perpendicular to the stem, we get what is shown in *Figure* 3. Here the number on the leaf refers to their order of appearance (i.e., a higher number implies later appearance) and lines are drawn through leaves that are in contact in this diagram. These lines trace out two families of spirals of opposite hands. This double-spiral pattern is more immediately evident when the primordia develop not into leaves but into florets in a flower head, since in that case they remain almost in one plane. *Figure* 4 shows them clearly for a sunflower. Similar double spirals are also seen in cork pine and vegetables like cauliflower [6].

*Figure 4. Double spirals in the head of a sunflower.*

Referring again to *Figure* 3, we can identify the 'Golden Angle' between successive leaves. Moreover, the numbers on leaves in contact along one spiral differ by 8 and along the other by 13, two adjacent numbers in the Fibonacci se-

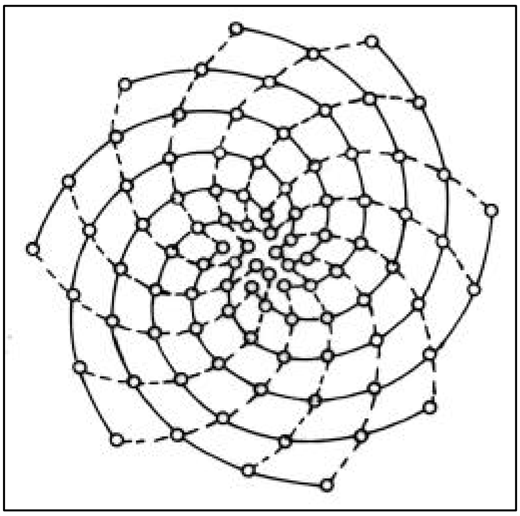


*Figure 5. Double spiral pattern generated by growth along a tightly wound spiral with divergence angle equal to Golden Angle.*

quence! Consequently, the number of right-hand spirals is 8 and that of left-hand spirals is 13! Examples from other trees show (5,8) and (3,5) relationships.

If we plot small circles with centres lying on a tightly wound spiral at angular separation of 137.5 degrees, then the diagram looks like *Figure* 5. The tightly wound spiral has not been shown in this figure. It is clear that the circles appear to lie on two spirals of opposite hands. Due to the relation between Fibonacci numbers and the Golden Angle, the numbers of these spirals are two consecutive numbers (8,13). These numbers depend on the tightness of the initial spiral (not shown) on which the centres were plotted. The florets of *Figure* 4 developed on such a tightly wound spiral at angular divergence of 137.5 degrees. Depending on the species of giant sunflowers the number of spirals could be (34, 55), (55, 89) or even (89, 134)! Why is the angle of deposition 137.5 degrees? If the angle is a rational fraction of 360 degrees, then the florets will be finally appearing along radial lines (rather than on two spirals) and there will be gap between these radial lines. So efficient packing while maintaining regularity (like atoms in a crystal) requires that the divergence angle (of deposition) must be an irrational multiple of 360 degrees. But why does that irrational multiple have to generate the 'Golden Angle'? Is it because the Golden Section is the 'worst' irrational number as noted earlier?

This divergence at Golden Angle was found as a consequence of simple dynamics during an experiment (with non-living objects obeying laws of physics). Droplets of magnetic fluid were dropped at regular intervals on a horizontal round dish coated with silicone oil. The dish was placed in a uniform vertical magnetic fluid and non-uniform horizontal field, which was stronger at the edge and weaker near the centre. The magnetic fluid drops (polarized by the magnetic fields) were repelled by each other while moving towards the edge and arranged themselves in a double spiral as shown in *Figure* 6. The divergence angle between the successive drops is close to 137.5 degrees. Of course, the angle of divergence depends on the rate of dropping, but this was quite a predominant pattern. So some dynamical model (valid for the physics experiment) may explain the floret pattern!

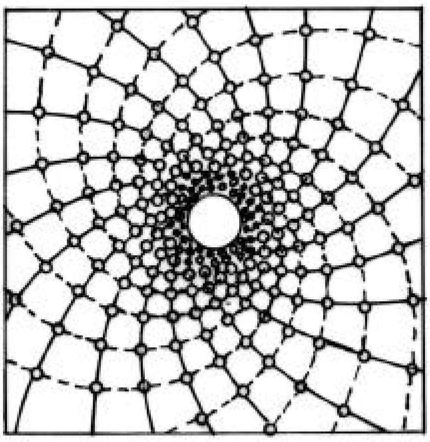
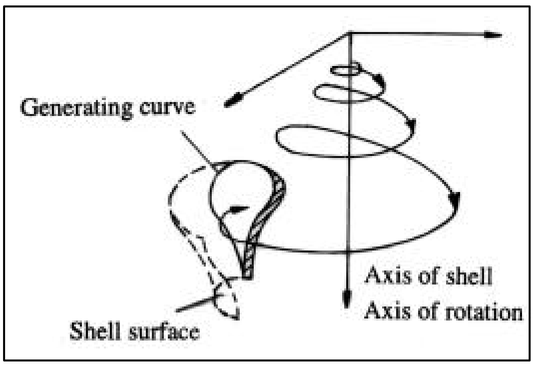


***Figure 6. Double spirals exhibited by moving magnetic fluid drops ( Adapted from [5]).***

If we look at the shell of a snail, or a chambered nautilus and other molluscs, horns of several species of sheep and buffalo, waves of calcium travelling across the surface of frog eggs when they are fertilized, the spiral galaxy M100 or NGC 5236 (for pictures see [5] and [1]), then we cannot fail to notice the preponderance of logarithmic spiral as the result of a growth process. No doubt, as compared to physics, botany and zoology are less amenable to "unreasonably successful mathematics". Scottish zoologist D'Arcy Wentworth Thompson (1917) in his beautiful book "*On growth and Form*", however, suggested such an engineer's answer against 'Darwinism'. Instead of million years of selective fine-tuning to explain the sheep's horn, possibly a very simple growth law based on proximate physical causes is worth looking for. A little imbalance between the growth rates on two sides of the horn will produce a logarithmic spiral. The shell of a mollusc will have a cone shape in the absence of this imbalance, just as one finds in some species of molluscs (like a limpet). The surface of any shell may be generated by the revolution about a fixed axis of a closed curve (called the generating curve), which grows exponentially (in time) but rotating at a constant rate (*Figure* 7). Following this prescription with different generating curves, some shells generated by computer simulation are found to be remarkably close to those found in nature [5]. It must be mentioned that this mathematical approach to biology does not go against 'Darwinism', rather it gives greater confidence in the verbal line of argument. One criticism of evolution by random mutation was that very complex structures (like an eye) have to evolve

***Figure 7. Simulation of shell surfaces by logarithmic helico-spiral (Adapted from [5]).***



fully formed or else they won't work properly (there is no use of half an eye!). However, a half-developed eye might be useful (e.g., only having a retina but no lens that will still collect light and detect movement). A mathematical model of a flat region of cells, permitting various random mutations, like some cells becoming more sensitive to light, the shape of a region becoming bent etc., was carried out through computer simulations. In this model any mutation giving improvement to 'vision' was retained. The simulation surprisingly shows the development of a spherical cavity, with an opening and most dramatically, a lens with a variation of refractive index very similar to that of our own eye [4]! The estimated time required for this evolution was found to be approximately 400,000 years.

We have discussed the Fibonacci sequence arising out of natural numbers and showed its connection to Golden Section, Golden Angle, Golden Rectangle and Logarithmic Spiral. We have also seen how simple growth laws can ultimately manifest into Golden Section and Logarithmic Spiral. Finally we gave examples of occurrences of these mathematical entities in nature. The investigation of this self-made tapestry in nature of various proportions, forms and patterns has recently become a fast growing subject transcending the boundaries of a number of traditionally different disciplines.

## Suggested Reading


[1] **E Maor, *e: The story of a Number*, Universities Press (India) Limited, Hyderabad, 1999.**

[2] **http://www.goldenmeangauge.co.uk/golden.htm**

[3] **http://www.ma.utexas.edu/~lefcourt/SP97/M302/projects/lefc062/FLOWERS.html.**

[4] **I Stewart, *Nature's Numbers – Discovering Order and Pattern in the Universe*, Phoenix Paperback, London, 1996**

[5] **P Ball, *The Self-Made Tapestry – Pattern Formation in Nature*, Oxford University Press, Paperback, Oxford, 2001.**

[6] **http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html**

[7] **C S Ogilvy, *Excursions in Geometry*, Dover Books, New York, 1990**

[8] **T A Cook, *The Curves of Life*, Dover Books, New York, 1979.**



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