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ON THE OSCILLATIONS OF THE MACLAURIN SPHEROID BELONGING TO THE THIRD HARMONICS

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ABSTRACT

The characteristic frequencies of oscillation of the Maclaurin spheroid belonging to the third harmonics are found Two further points of neutral stability, beyond the first at eccentricity $e = 0.81267$, are isolated They occur at $e = 0.89926$ and $e = 0.96937$; it is the second of these that is the analogue of the point of bifurcation along the Jacobian sequence.

I. INTRODUCTION

It is well known that the Maclaurin spheroids become unstable by particular modes of oscillation belonging to the second harmonics. On this account, the modes of oscillation belonging to these harmonics have been investigated at various times (see Lamb 1932 for historical references). And recently the problem has been reconsidered on the basis of the second-order virial equations and the different characteristic frequencies (five in all) have been explicitly obtained and exhibited (Lebovitz 1961).

Since the Maclaurin spheroids become unstable with respect to modes belonging to the second harmonics, there has not been much interest in the oscillations belonging to the third harmonics. Nevertheless, since the Jacobi ellipsoids become unstable with respect to a mode of oscillation belonging to the third harmonics, a comparison of the reactions of the Maclaurin spheroids and the Jacobi ellipsoids, to perturbations involving the third harmonics, is of some interest. Such a comparison can be made quite readily now, since all that is needed is a specialization of the formulae (by putting $a_1 = a_2$) which have already been derived in connection with the investigation on the stability of the Jacobi ellipsoids in the paper preceding this one (Chandrasekhar and Lebovitz 1963; this paper will be referred to hereafter as "Paper I").

II. THE EVEN MODES

We have seen in Paper I (Secs. III and IV) that the eighteen equations governing the third-order virials fall into two non-combining groups of ten and eight equations, respectively, distinguished by their "parity" with respect to the index 3 (the direction of the axis of rotation); that the even equations can be reduced to a set of four equations involving the six symmetric even virials, while the odd equations can be reduced to a set of three equations involving the four symmetric odd virials; and, finally, that these reduced equations, supplemented by the divergence conditions (two for the even set and one for the odd set), provide a complete set of equations for determining *all* the characteristic frequencies belonging to the third harmonics.

In this section we shall consider the even modes. The relevant equations are the same as for the Jacobi ellipsoid and are given by Paper I, equations (76)–(79), (132), and (133). However, since a_1 is now equal to a_2, the coefficients of the virials in the expansions of $\delta\mathfrak{W}_{ij;k}$ and δS_{ijj} are much simplified. Indeed, from the definitions of the symbols $A_{ijk}\ldots$ and $B_{ijk}\ldots$ it follows (from the present equality of a_1 and a_2) that the value of any of these symbols is unaltered if the index 1 (wherever it may occur) is replaced by the index 2, and conversely. On this account, the two parts of Table 2 in Paper I referring to the even equations are identical (except for the labeling of the rows and the columns). And there are some further simplifications. Since the extent and the nature of these simplifications are essential to our present purposes, we give in Table 1 the coefficients of the virials in the expansions of the relevant quantities. Table 1 is a transcription of

TABLE 1

THE COEFFICIENTS OF THE VIRIALS IN THE EXPANSIONS OF THE $\delta\mathfrak{W}_{ij;k}$'S AND THE δS_{ij}'S WHICH OCCUR IN THE EVEN EQUATIONS*

Element	V_{111} (V_{222})	V_{122} (V_{112})	V_{133} (V_{233})
$\delta S_{122}(\delta S_{112})$	$-2(B_{11}+a_1^2B_{111})$	$6(B_{11}+a_1^2B_{111})$	0
$\delta S_{133}(\delta S_{233})$	$(2a_1^2+a_3^2)B_{113}-5a_1^2B_{111}-2B_{11}$	$(2a_1^2+a_3^2)\ B_{113}-3a_1^2B_{111}$	$3[B_{13}+B_{33}+(a_1^2+2a_3^2)B_{133}-a_1^2B_{113}]$
$-2\delta\mathfrak{W}_{12;2}(-2\delta\mathfrak{W}_{12;1})$	$a_1^2B_{111}$	$2B_{11}+3a_1^2B_{111}$	$a_1^2B_{113}$
$-2\delta\mathfrak{W}_{13;3}(-2\delta\mathfrak{W}_{23;3})$	$a_3^2B_{113}$	$a_3^2B_{113}$	$2B_{13}+3a_3^2B_{133}$

* A common factor, $\pi G\rho a_1^2a_3$, in all the symbols has been omitted.

TABLE 2

THE COEFFICIENTS OF THE VIRIALS IN THE EXPANSIONS OF THE $\delta\mathfrak{W}_{ij;k}$'S AND δS_{ij}'S WHICH OCCUR IN EQUATIONS (13) AND (14)

Element	V_{122} (V_{112})	V_{133} (V_{233})
$\delta S_{133}(\delta S_{233})$	$4(2a_1^2+a_3^2)B_{113}-18a_1^2B_{111}-6B_{11}$	$3[B_{13}+B_{33}+(a_1^2+2a_3^2)B_{133}-a^2B_{113}]$
$-2\delta\mathfrak{W}_{12;2}(-2\delta\mathfrak{W}_{12;1})$	$2B_{11}+6a_1^2B_{111}$	$a_1^2B_{113}$
$-2\delta\mathfrak{W}_{13;3}(-2\delta\mathfrak{W}_{23;3})$	$4a_3^2B_{113}$	$2B_{13}+3a_3^2B_{133}$

TABLE 3

THE COEFFICIENTS OF THE VIRIALS IN THE EXPANSIONS OF THE $\delta\mathfrak{W}_{ij;k}$'S AND THE δS_{ijk}'S WHICH OCCUR IN THE ODD EQUATIONS

Element	V_{333}	V_{113}	V_{223}
δS_{113}	$(a_1^2+2a_3^2)B_{133}-5a_3^2B_{333}-2B_{33}$	$3[B_{11}+B_{13}+(2a_1^2+a_3^2)B_{113}-a_3^2B_{133}]$	$(a_1^2+2a_3^2)B_{113}-3a_3^2B_{133}$
δS_{223}	$(a_1^2+2a_3^2)B_{133}-5a_3^2B_{333}-2B_{33}$	$(a_1^2+2a_3^2)B_{113}-3a_3^2B_{133}$	$3[B_{11}+B_{13}+(2a_1^2+a_3^2)B_{113}-a_3^2B_{133}]$
$-2\delta\mathfrak{W}_{13;1}$	$a_1^2B_{133}$	$2B_{13}+3a_1^2B_{113}$	$a_1^2B_{113}$
$-2\delta\mathfrak{W}_{23;2}$	$a_1^2B_{133}$	$a_1^2B_{113}$	$2B_{13}+3a_1^2B_{113}$

$$-2\delta\mathfrak{W}_{13;2} = -2\delta\mathfrak{W}_{23;1} = 2(B_{13}+a_1^2B_{113})V_{123}\,;\quad \delta S_{123} = 2[B_{11}+2B_{13}+(2a_1^2+a_3^2)B_{113}]V_{123}\,.$$

the first two parts of Table 2 of Paper I with the simplifications arising from the equality of a_1 and a_2.

According to Table 1, the expansions of δS_{122} and δS_{112} contain, respectively, no terms in V_{133} and V_{233}; and, moreover,

$$\delta S_{122} = 6(B_{11} + a_1^2 B_{111})(V_{122} - \tfrac{1}{3}V_{111})$$

and (1)

$$\delta S_{112} = 6(B_{11} + a_1^2 B_{111})(V_{112} - \tfrac{1}{3}V_{222}) .$$

The first pair of even equations (Paper I, eqs. [76] and [77]), therefore, takes the form

$$[\lambda^2 - 3\Omega^2 + 6(B_{11} + a_1^2 B_{111})](V_{122} - \tfrac{1}{3}V_{111}) + 2\lambda\Omega(V_{112} - \tfrac{1}{3}V_{222}) = 0 \tag{2}$$

and

$$[\lambda^2 - 3\Omega^2 + 6(B_{11} + a_1^2 B_{111})](V_{112} - \tfrac{1}{3}V_{222}) - 2\lambda\Omega(V_{122} - \tfrac{1}{3}V_{111}) = 0 . \tag{3}$$

It is clear that these equations can be considered independently of the divergence condition and of the remaining pair of equations.

Equations (2) and (3) lead to the characteristic equation

$$[\lambda^2 - 3\Omega^2 + 6(B_{11} + a_1^2 B_{111})]^2 + 4\lambda^2\Omega^2 = 0 . \tag{4}$$

It is now convenient to write

$$\lambda^2 = -\sigma^2 \tag{5}$$

so that a real σ implies stability. With the substitution (5), equation (4) can be factorized to give

$$\sigma^2 - 2\Omega\sigma + 3\Omega^2 - 6(B_{11} + a_1^2 B_{111}) = 0 , \tag{6}$$

and a similar equation with $-\Omega$ in place of Ω. The roots of equation (6) are

$$\sigma = \Omega \pm [6(B_{11} + a_1^2 B_{111}) - 2\Omega^2]^{1/2} . \tag{7}$$

From equation (7) it is apparent that a neutral mode occurs when

$$\frac{\Omega^2}{\pi G\rho} = 2(B_{11} + a_1^2 B_{111})a_1^2 a_3 \qquad (\sigma = 0) ; \tag{8}$$

and, further, that overstability with a frequence Ω occurs when

$$\frac{\Omega^2}{\pi G\rho} > 3(B_{11} + a_1^2 B_{111})a_1^2 a_3 \qquad (\sigma \text{ complex}) . \tag{9}$$ [1]

[The factor $\pi G\rho a_1^2 a_3$, which was suppressed in writing the symbols in Table 1, has been restored in (8) and (9).]

It is found that condition (8) is met when

$$e = 0.89926 \qquad (\sigma = 0) \tag{10}$$

and that overstability in accordance with the condition (9) occurs when

$$e > 0.96696 \qquad (\sigma \text{ complex}) . \tag{11}$$

[1] Conditions equivalent to these appear to be included in an early paper by Cartan (1922). The methods of Cartan (cf. Lyttleton 1953) are very different from ours and we have not attempted any comparisons.

It is to be particularly noted that the neutral mode at $e = 0.89926$ occurs *before* the Maclaurin spheroid first becomes overstable at $e = 0.95289$ by a mode of oscillation belonging to the second harmonic.

Since a neutral mode occurs at $e = 0.89926$, it would seem that the Maclaurin sequence has its *second* point of bifurcation here.[2]

The characteristic frequencies given by equation (7) are designated σ_2 and σ_5; for their behavior along the Maclaurin sequence see Table 4 and Figure 3 in Section IV below.

There are altogether six even modes; and we have accounted for two of them. The remaining four modes are determined by Paper I, equations (78) and (79) supplemented by the two divergence conditions.

In Paper I, equations (78) and (79), we can now put

$$V_{111} = 3V_{122} \quad \text{and} \quad V_{222} = 3V_{112}\,, \tag{12}$$

so that the roots given by equation (4) may be excluded and there may be no inconsistency with equations (2) and (3). With the substitution (12), the relevant coefficients given in Table 1 can be further simplified and reduced. Table 2 is the required simplified version.

With the coefficients given in Table 2 and with the substitutions (12), Paper I, equations (78) and (79) become

$$\begin{aligned}\lambda(\lambda^2 + 4\Omega^2)[(\lambda^2 - \Omega^2)V_{133} - (\lambda^2 + \Omega^2)V_{122} + \delta S_{133}]\\ + 2\Omega(\lambda^2 + 4\Omega^2)(\Omega^2 V_{112} + 2\delta\mathfrak{W}_{12;1}) - 2\Omega\lambda^2(\Omega^2 V_{233} + 2\delta\mathfrak{W}_{23;3})\\ + 4\Omega^2\lambda(\Omega^2 V_{133} + 2\delta\mathfrak{W}_{13;3}) = 0\,,\end{aligned} \tag{13}$$

$$\begin{aligned}\lambda(\lambda^2 + 4\Omega^2)[(\lambda^2 - \Omega^2)V_{233} - (\lambda^2 + \Omega^2)V_{112} + \delta S_{233}]\\ -2\Omega(\lambda^2 + 4\Omega^2)(\Omega^2 V_{122} + 2\delta\mathfrak{W}_{12;2}) + 2\Omega\lambda^2(\Omega^2 V_{133} + 2\delta\mathfrak{W}_{13;3})\\ + 4\Omega^2\lambda(\Omega^2 V_{233} + 2\delta\mathfrak{W}_{23;3}) = 0\,.\end{aligned} \tag{14}$$

Equations (13) and (14) provide two linear relations among the four virials V_{122}, V_{133}, V_{112}, and V_{233}. Two further relations are provided by the divergence conditions (Paper I, eqs. [132] and [133]); these conditions, in view of equations (12), become

$$V_{122} = -\frac{a_1^2}{4a_3^2}V_{133} \quad \text{and} \quad V_{112} = -\frac{a_1^2}{4a_3^2}V_{233}\,. \tag{15}$$

The characteristic equation which follows from equations (13)–(15) is of degree five in λ^2. But the five roots, determined by this characteristic equation, include one, namely, $\lambda^2 = -4\Omega^2$, which is spurious. The origin of the occurrence of this spurious root has been explained in Paper I, Section IV*a*.

The even modes include the analogue of the one with respect to which the Jacobi ellipsoids become unstable. This is the mode which belongs to the root which is designated σ_1^2 in Table 4 and Figure 3 in Section IV. It will be noticed that, in contrast to

[2] The contrary statement in an earlier paper (Chandrasekhar 1962, Appendix II) that the second point of bifurcation occurs at $e = 0.96937$, where a pear-shaped sequence (similar to the one along the Jacobian sequence) branches off, is an error. This entire matter of the occurrence of points of bifurcation along the various sequences and their relationship with the conditions for equilibrium which follow from the virial equations of the different orders are clarified in a later paper (Chandrasekhar 1963).

the Jacobi ellipsoids, the Maclaurin spheroids continue to be stable (i.e., in the absence of any dissipative mechanisms) beyond the point ($e = 0.96937$) at which this mode becomes neutrally stable and a pear-shaped sequence presumably[3] branches off. The mode belonging to σ_1^2 subsequently becomes overstable.

III. THE ODD MODES

In Table 3 we list the coefficients of the virials in the expansions of the $\delta\mathfrak{W}_{ij;k}$'s and the δS_{ijk}'s which occur in the equations for the odd modes; this table is a transcription of the last part of Table 2 in Paper I with the simplifications arising from the present equality of a_1 and a_2. And the relevant equations are (cf. Paper I, eqs. [101]–[103])

$$\lambda^3 V_{123} + \lambda^2\Omega(V_{113} - V_{223}) + \lambda(\delta S_{123} - 2\Omega^2 V_{123}) + 2\Omega(\delta\mathfrak{W}_{23;2} - \delta\mathfrak{W}_{13;1}) = 0\,, \tag{16}$$

$$\lambda^3(V_{113} - V_{223}) - 4\lambda^2\Omega V_{123} + \lambda[-2\Omega^2(V_{113} - V_{223}) + \delta S_{113} - \delta S_{223}] + 4\Omega(\delta\mathfrak{W}_{13;2} + \delta\mathfrak{W}_{23;1}) = 0\,, \tag{17}$$

$$\lambda^4(V_{113} + V_{223} - \tfrac{2}{3}V_{333}) + \lambda^2[2\Omega^2(V_{113} + V_{223}) + \delta S_{113} + \delta S_{223}] - 8\Omega^2(\delta\mathfrak{W}_{13;1} + \delta\mathfrak{W}_{23;2}) = 0\,. \tag{18}$$

It will be observed that in equation (18) the term in λV_{123} is absent; the reason is that $\delta\mathfrak{W}_{13;2} = \delta\mathfrak{W}_{23;1}$ for spheroids (see Table 3).

Equations (16)–(18) must be further supplemented by the divergence condition (cf. Paper I, eq. [129])

$$V_{113} + V_{223} = -\frac{a_1^2}{a_3^2} V_{333}\,. \tag{19}$$

Considering equations (16)–(19), we first observe that equations (16) and (17) are independent of the remaining two equations; for, according to Table 3,

$$\delta S_{113} - \delta S_{233} = [3(B_{11} + B_{13}) + (5a_1^2 + a_3^2)B_{113}](V_{113} - V_{223})$$

and

$$-2\delta\mathfrak{W}_{13;1} + 2\delta\mathfrak{W}_{23;2} = 2(B_{13} + a_1^2 B_{113})(V_{113} - V_{223})\,. \tag{20}$$

Equations (16) and (17), therefore, involve only V_{113}–V_{223} and V_{123}; thus

$$\{\lambda^3 + \lambda[3(B_{11} + B_{13}) + (5a_1^2 + a_3^2)B_{113} - 2\Omega^2]\}(V_{113} - V_{223}) - 4\Omega[\lambda^2 + 2(B_{13} + a_1^2 B_{113})]V_{123} = 0\,, \tag{21}$$

and

$$\{\lambda^3 + \lambda[2B_{11} + 4B_{13} + (4a_1^2 + 2a_3^2)B_{113} - 2\Omega^2]\}V_{123} + \Omega[\lambda^2 + 2(B_{13} + a_1^2 B_{113})](V_{113} - V_{223}) = 0\,. \tag{22}$$

Before writing down the characteristic equation which follows from these two equations, we may note that the coefficient of $(V_{113} - V_{223})$ in equation (21) and of V_{123} in equation (22) are the same; for their difference clearly vanishes:

$$B_{11} - B_{13} + (a_1^2 - a_3^2)B_{113} = 0\,. \tag{23}$$

[3] For this expression of implied doubt see Chandrasekhar (1963, Sec. II).

Accordingly, we may write

$$\lambda^2[\lambda^2 + 3(B_{11} + B_{13}) + (5a_1^2 + a_3^2)B_{113} - 2\Omega^2]^2 + 4\Omega^2[\lambda^2 + 2(B_{13} + a_1^2B_{113})]^2 = 0 . \tag{24}$$

Writing $-\sigma^2$ in place of λ^2 (so that a real σ implies stability), we can factorize equation (24) to give

$$\sigma^3 \pm 2\Omega\sigma^2 - [3(B_{11} + B_{13}) + (5a_1^2 + a_3^2)B_{113} - 2\Omega^2]\sigma \mp 4\Omega(B_{13} + a_1^2B_{113}) = 0 . \tag{25}$$

It should be noted that the three characteristic frequencies determined by equation (25) are independent of the divergence condition.

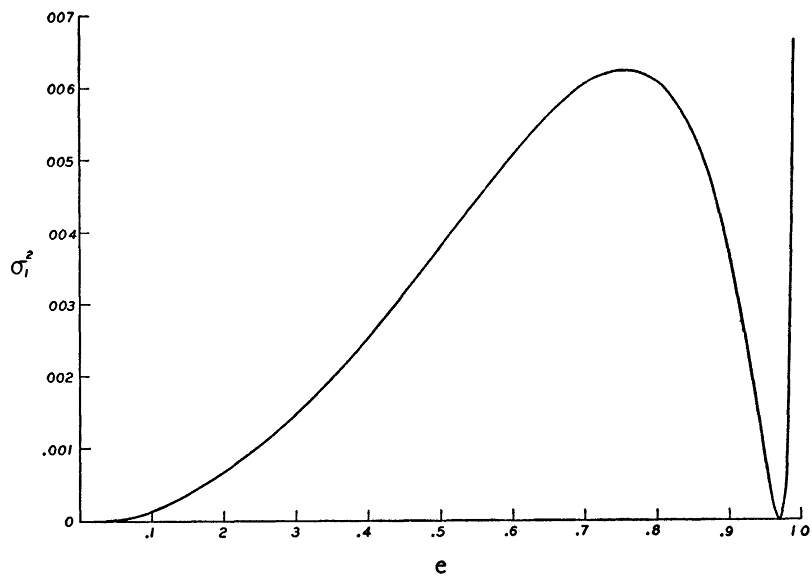

FIG. 1.—The square of the characteristic frequency belonging to a mode which becomes neutral at $e = 0\,96937$, where a pear-shaped sequence branches off This figure should be contrasted with Fig 1 in Paper I.

Equations (18) and (19) determine the two remaining characteristic frequencies belonging to the odd modes. Since the equality of V_{113} and V_{223} is now required for consistency with equations (21) and (22), the divergence condition takes the form

$$V_{113} = V_{223} = -\frac{a_1^2}{2a_3^2} V_{333} , \tag{26}$$

and equation (18) gives (cf. Table 3)

$$\left(\tfrac{2}{3} + \frac{a_1^2}{a_3^2}\right)\sigma^4 + \Big\{2[(a_1^2 + 2a_3^2)B_{133} - 5a_3^2B_{333} - 2B_{33}] - \frac{a_1^2}{a_3^2}[2\Omega^2 + 3(B_{11} + B_{13}) + (7a_1^2 + 5a_3^2)B_{113} - 6a_3^2B_{113}]\Big\}\sigma^2 - 8\Omega^2\left[a_1^2B_{133} - \frac{a_1^2}{a_3^2}(B_{13} + 2a_1^2B_{113})\right] = 0 , \tag{27}$$

where we have again written $-\sigma^2$ in place of λ^2.

IV. THE CHARACTERISTIC FREQUENCIES

The characteristic equations derived in Sections II and III have been solved for all their roots for some twenty-five members of the Maclaurin sequence. In Tables 4 and 5 the results of the calculation are given. And in Figures 1, 2, and 3 the variation of the squares of the different characteristic frequencies along the sequence is illustrated.

The enumeration of the roots in the tables and in the figures has been chosen to agree, at the point of bifurcation ($e = 0.8127$), with that adopted for the Jacobian sequence in Paper I. Thus, with the adopted enumeration, the entries in the lines opposite $e = 0.81267$ in Tables 4 and 5 agree with those in the first lines in Tables 3 and 4 in Paper I.

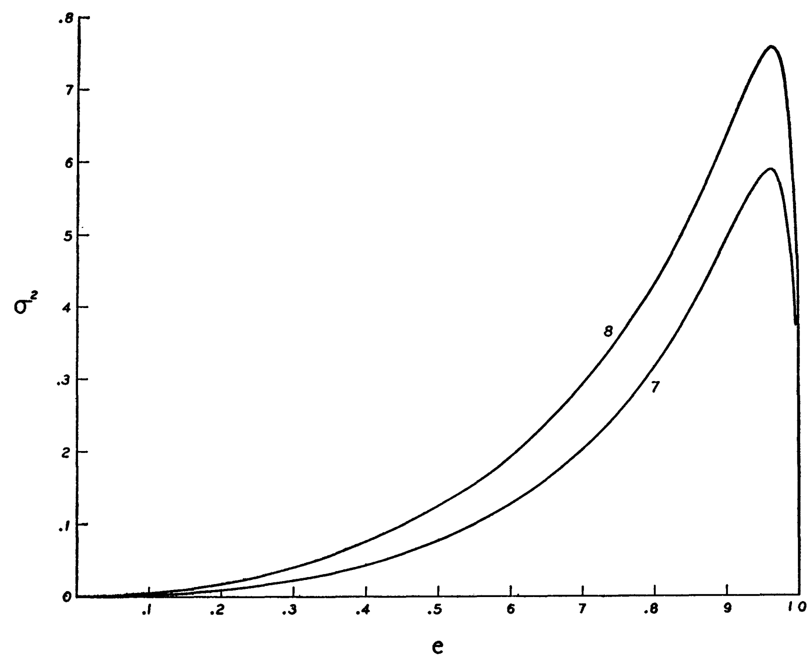

FIG. 2 —The squares of the characteristic frequencies belonging to the odd modes "7" and "8." (The labeling of the curves corresponds to the enumeration in Tables 4 and 5.)

It may be recalled here that the six even modes include two (belonging to σ_2^2 and σ_5^2) which are independent of the divergence conditions. One of these two modes (the one belonging to σ_2^2) becomes neutral at $e = 0.8993$; they coincide at $e = 0.9669$, at which point they become overstable.

The "lowest" of the even modes (belonging to σ_1^2) becomes neutral at $e = 0.96937$. This mode is the analogue of the one with respect to which the Jacobi ellipsoids become unstable. But, unlike the Jacobi ellipsoids, the Maclaurin spheroids continue to be stable (in the absence of any dissipative mechanism) beyond the point at which it becomes neutral. However, the mode belonging to σ_1^2 eventually becomes overstable by first coinciding with the mode belonging to σ_3^2 (see the entries in Table 4 opposite to $e = 0.995$, 0.999, and 0.9999).

Turning to the five odd modes, we may again recall that among them there are three which are independent of the divergence condition. And none of them exhibits any instability or neutrality.

TABLE 4

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE SIX EVEN MODES

(σ^2 Is Listed in the Unit $\pi G\rho$)

e	σ_2^2	σ_5^2	σ_1^2	σ_3^2	σ_4^2	σ_6^2
0 .	2 2857	2 2857	0	0	2 2857	2 2857
0 2	1 8031	2 6756	0 0006580	0 04947	2 1596	2 4515
3	1 5358	2 8233	0014569	0 11376	2 1145	2 5472
4	1 2554	2 9284	0025216	0 20872	2 0815	2 6468
5	0 96562	2 97773	0037716	0 34029	2 06385	2 74774
6	0 67256	2 94996	0050513	0 51768	2 06271	2 84001
7	0 38727	2 80756	0060504	0 75473	2 07634	2 90106
8. . .	0 13539	2 47500	0061138	1 06894	2 08672	2 86996
81267	0 10863	2 41205	0059941	1 11478	2 08452	2 85227
82	0 09397	2 37278	0059063	1 14186	2 08244	2 83982
84	0 05765	2 25370	0055876	1 21756	2 07269	2 79598
86	0 02806	2 11425	0051348	1 29509	2 05444	2 73403
88	0 00759	1 94957	0045207	1 37249	2 02303	2 64765
90	0 00001	1 75257	0037156	1 44598	1 97096	2 52756
92	0 01195	1 51185	0026947	1 50764	1 88564	2 35969
94	0 05778	1 20576	0014712	1 53862	1 74463	2 12353
95	0 10508	1 01297	.....	....	....	....
96 . .	0 19248	0 76686	0002634	1 47191	1 50443	1 80677
966956	0 41964	±0 006682i	0000225	1 37825	1 38590	1 69349
97	0.39187	±0 18482i	0000017	1 32124	1 32124	1 64709
98	0 29252	±0 35955i	0008046	1 06879	1 02956	1 48787
99	0 17511	±0 40964i	0066392	0 70002	0 58020	1 22690
995	0 10415	±0 36886i	0 022752	0 44050	0 26623	0 96870
999	0 03210	±0 21634i	+0 005045	±0 069545i	0 14918	0 50094
0 9999	0 007105	±0 07829i	−0 013570	±0 02932i	0 03747	0 17141

TABLE 5

THE SQUARES OF THE CHARACTERISTIC FREQUENCIES BELONGING TO THE FIVE ODD MODES

(σ^2 Is Listed in the Unit $\pi G\rho$)

e	σ_7^2	σ_8^2	σ_9^2	σ_{10}^2	σ_{11}^2
0 .	0	0	2 2857	2 2857	2 2857
0 2	0 00989	0 01746	1 9891	2 3138	2 5722
3	02350	04047	1 8413	2 3483	2 7048
4	04510	07510	1 6964	2.3985	2 8255
5	07792	12433	1 55478	2 46126	2 92655
6	12729	19307	1 41716	2 53361	2 99676
7	20202	28950	1 28285	2 60493	3 01517
8 . .	31659	42767	1 14689	2 64104	2 93615
81267	33494	44920	1 12881	2 63888	2 91452
82	34602	46214	1 11817	2 63645	2 90033
84	37798	49933	1 08824	2 62446	2 85443
86	41253	53939	1 05653	2 60238	2 79595
88	44956	58233	1 02216	2 56629	2 72134
90	48855	62780	0 98365	2 51026	2 62529
92	52813	67468	0 93850	2 42467	2 49935
94	56476	71968	0 88182	2 29256	2 32863
.95	57944	73894	0 84622	2 19954	2 21808
96 . .	58875	75290	0 80267	2 07924	2 08240
966956.	58968	75686	0 76535	1 97283	1 96719
97	58790	75622	0 74647	1 91840	1 90956
.98	56629	73643	0 66758	1 69049	1 67495
99.	49319	65502	0 53745	1 32462	1 31088
.995	40165	54388	0 41983	1 01119	1 00306
999.	21483	29971	0 21681	0 50595	0 50463
0 9999	0 07458	0 10578	0 07465	0 17159	0 17154

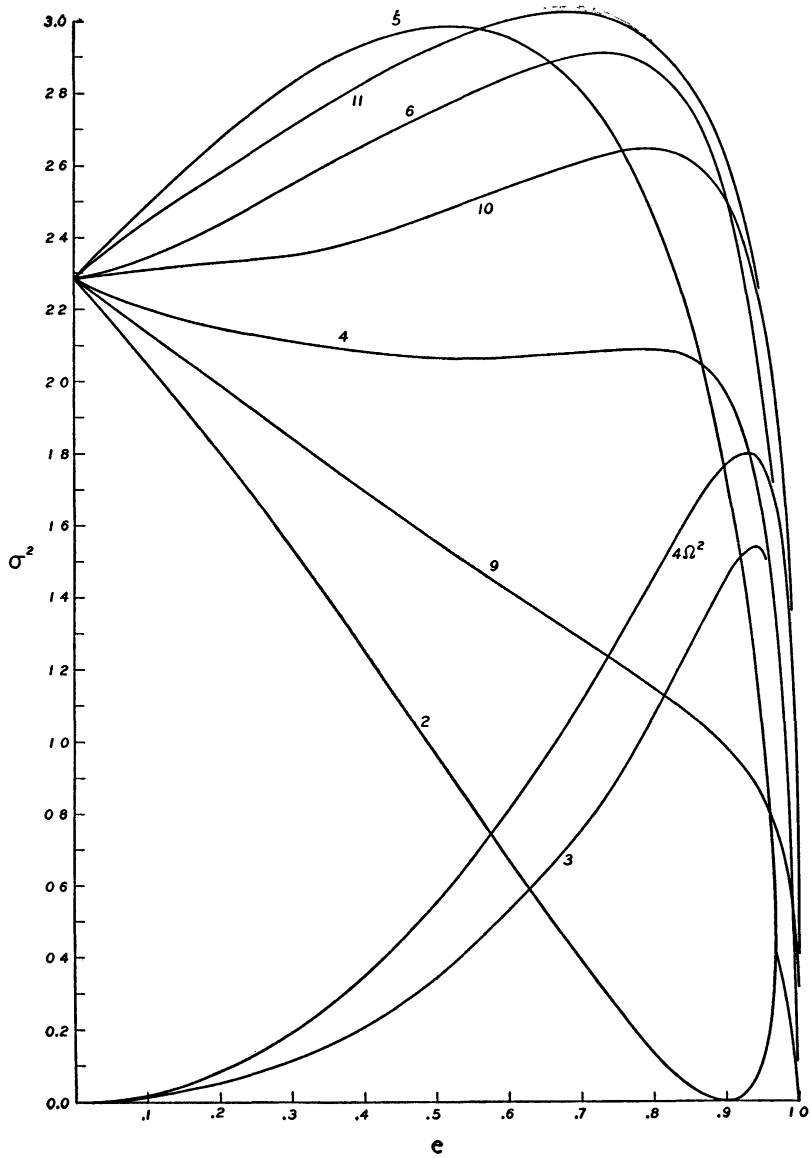

FIG. 3.—The squares of the characteristic frequencies belonging to the modes by which they are labeled. The mode labeled "2" becomes neutral at $e = 0.89926$; and this mode subsequently becomes coincident with the mode "5" at $e = 0\ 96696$. The mode labeled "3" subsequently becomes coincident with the mode "1" (see Fig. 1).

We are again grateful to Miss Donna Elbert for having carried out the solution of the many equations whose roots are listed in Tables 4 and 5.

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