ON THE OSCILLATIONS AND THE STABILITY OF ROTATING GASEOUS MASSES

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ABSTRACT

In this paper the oscillations and the stability of a rotating gaseous mass are considered on the basis of an appropriate tensor form of the virial theorem. On the assumption that the Lagrangian displacement $\boldsymbol{\xi}$ can be expressed in the form

$$\xi_j = X_{jr} x_r e^{\lambda t} \qquad (X_{jr} \text{ constants}),$$

a characteristic equation for λ (of order eighteen) is derived from the nine integral relations provided by the virial theorem. An examination of the roots of this characteristic equation enables the enumeration of the properties of all the natural modes of oscillation belonging essentially to harmonics not higher than the second. It is shown that there are three principal groups among these modes: a group of three modes, each of which exhibits a doublet character; a group of two modes, one of which becomes neutral at a point where the condition for the occurrence of a point of bifurcation is satisfied and both of which become overstable at a higher angular velocity; and a group which represents the coupling of two modes, one of which is purely radial and the other of which is purely non-radial in the absence of rotation. In addition to these modes, there are two "trivial" modes, one of which is neutral and the other of which has a characteristic frequency equal to the angular velocity.

I. INTRODUCTION

The stability of rotating incompressible fluid masses has been the subject of many investigations; and the role of the point of bifurcation (at which the Jacobi ellipsoids branch off from the sequence of the Maclaurin spheroids) in determining stability has attracted much attention. In a recent paper (Lebovitz 1961; this paper will be referred to hereafter as "Paper I") the principal results of the classical investigations bearing on the point of bifurcation were clarified by an explicit evaluation of the frequencies of all the normal modes belonging to the second harmonic. The relative simplicity of the methods used in Paper I has encouraged us to attempt, by similar methods, the more general problem of the stability of rotating gaseous masses. The importance of this general problem for the "wider aspects of cosmogony" requires no emphasis, but it has been emphasized by Ledoux (1951; see also Ledoux and Walraven 1958) that the same problem is relevant for certain specific and practical questions raised by variable stars which exhibit the phenomena of multiple periods and beats. In this paper we shall survey the fundamental problems; in later papers we shall return to detailed examinations of specific questions.

II. THE VIRIAL THEOREM FOR SMALL PERTURBATIONS ABOUT EQUILIBRIUM OF ROTATING BODIES

In a frame of reference rotating with a constant angular velocity $\boldsymbol{\Omega}$, the virial theorem takes the form (Chandrasekhar 1960, 1961*a*, *b;* and Paper I)

$$\frac{d}{dt}\int_V \rho x_i u_j d\boldsymbol{x} = 2\mathfrak{T}_{ij} + \mathfrak{W}_{ij} + \delta_{ij}\int_V p\, d\boldsymbol{x} + \Omega^2 I_{ij} - I_{il}\Omega_l\Omega_j + 2\int_V \rho x_i \epsilon_{jlm} u_l \Omega_m d\boldsymbol{x}, \quad (1)$$

where

$$\mathfrak{T}_{ij} = \frac{1}{2}\int_V \rho u_i u_j d\boldsymbol{x}, \quad (2)$$

$$\mathfrak{W}_{ij} = -\frac{1}{2}\int_V \rho \mathfrak{B}_{ij} d\boldsymbol{x}, \quad (3)$$

and

$$I_{ij}=\int_V \rho x_i x_j dx \tag{4}$$

are the kinetic energy, the potential energy, and the moment of inertia tensors, respectively. In equation (3) $\mathfrak{B}_{ij}$ is the tensor potential discussed at length in the preceding paper (Chandrasekhar and Lebovitz 1962; this paper will be referred to hereafter as "Paper II"), and the remaining symbols have their usual meanings.

Before we write down the form which equation (1) takes for infinitesimal perturbations about equilibrium, we may note the relations that obtain in equilibrium. They are (since u_i is now zero)

$$\mathfrak{W}_{ij}+\Omega^2 I_{ij}-I_{il}\Omega_l\Omega_j=-\delta_{ij}\int_V p\,dx\,. \tag{5}$$

If the x_3-axis is chosen in the direction of $\boldsymbol{\Omega}$,

$$\Omega_j=\Omega\,\delta_{j3}\,, \tag{6}$$

and equation (5) gives

$$\mathfrak{W}_{11}+\Omega^2 I_{11}=\mathfrak{W}_{22}+\Omega^2 I_{22}=\mathfrak{W}_{33}=-\int_V p\,dx\,. \tag{7}$$

It has been shown in a different connection (Chandrasekhar 1961*b*) how we can deduce from equations (7) explicit expressions for the different components of the potential energy tensor in case the equation of state is polytropic.

Now suppose that the equilibrium is disturbed and that, as a result, each element of mass dm $(=\rho dx)$ suffers a Lagrangian displacement $\boldsymbol{\xi}(\boldsymbol{x}, t)$. Let δI_{ij}, $\delta\mathfrak{W}_{ij}$, and

$$\delta\int_V p\,dx$$

be the first-order changes in the respective quantities caused by the perturbation; then, equation (1) gives

$$\frac{d^2}{dt^2}\int_V \rho x_i \xi_j dx=\delta\mathfrak{W}_{ij}+\delta_{ij}\left(\delta\int_V p\,dx\right)+\Omega^2\delta I_{ij} \\ -\delta I_{il}\Omega_l\Omega_j+2\,\frac{d}{dt}\int_V \rho x_i \epsilon_{jlm}\xi_l\Omega_m dx\,. \tag{8}$$

The required first-order changes in I_{ij}, etc., are given by (cf. Chandrasekhar 1961*a*, § 118, and Paper I, § III)

$$\delta I_{ij}=\int_V \rho(x_i\xi_j+\xi_i x_j)\,dx\,, \tag{9}$$

$$\delta\mathfrak{W}_{ij}=-\int_V \rho\,\xi_l\,\frac{\partial\mathfrak{B}_{ij}}{\partial x_l}\,dx\,, \tag{10}$$

and

$$\delta\int_V p\,dx=\int_V \delta\left(\frac{p}{\rho}\right)\rho\,dx=-(\gamma-1)\int_V p\,\operatorname{div}\boldsymbol{\xi}\,dx\,, \tag{11}$$

where γ denotes the ratio of the specific heats. In deriving the last of the foregoing equations, the assumption has been made that the perturbation is accompanied by adiabatic changes, so that

$$\frac{\delta p}{p}=\gamma\,\frac{\delta\rho}{\rho}=-\gamma\,\operatorname{div}\boldsymbol{\xi}\,. \tag{12}$$

Inserting the results (9)–(11) in equation (8), we obtain the desired equation:

$$\frac{d^2}{dt^2}\int_V \rho x_i \xi_j dx = -\int_V \rho \xi_l \frac{\partial \mathfrak{B}_{ij}}{\partial x_l} dx - (\gamma - 1)\,\delta_{ij}\int_V p \operatorname{div} \xi dx$$
$$+\Omega^2\int_V \rho(x_i\xi_j + \xi_i x_j)\,dx - \Omega_j\Omega_l\int_V \rho(x_l\xi_i + \xi_l x_i)\,dx + 2\,\frac{d}{dt}\int_V \rho x_i \epsilon_{jlm}\xi_l\Omega_m dx\,. \qquad (13)^1$$

III. THE METHOD OF APPROXIMATION

Equation (13) is an *exact* integral relation which must be satisfied in all cases of infinitesimal perturbations; in particular, it must be satisfied by the proper solutions belonging to the natural modes of oscillation of the system. Since the solutions belonging to the natural modes will have a time dependence of the form

$$e^{\lambda t}\,, \qquad (14)$$

it would appear that, by inserting for ξ in equation (13) a "trial function" with a space dependence which one might use, for example, in a variational treatment of the problem (i.e., if the problem should allow one), we should obtain an equation which would enable us to determine the frequencies of oscillation[2] with some precision. Clearly, this method has no "absolute" basis such as those which are derived from a strict minimal (or maximal) principle. And, moreover, when basing on equation (13), we cannot use "trial" functions with as many variational parameters as we like. In the present instance, since equation (13) provides nine equations, a "trial" function which we may wish to insert cannot involve more than nine constants. And the simplest "trial" function that suggests itself in the present connection is (cf. Chandrasekhar 1961*a*)

$$\xi_j = X_{jr} x_r e^{\lambda t}\,; \qquad (15)$$

and the nine coefficients of this linear transformation play, in the present treatment, the role of the variational parameters in a proper variational treatment.

In assuming a "trial" function of the form (15), we have been guided by two considerations: first, it is known from the investigations of Ledoux (1945) that the purely radial oscillations of a non-rotating gas sphere treated on the basis of the usual contracted version of the virial theorem and the simple substitution,

$$\xi_j = \text{Constant}\; x_j e^{\lambda t}\,, \qquad (16)$$

[1] In this equation it is customary (cf. Chandrasekhar 1961*a*) to transform the integral

$$\int_V p \operatorname{div} \xi dx$$

by an integration by parts to

$$-\int_V \xi \cdot \operatorname{grad} p dx\,;$$

then, after substitution for grad p from the equation of hydrostatic equilibrium, express the integral in terms of equilibrium quantities. However, in the present connection it is more convenient to leave the term as it is.

[2] For reasons which have been explained in Paper I, by the use of the virial theorem in the form of eq. (13), we are, in the case of a configuration of uniform density, limited to the modes of oscillation that belong to the second harmonic.

leads to a formula for λ^2 (eq. [44] below) which agrees exactly with what follows from a strict variational treatment based on the same trial function (cf. Ledoux and Pekeris 1941); and, second, the substitution (15), in fact, corresponds to the *exact* solution in case the density is uniform. For these reasons the results to be derived on the basis of the substitution (15) are not likely to be seriously in error: certainly, the qualitative features of the theory should be trustworthy.

Inserting, then, in equation (13) the form (15) for ξ, we obtain

$$\lambda^2 X_{jl} I_{li} = 2\lambda\Omega\epsilon_{jl3} X_{lm} I_{mi} + \Omega^2 (X_{jl} I_{li} + X_{il} I_{lj}) - \Omega^2 \delta_{j3} (X_{3l} I_{li} + X_{il} I_{l3}) - X_{lr} \mathfrak{W}_{rl;ij} + J X_{rr} \delta_{ij} , \tag{17}$$

where the choice of the co-ordinates appropriate for the direction of $\boldsymbol{\Omega}$ to coincide with the x_3-axis has been made. Moreover, in equation (17),

$$\mathfrak{W}_{rl;ij} = \int_V \rho x_r \frac{\partial \mathfrak{B}_{ij}}{\partial x_l} dx \tag{18}$$

is the super-matrix considered in Paper II, X_{rr} is the trace of $\boldsymbol{X}$, and

$$J = -(\gamma - 1) \int_V p\, dx . \tag{19}$$

If use is made of the last of the three equilibrium conditions given in equations (7), we can write

$$J = (\gamma - 1) \mathfrak{W}_{33} . \tag{20}$$

In our further considerations, we shall suppose that the equilibrium configuration has axial symmetry about the x_3-axis (i.e., the axis of rotation) and also that the (x_1, x_2)-plane is a plane of symmetry. Under these circumstances the tensors $\mathfrak{W}_{ij}$ and $\mathfrak{W}_{pq;ij}$ have the symmetries listed in Paper II (§ 6); and the moment-of-inertia tensor is, of course, also diagonal with $I_{11} = I_{22}$. With the simplifications introduced by these various symmetries, the nine equations which equation (17) represents take the following explicit forms:

$$\lambda^2 X_{11} I_{11} = +2\lambda\Omega X_{21} I_{11} + 2\Omega^2 X_{11} I_{11} + J X_{rr} - (X_{11} \mathfrak{W}_{11;11} + X_{22} \mathfrak{W}_{22;11} + X_{33} \mathfrak{W}_{33;11}) , \tag{21}$$

$$\lambda^2 X_{22} I_{11} = -2\lambda\Omega X_{12} I_{11} + 2\Omega^2 X_{22} I_{11} + J X_{rr} - (X_{11} \mathfrak{W}_{22;11} + X_{22} \mathfrak{W}_{11;11} + X_{33} \mathfrak{W}_{33;11}) , \tag{22}$$

$$\lambda^2 X_{33} I_{33} = J X_{rr} - (X_{11} + X_{22}) \mathfrak{W}_{33;11} - X_{33} \mathfrak{W}_{33;33} , \tag{23}$$

$$\lambda^2 X_{21} I_{11} = -2\lambda\Omega X_{11} I_{11} + (\Omega^2 I_{11} - \mathfrak{W}_{12;12}) (X_{21} + X_{12}) , \tag{24}$$

$$\lambda^2 X_{12} I_{11} = +2\lambda\Omega X_{22} I_{11} + (\Omega^2 I_{11} - \mathfrak{W}_{12;12}) (X_{12} + X_{21}) , \tag{25}$$

$$\lambda^2 X_{31} I_{11} = -X_{13} \mathfrak{W}_{31;13} - X_{31} \mathfrak{W}_{13;13} , \tag{26}$$

$$\lambda^2 X_{32} I_{11} = -X_{23} \mathfrak{W}_{31;13} - X_{32} \mathfrak{W}_{13;13} , \tag{27}$$

$$\lambda^2 X_{13} I_{33} = +2\lambda\Omega X_{23} I_{33} + (\Omega^2 I_{33} - \mathfrak{W}_{31;13}) X_{13} + (\Omega^2 I_{11} - \mathfrak{W}_{13;13}) X_{31} , \tag{28}$$

$$\lambda^2 X_{23} I_{33} = -2\lambda\Omega X_{13} I_{33} + (\Omega^2 I_{33} - \mathfrak{W}_{31;13}) X_{23} + (\Omega^2 I_{11} - \mathfrak{W}_{13;13}) X_{32} . \tag{29}$$

IV. THE MODES OF OSCILLATION OF A SPHERICAL MASS OF GAS IN THE ABSENCE OF ROTATION

Before we proceed to a general consideration of equations (21)–(29), it will be instructive to examine them in the special case when $\Omega = 0$ and the unperturbed configuration is spherical. Then

$$I_{11} = I_{22} = I_{33} = \tfrac{1}{3} I \text{ (say)}, \tag{30}$$

and

$$\mathfrak{W}_{11} = \mathfrak{W}_{22} = \mathfrak{W}_{33} = \tfrac{1}{3}\mathfrak{W} ; \tag{31}$$

and there will be only three distinct components of the tensor $\mathfrak{W}_{pq;ij}$. The typical elements of $\mathfrak{W}_{pq;ij}$ are (cf. Paper II, eqs. [74]–[76])

$$\mathfrak{W}_{11;11} = -\tfrac{1}{15}\mathfrak{W}, \qquad \mathfrak{W}_{11;22} = \tfrac{1}{5}\mathfrak{W}, \qquad \text{and} \qquad \mathfrak{W}_{12;12} = -\tfrac{2}{15}\mathfrak{W} . \tag{32}$$

The equations governing the non-diagonal elements of $\boldsymbol{X}$ (namely, eqs. [24]–[29]) now become

$$\tfrac{1}{3}\lambda^2 I X_{21} = \tfrac{2}{15}\mathfrak{W}(X_{12} + X_{21}) \tag{33}$$

and

$$\tfrac{1}{3}\lambda^2 I X_{12} = \tfrac{2}{15}\mathfrak{W}(X_{21} + X_{12}) ; \tag{34}$$

and two similar pairs which are obtained by cyclically permuting the indices. From equations (33) and (34) it follows that

$$\lambda^2 = 0 \qquad (\text{if } X_{21} \neq X_{12}), \tag{35}$$

and

$$\lambda^2 = \tfrac{4}{5}\frac{\mathfrak{W}}{I} \qquad (\text{if } X_{21} = X_{12}); \tag{36}$$

and both these roots are repeated three times.

Turning next to equations (21)–(23) we have

$$\tfrac{1}{3}\lambda^2 I X_{11} = J X_{rr} + \tfrac{1}{15}\mathfrak{W} X_{11} - \tfrac{1}{5}\mathfrak{W}(X_{22} + X_{33}), \tag{37}$$

$$\tfrac{1}{3}\lambda^2 I X_{22} = J X_{rr} + \tfrac{1}{15}\mathfrak{W} X_{22} - \tfrac{1}{5}\mathfrak{W}(X_{33} + X_{11}), \tag{38}$$

and

$$\tfrac{1}{3}\lambda^2 X_{33} = J X_{rr} + \tfrac{1}{15}\mathfrak{W} X_{33} - \tfrac{1}{5}\mathfrak{W}(X_{11} + X_{22}), \tag{39}$$

where it may be recalled that now (cf. eqs. [20] and [31])

$$J = \tfrac{1}{3}(\gamma - 1)\mathfrak{W} \qquad \text{and} \qquad X_{rr} = X_{11} + X_{22} + X_{33} . \tag{40}$$

By subtracting equation (38) from equation (37), we have

$$\tfrac{1}{3}\lambda^2 I (X_{11} - X_{22}) = \tfrac{4}{15}\mathfrak{W}(X_{11} - X_{22}) . \tag{41}$$

The same equation governs $X_{22} - X_{33}$ and $X_{33} - X_{11}$; but, of these three equations, only two are linearly independent. Therefore, the root

$$\lambda^2 = \tfrac{4}{5}\frac{\mathfrak{W}}{I} \qquad (X_{11} \neq X_{22}), \tag{42}$$

which follows from these equations, is of multiplicity 2.

Finally, adding all three equations (37)–(39), we obtain

$$\tfrac{1}{3}\lambda^2 I X_{rr} = \left(3J + \tfrac{1}{15}\mathfrak{W} - \tfrac{2}{5}\mathfrak{W}\right) X_{rr} = \left(\gamma - \tfrac{4}{3}\right)\mathfrak{W} X_{rr}\,; \tag{43}$$

and this equation leads to the root

$$\lambda^2 = (3\gamma - 4)\frac{\mathfrak{W}}{I}\,. \tag{44}$$

This last root represents a purely radial mode of oscillation; and the formula for λ^2 agrees with the one first derived by Ledoux (1945).

The foregoing discussion has shown that a spherical mass of gas in equilibrium under its own gravitation has two fundamental frequencies of oscillation corresponding to the two roots $\lambda^2 = \tfrac{4}{5}\mathfrak{W}/I$ and $\lambda^2 = (3\gamma - 4)\mathfrak{W}/I$; the former root is of multiplicity 5 and represents all the non-radial modes belonging to the second harmonic; and the latter root is non-degenerate and represents the fundamental of the purely radial modes. In addition, we have three neutral modes belonging to $\lambda^2 = 0$; the existence of three such neutral modes corresponds to the possibility of arbitrary rotations about three perpendicular axes. And, finally, it should be noted that when $\gamma = 1.6$ the two non-vanishing roots coincide and a case of *accidental degeneracy* arises.[3]

V. THE TRANSVERSE SHEAR MODES

Returning to equations (21)–(29), we observe that equations (26)-(29) involving X_{31}, X_{32}, X_{13}, and X_{23} are independent of the others; we have, therefore, four modes of oscillation in which all the elements of the transformation (15), except these four, vanish. The transformation appropriate to these modes is

$$\xi_1 = X_{13}x_3\,, \qquad \xi_2 = X_{23}x_3\,, \qquad \text{and} \qquad \xi_3 = X_{31}x_1 + X_{32}x_2\,, \tag{45}$$

where we have suppressed the time-dependent factor $e^{\lambda t}$. The predominant feature of these modes is the relative shearing of the northern and the southern hemispheres. For this reason, we shall call them the *transverse shear modes*, where the qualification "transverse" is with respect to the direction of $\mathbf{\Omega}$.

The four equations governing X_{31}, X_{32}, X_{13}, and X_{23} can be written in the matrix form

$$\begin{vmatrix} \lambda^2 I_{11} + \mathfrak{W}_{13;13} & 0 & \mathfrak{W}_{31;13} & 0 \\ 0 & \lambda^2 I_{11} + \mathfrak{W}_{13;13} & 0 & \mathfrak{W}_{31;13} \\ -\Omega^2 I_{11} + \mathfrak{W}_{13;13} & 0 & (\lambda^2 - \Omega^2) I_{33} + \mathfrak{W}_{31;13} & -2\lambda\Omega I_{33} \\ 0 & -\Omega^2 I_{11} + \mathfrak{W}_{13;13} & +2\lambda\Omega I_{33} & (\lambda^2 - \Omega^2) I_{33} + \mathfrak{W}_{31;13} \end{vmatrix} \begin{vmatrix} X_{31} \\ X_{32} \\ X_{13} \\ X_{23} \end{vmatrix} = 0\,. \tag{46}$$

The determinant of the matrix on the right-hand side of equation (46) must vanish; and the characteristic equation which follows is

$$\begin{aligned} \{ (\sigma^2 I_{11} - \mathfrak{W}_{13;13}) [(\sigma^2 + \Omega^2) I_{33} - \mathfrak{W}_{31;13}] + (\Omega^2 I_{11} - \mathfrak{W}_{13;13}) \mathfrak{W}_{31;13} \}^2 \\ = 4\sigma^2 \Omega^2 I_{33}^2 (\sigma^2 I_{11} - \mathfrak{W}_{13;13})^2\,, \end{aligned} \tag{47}$$

[3] This occurrence of accidental degeneracy has been deduced on the basis of the present approximative treatment An exact treatment should reveal the same phenomenon; but the value of γ at which it will occur will probably differ from $\gamma = 1.6$.

where we have written

$$\sigma^2 = -\lambda^2 ; \tag{48}$$

so that, for stable oscillations, σ should be real. Equation (47) factorizes to give

$$(\sigma^2 - L)(\sigma^2 + \Omega^2 - M) + (\Omega^2 - L)M = \pm 2\sigma\Omega(\sigma^2 - L), \tag{49}$$

where, for the sake of brevity, we have written

$$L = \frac{\mathfrak{W}_{13;13}}{I_{11}} \quad \text{and} \quad M = \frac{\mathfrak{W}_{31;13}}{I_{33}} . \tag{50}$$

The occurrence of two signs on the right-hand side of equation (49) means that these modes have a doublet character, with $+\Omega$ and $-\Omega$ playing equivalent roles (as in the normal Zeeman effect; cf. Pekeris, Alterman, and Jarosch 1961).

On further simplification, equation (49) reduces to the form

$$(\sigma \mp \Omega)[(\sigma^2 - L)(\sigma \mp \Omega) - M(\sigma \pm \Omega)] = 0 . \tag{51}$$

Therefore,

$$\sigma^2 = \Omega^2 \tag{52}$$

is an allowed characteristic root. The remaining roots are given by

$$(\sigma^2 - L)(\sigma \mp \Omega) - M(\sigma \pm \Omega) = 0 \tag{53}$$

and, expanding this equation, we have

$$\sigma^3 \mp \sigma^2\Omega - \sigma\mu \pm \Omega\nu = 0 , \tag{54}$$

where we have introduced the abbreviations

$$\mu = L + M \quad \text{and} \quad \nu = L - M . \tag{55}$$

With the substitution

$$\sigma = \zeta \pm \tfrac{1}{3}\Omega , \tag{56}$$

equation (54) becomes

$$\zeta^3 - (\tfrac{1}{3}\Omega^2 + \mu)\zeta \mp \tfrac{2}{27}\Omega[\Omega^2 + \tfrac{9}{2}(\mu - 3\nu)] = 0 . \tag{57}$$

The necessary and sufficient conditions for the reality of the roots of this cubic equation are

$$\tfrac{1}{3}\Omega^2 + \mu \geq 0 \tag{58}$$

and

$$4(\tfrac{1}{3}\Omega^2 + \mu)^3 \geq \tfrac{4}{27}\Omega^2[\Omega^2 + \tfrac{9}{2}(\mu - 3\nu)]^2 . \tag{59}$$

On expanding this last inequality and simplifying, we are left with

$$\Omega^4\nu + \Omega^2[\mu^2 - \tfrac{3}{4}(\mu - 3\nu)^2] + \mu^3 \geq 0 . \tag{60}$$

Sufficient conditions for the reality of the roots of equation (51) are, therefore,

$$\mu \geq 0 , \quad \nu \geq 0 , \quad \text{and} \quad 4\nu\mu^3 \geq [\mu^2 - \tfrac{3}{4}(\mu - 3\nu)^2]^2$$

or (61)

$$\frac{1}{2\nu}\{-[\mu^2 - \tfrac{3}{4}(\mu - 3\nu)^2] \pm [\![\{\mu^2 - \tfrac{3}{4}(\mu - 3\nu)^2\}^2 - 4\nu\mu^3]\!]^{1/2}\} < 0 .$$

It can be verified that when $\mu \geq 0$ and $\nu \geq 0$, the last inequalities in (61) will be satisfied if

$$\mu \geq \tfrac{9}{7}\nu . \tag{62}$$

In terms of L and M, sufficient conditions for the stability of these modes are

$$L \geq 0 , \quad M \geq 0 , \quad \text{and} \quad M \leq L \leq 8M . \tag{63}$$

According to the results of Paper II (§ 6), the conditions enumerated in (63) imply

$$\mathfrak{W}_{13;13} = C - \mathfrak{W}_{11} \geq 0 , \quad \mathfrak{W}_{31;13} = C - \mathfrak{W}_{33} \geq 0 , \tag{64}$$

and

$$\frac{C - \mathfrak{W}_{11}}{I_{11}} \geq \frac{C - \mathfrak{W}_{33}}{I_{33}} . \tag{65}$$

By making use of the equilibrium condition

$$\mathfrak{W}_{11} + \Omega^2 I_{11} = \mathfrak{W}_{33} , \tag{66}$$

the inequality (65) can be brought to the form

$$\Omega^2 I_{11}^2 + (I_{33} - I_{11})(C - \mathfrak{W}_{11}) \geq 0 . \tag{67}$$

It is evident that in the limit of zero angular velocity (when $\mathfrak{W}_{11} \to \frac{1}{3}\mathfrak{W}$, $\mathfrak{W}_{33} \to \frac{1}{3}\mathfrak{W}$, and $C \to \frac{1}{5}\mathfrak{W}$) conditions (64) and (67) are fulfilled. Therefore, these modes certainly start being stable.

Finally, we may note that when $\Omega \to 0$, the roots of equation (54) have the limiting behaviors:

$$\sigma_1 = + \sqrt{\mu} \pm \tfrac{1}{2}\Omega\left(1 - \frac{\nu}{\mu}\right), \quad \sigma_2 = - \sqrt{\mu} \pm \tfrac{1}{2}\Omega\left(1 - \frac{\nu}{\mu}\right), \quad \text{and } \sigma_3 = \pm\Omega\frac{\nu}{\mu} . \tag{68}$$

These formulae exhibit in a striking manner the "doublet" character of these modes.

VI. THE TOROIDAL MODES

Returning to the remaining equations (21)–(25), we obtain, on subtracting equation (22) from equation (21),

$$\lambda^2 I_{11}(X_{11} - X_{22}) = 2\lambda\Omega I_{11}(X_{21} + X_{12}) + 2\Omega^2 I_{11}(X_{11} - X_{22}) - (\mathfrak{W}_{11;11} - \mathfrak{W}_{22;11})(X_{11} - X_{22}) . \tag{69}$$

By making use of the relation (cf. Paper II, eq. [60])

$$\mathfrak{W}_{11;11} - \mathfrak{W}_{22;11} = 2\mathfrak{W}_{12;12} , \tag{70}$$

we can rewrite equation (69) in the form

$$[\lambda^2 I_{11} - 2(\Omega^2 I_{11} - \mathfrak{W}_{12;12})](X_{11} - X_{22}) - 2\lambda\Omega I_{11}(X_{21} + X_{12}) = 0 . \tag{71}$$

Next, by the addition of equations (24) and (25), we obtain

$$\lambda^2 I_{11}(X_{12} + X_{21}) = - 2\lambda\Omega I_{11}(X_{11} - X_{22}) + 2(\Omega^2 I_{11} - \mathfrak{W}_{12;12})(X_{21} + X_{12}) , \tag{72}$$

or

$$[\lambda^2 I_{11} - 2(\Omega^2 I_{11} - \mathfrak{W}_{12;12})](X_{12} + X_{21}) + 2\lambda\Omega I_{11}(X_{11} - X_{22}) = 0 . \tag{73}$$

Equations (71) and (73) can be treated together, and they lead to the characteristic equation

$$[\lambda^2 I_{11} - 2(\Omega^2 I_{11} - \mathfrak{W}_{12;12})]^2 + 4\lambda^2\Omega^2 I_{11}^2 = 0 \ . \tag{74}$$

On simplification, equation (74) becomes

$$I_{11}^2\lambda^4 + 4I_{11}\mathfrak{W}_{12;12}\lambda^2 + 4(\Omega^2 I_{11} - \mathfrak{W}_{12;12})^2 = 0 \ . \tag{75}$$

The roots of this equation are

$$\lambda^2 = -2\frac{\mathfrak{W}_{12;12}}{I_{11}} \pm 2\Omega\left(2\frac{\mathfrak{W}_{12;12}}{I_{11}} - \Omega^2\right)^{1/2} . \tag{76}$$

From equation (76) it now follows that

$$\lambda^2 = 0 \text{ is a root when } \Omega^2 = \frac{\mathfrak{W}_{12;12}}{I_{11}} . \tag{77}$$

But we have already seen in Paper II, § VII, that if the sequence of axially symmetric configurations should have a point of bifurcation at which objects with genuine triplanar symmetry branch off, then at such a point the equality, $\Omega^2 I_{11} = \mathfrak{W}_{12;12}$, must obtain. Therefore, in analogy with what happens along the sequence of the Maclaurin spheroids at the point of bifurcation where the Jacobi ellipsoids branch off (cf. Paper I, § VII [d]), we may now conclude that the occurrence of a neutral mode at $\Omega^2 = \mathfrak{W}_{12;12}/I_{11}$ means only that, at the point of bifurcation (should one occur), the associated neutral mode simply carries the axially symmetric configuration over into a neighboring equilibrium configuration of genuine triplanar symmetry.

In the limit $\Omega^2 = 0$, $\mathfrak{W}_{12;12}$ has a finite positive limit, so that in this limit the modes are definitely stable. Should $\mathfrak{W}_{12;12}$ continue to be positive, then, when

$$\Omega^2 I_{11} > 2\mathfrak{W}_{12;12} , \tag{78}$$

the configuration becomes unstable. Also, it can be verified that, should these unstable modes occur, the real part of the frequencies will be Ω, so that instability occurs as overstability.

VII. THE PULSATION MODES

Of the five equations (21)–(25), we have considered in Section VI two linear combinations of them. It remains to consider three other linear combinations; and we shall select them in the following manner.

We have already considered in Section VI the equation resulting from the addition of equations (24) and (25). Now, by the subtraction of one from the other, we obtain

$$\lambda^2(X_{21} - X_{12})I_{11} = -2\lambda\Omega I_{11}(X_{11} + X_{22}) . \tag{79}$$

Next, adding equations (21) and (22) and subtracting from the result equation (23) twice, we obtain

$$\begin{aligned} \lambda^2[(X_{11} + X_{22})I_{11} - 2X_{33}I_{33}] &= 2\lambda\Omega I_{11}(X_{21} - X_{12}) + 2\Omega^2 I_{11}(X_{11} + X_{22}) \\ &- 2(\mathfrak{W}_{33;11} - \mathfrak{W}_{33;33})X_{33} - (\mathfrak{W}_{11;11} + \mathfrak{W}_{22;11} - 2\mathfrak{W}_{33;11})(X_{11} + X_{22}) . \end{aligned} \tag{80}$$

And we shall retain equation (23) as it is:

$$\lambda^2 I_{33}X_{33} = J[(X_{11} + X_{22}) + X_{33}] - \mathfrak{W}_{33;11}(X_{11} + X_{22}) - \mathfrak{W}_{33;33}X_{33} . \tag{81}$$

First, we observe that, according to equation (79),

$$\lambda^2 = 0 \text{ is a root if } X_{21} \neq X_{12}\,, \qquad X_{11} + X_{22} = X_{33} = 0\,. \tag{82}$$

This neutral mode is, in fact, the same one that occurred, under the same circumstances, in the absence of rotation (Sec. IV, eq. [35]); its continued existence in the presence of rotation is clearly to be expected on symmetry grounds.

If we ignore the root $\lambda^2 = 0$, we can eliminate $(X_{12} - X_{21})$ in equation (80) by making use of equation (79); and the equations we must now consider are

$$[\lambda^2 I_{11} + 2\Omega^2 I_{11} + (\mathfrak{W}_{11;11} + \mathfrak{W}_{22;11} - 2\mathfrak{W}_{33;11})](X_{11} + X_{22}) - 2[\lambda^2 I_{33} - (\mathfrak{W}_{33;11} - \mathfrak{W}_{33;33})] X_{33} = 0 \tag{83}$$

and

$$(\lambda^2 I_{33} - J + \mathfrak{W}_{33;33}) X_{33} + (\mathfrak{W}_{33;11} - J)(X_{11} + X_{22}) = 0\,. \tag{84}$$

In the notation of Paper II, § VI,

$$\mathfrak{W}_{33;11} = C \qquad \text{and} \qquad \mathfrak{W}_{33;33} = D\,; \tag{85}$$

and, making use of the various relations listed in the same section, as well as equation (66), we verify that

$$\begin{aligned} -2\Omega^2 I_{11} - (\mathfrak{W}_{11;11} + \mathfrak{W}_{22;11} - 2\mathfrak{W}_{11;33}) &= -2\Omega^2 I_{11} - (A + B - 2C) \\ &= -\Omega^2 I_{11} + C - D\,. \end{aligned} \tag{86}$$

Equations (83) and (84) can now be written in the matrix form

$$\begin{vmatrix} \lambda^2 I_{11} + \Omega^2 I_{11} - C + D & -2\lambda^2 I_{33} + 2(C - D) \\ -J + C & \lambda^2 I_{33} - J + D \end{vmatrix} \begin{vmatrix} X_{11} + X_{22} \\ X_{33} \end{vmatrix} = 0\,. \tag{87}$$

The determinant of the matrix on the left-hand side of equation (87) must vanish, and we find that this leads to the characteristic equation

$$I_{11} I_{33} \lambda^4 - [\beta I_{11} + (\beta + \alpha) I_{33} - \Omega^2 I_{11} I_{33}] \lambda^2 - \Omega^2 I_{11} \beta + (\beta + 2\alpha)(\beta - \alpha) = 0\,, \tag{88}$$

where we have introduced the abbreviations

$$\alpha = J - C \qquad \text{and } \beta = J - D\,. \tag{89}$$

Note that, with the foregoing definition of α and β (cf. eq. [20] and Paper II, eq. [70]),

$$2\alpha + \beta = 3J - (2C + D) = (3\gamma - 4)\mathfrak{W}_{33} \tag{90}$$

and

$$\beta - \alpha = C - D\,. \tag{91}$$

The roots of equation (88) can be readily written down; but greater interest attaches to the coupling between the radial and the non-radial modes of oscillation which equation (88) predicts. The nature of this coupling is best clarified by considering the limit $\Omega^2 \to 0$.

When $\Omega^2 = 0$,

$$I_{11} = I_{33} = \tfrac{1}{3} I\,, \qquad \mathfrak{W}_{33} = \tfrac{1}{3}\mathfrak{W}\,, \qquad \text{and } C - D = \tfrac{4}{15}\mathfrak{W}\,, \tag{92}$$

and equation (88) becomes

$$\tfrac{1}{9} I^2 \lambda^4 - \tfrac{1}{3}(2\beta^0 + \alpha^0) I \lambda^2 + (\beta^0 + 2\alpha^0)(\beta^0 - \alpha^0) = 0\,, \tag{93}$$

where the superscript "0" distinguishes the value of the quantity in the spherically symmetric state. The roots of equation (93) are

$$\lambda^2 = 3\,\frac{\beta^0 + 2\alpha^0}{I} \quad \text{and} \quad \lambda^2 = 3\,\frac{\beta^0 - \alpha^0}{I}; \tag{94}$$

or, making use of equations (90), (91), and (92), we have

$$\lambda^2 = (3\gamma - 4)\frac{\mathfrak{W}}{I} = \lambda_r^2\,(\text{say})$$

and (95)

$$\lambda^2 = \frac{4}{5}\,\frac{\mathfrak{W}}{I} = \lambda_s^2\,(\text{say});$$

these agree with the results for the radial and the non-radial modes derived in Section IV (eqs. [42] and [44]). We now see that *rotation couples these modes*. An important aspect of this coupling is that the non-radial mode, which in the absence of rotation is volume-conserving, is no longer so in the presence of rotation. This fact is apparent from the incompatibility of the equations (87) with the condition $X_{rr} = 0$ if $\Omega \neq 0$.

We shall now determine the extent of the coupling between the radial and the non-radial modes in the limit $\Omega^2 \to 0$. For this purpose we shall first rewrite equation (88) in the form

$$I_{11}I_{33}\lambda^4 - [(\beta - \alpha)I_{11} + (\beta + 2\alpha)I_{33} + \alpha(I_{11} - I_{33}) - \Omega^2 I_{11}I_{33}]\,\lambda^2 - \Omega^2 I_{11}\beta + (\beta + 2\alpha)(\beta - \alpha) = 0\,, \tag{96}$$

or

$$[\lambda^2 I_{11} - (\beta + 2\alpha)][\lambda^2 I_{33} - (\beta - \alpha)] = \lambda^2(\alpha \Delta I - \Omega^2 I_{11} I_{33}) + \Omega^2 I_{11}\beta\,, \tag{97}$$

where

$$\Delta I = I_{11} - I_{33} \tag{98}$$

is the change in the components of the moment of inertia caused by the rotation; ΔI is clearly of order Ω^2.

Now the terms on the right-hand side of equation (97) are all of order Ω^2; therefore, to this order, we can replace the various coefficients (such as I_{11}, α, etc.,) by their values in the absence of rotation, namely,

$$\alpha^0 = J^0 - C^0 = \tfrac{1}{3}(\gamma - 1)\mathfrak{W} - \tfrac{1}{5}\mathfrak{W} = \tfrac{1}{15}(5\gamma - 8)\mathfrak{W},$$
$$\beta^0 = J^0 - D^0 = \tfrac{1}{3}(\gamma - 1)\mathfrak{W} + \tfrac{1}{15}\mathfrak{W} = \tfrac{1}{15}(5\gamma - 4)\mathfrak{W}, \tag{99}$$

and

$$I_{11}^0 = I_{33}^0 = \tfrac{1}{3}I\,.$$

Substituting the foregoing values on the right-hand side of equation (97), we obtain

$$(\lambda^2 - \lambda_R^2)(\lambda^2 - \lambda_S^2) = \tfrac{3}{5}(5\gamma - 8)\frac{\mathfrak{W}}{I}\frac{\Delta I}{I}\,\lambda^2 - \tfrac{1}{5}\Omega^2\left[5\lambda^2 - (5\gamma - 4)\frac{\mathfrak{W}}{I}\right] + O(\Omega^4)\,, \tag{100}$$

where

$$\lambda_R^2 = \frac{\beta + 2\alpha}{I_{11}} \quad \text{and} \quad \lambda_S^2 = \frac{\beta - \alpha}{I_{33}}\,. \tag{101}$$

(Note that the values of λ_R^2 and λ_S^2 differ from their zero-order values, namely, λ_r^2 and λ_s^2 given by equations (95), by quantities of order Ω^2, i.e., by the same order as those retained in the present calculations.)

If the terms on the right-hand side of equation (100) are ignored (which we may not!) the roots of the equation are, of course, λ_R^2 and λ_S^2. Therefore, to the first order in Ω^2, the change in the root λ_R^2, for example, due to the presence of the terms on the right-hand side, is given by

$$\delta\lambda_R^2(\lambda_R^2-\lambda_S^2)=\tfrac{3}{5}(5\gamma-8)\frac{\mathfrak{W}}{I}\frac{\Delta I}{I}\lambda_R^2-\tfrac{1}{5}\Omega^2\left[5\lambda_R^2-(5\gamma-4)\frac{\mathfrak{W}}{I}\right]. \tag{102}$$

In this equation we can clearly replace λ_R^2 and λ_S^2 by their zero-order values λ_r^2 and λ_s^2; and we find

$$\delta\lambda_R^2=\frac{3(5\gamma-8)}{5(\lambda_r^2-\lambda_s^2)}\frac{\mathfrak{W}}{I}\left(\frac{\Delta I}{I}\lambda_r^2-\tfrac{2}{3}\Omega^2\right). \tag{103}$$

In the same way we find

$$\delta\lambda_S^2=-\frac{3(5\gamma-8)}{5(\lambda_r^2-\lambda_s^2)}\frac{\mathfrak{W}}{I}\left(\frac{\Delta I}{I}\lambda_s^2+\tfrac{1}{3}\Omega^2\right). \tag{104}$$

The foregoing formulae do not apply when $\gamma = 1.6$; for, in this case,

$$\lambda_r^2=\lambda_s^2=\tfrac{4}{5}\frac{\mathfrak{W}}{I}, \tag{105}$$

and the terms of order Ω^2 on the right-hand side of equation (100) vanish identically. Therefore, in this case, the required roots are given by equations (101) correctly to the first order in Ω^2. Because α, β, I_{11}, and I_{33} differ (on account of rotation) from their zero-order values, the two roots, λ_R^2 and λ_S^2, which are coincident in the absence of rotation, become separate. The accidental degeneracy which exists when $\gamma = 1.6$ is thus lifted by rotation.

VIII. SUMMARY OF RESULTS

The principal results of the preceding sections are summarized in Table 1, in which a comparison is further made with the corresponding results for incompressible fluids.

IX. CONCLUDING REMARKS

To some extent the theory presented in this paper is a formal one, since it presupposes a knowledge of the structure of the equilibrium configuration; and, except in the case of incompressible fluids, this knowledge is, in large measure, lacking. However, there is one case in which the formulae of this paper can be used to derive concrete results; this is the case of the rotationally distorted polytropes (Chandrasekhar 1933). In this theory of the distorted polytropes, the effect of rotation is treated by a perturbation method valid for small Ω^2. The first-order changes in the pressure, the density, and the gravitational potential have been evaluated and expressed in terms of two functions (ψ_0 and ψ_2 in the theory) which have been tabulated. For our present purposes, this information will have to be further completed by determining the super-potential χ to the same order as the other quantities; but this is a straightforward matter, and we shall return to it in another paper.

TABLE 1

THE CLASSIFICATION OF THE MODES

Modes	Parameters	Non-rotating Mass	Rotating Incompressible Mass	Rotating Compressible Mass
	X_{12}, X_{21}	$X_{12} \neq X_{21}$: $\lambda^2=0$	$\lambda^2=0$	$\lambda^2=0$
Transverse shear modes	X_{13} X_{23} X_{31} X_{32}	$X_{13} \neq X_{31}$: $\lambda^2=0$ $X_{23} \neq X_{32}$: $\lambda^2=0$ $X_{13}=X_{31}$: $\lambda^2=\frac{4}{5}\mathfrak{W}/I$ $X_{23}=X_{32}$: $\lambda^2=\frac{4}{5}\mathfrak{W}/I$	$\lambda^2=-\Omega^2$ $\lambda^2=0$ Two stable modes	$\lambda^2=-\Omega^2$ Three coupled modes, each having a doublet character; probably stable
Toroidal modes	$X_{12}+X_{21}$ $X_{11}-X_{22}$	$X_{21}=X_{12}$: $\lambda^2=\frac{4}{5}\mathfrak{W}/I$ $X_{11}=-X_{22}$: $\lambda^2=\frac{4}{5}\mathfrak{W}/I$	Two modes, one of which becomes neutral at the point of bifurcation ($e=$ 0 81); both become unstable when $e=$ 0.95; real part of frequency beyond instability Ω	Two modes, one of which becomes neutral when $\Omega^2 I_{11} = \mathfrak{W}_{12;12}$; both become unstable when $\Omega^2 I_{11} = 2\mathfrak{W}_{12;12}$; real part of frequency beyond instability Ω
Pulsation modes	$X_{11}+X_{22}$ X_{33}	$X_{22}=-X_{33}$, $X_{11}=0$: $\lambda^2=\frac{4}{5}\mathfrak{W}/I$ $X_{11}=X_{22}=X_{33}$: $\lambda^2=(3\gamma-4)\mathfrak{W}/I$* $X_{11}+X_{22}+X_{33}=0$; $\lambda^2=0$†	Stable mode . . $\lambda^2=0$	The radial and the non-radial modes are coupled

* Compressible case
† Incompressible case

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