AXISYMMETRIC MAGNETIC FIELDS AND FLUID MOTIONS*

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Received March 27, 1956

ABSTRACT

The equations of hydromagnetics appropriate for an incompressible inviscid fluid of finite electrical conductivity are considered in case the magnetic fields and the fluid motions have symmetry about an axis By using a known theorem that a solenoidal vector field can be expressed as a superposition of a poloidal and a toroidal field, the equations are reduced to four equations for four governing scalars. In the limiting case of infinite electrical conductivity, a general relation is derived which includes, as special cases, the known theorems on force-free fields, isorotation, and conditions for hydrostatic equilibrium. Some integral relations which follow from the four equations are also derived.

1. INTRODUCTION

The equations of hydromagnetics governing the velocity ($\boldsymbol{v}$) and the magnetic ($\boldsymbol{H}$) fields in a fluid conductor present problems of considerable mathematical complexity. The complexity arises, in part, from the nonlinearity of the equations. The nonlinear terms are of three kinds: the terms $(\boldsymbol{v}\cdot\text{grad})\boldsymbol{v}$ and $\boldsymbol{H} \times \text{curl}\, \boldsymbol{H}$, representing the inertia and the electromagnetic forces in the equation for $\boldsymbol{v}$, and the term curl $(\boldsymbol{v} \times \boldsymbol{H})$, representing electromagnetic induction in the equation for $\boldsymbol{H}$. It is on this account that most of the hydromagnetic problems which have been solved are those which allow a linearization of the equations. The large class of stability problems which have been successfully investigated in recent years is in this category. In these problems one supposes that an external magnetic field is present, and one examines the behavior of the system in the neighborhood of a well-defined stationary solution of the equations; this leads one to characteristic value problems in linear equations (albeit of high orders). Similarly, in current investigations of the dynamo problem (cf. Bullard and Gellman 1954; Elsasser 1955, 1956) by ignoring the equation for $\boldsymbol{v}$ and assuming $\boldsymbol{v}$ to be given, one reduces the problem to a linear one for characterizing the magnetic field. While the class of such linear problems which one might usefully investigate is by no means exhausted (cf. Chandrasekhar 1956*c*), it would appear that hydromagnetics has now reached a stage of development at which further understanding depends on considering the complete set of equations. In this connection the possibility should not be overlooked that, by not considering all the equations of a problem, we may miss discovering certain essential novelties which result as a consequence of imposing on a system conformity with two different sets of laws: such as conformity with the laws of electrodynamics and hydrodynamics. Thus there is some danger in being too exclusively guided by analogy with the behavior of the system when it obeys only one or the other set of laws. An example of this has recently been found (Chandrasekhar 1956*b*).

It is the object of this paper to make a systematic beginning in the study of the complete set of equations of hydromagnetics. However, the treatment will be restricted to the case in which the fluid motions and the magnetic fields have symmetry about an axis. It should be realized that this restriction to axisymmetry may have far-reaching consequences, such as Cowling's theorem (Cowling 1934; Backus and Chandrasekhar 1956) on the impossibility of a self-excited homogeneous dynamo.

In addition to axisymmetry, it will be supposed that the fluid is incompressible and

* The research reported in this paper has been supported in part by the Geophysics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command, under Contract AF 19(604)-299 with the University of Chicago.

inviscid. The neglect of viscosity is justifiable in most cases (cf. Bullard 1956); but the restriction to incompressibility (except in the context of the earth's core) is not. Nevertheless, the latter assumption is essential if we are not to complicate a problem already beset with considerable mathematical difficulties.

It will be shown that the equations derived under the restrictions stated enable a unified treatment of the diverse "laws" (such as the law of isorotation) which have been discussed in the literature. Also certain general integral relations will be derived which exhibit the net effects of the interaction between the fluid motions and the magnetic field.

2. THE EQUATIONS OF THE PROBLEM

The equations of hydromagnetics appropriate for an incompressible, inviscid fluid of finite electrical conductivity (σ) are well known. They are (cf. Bullard 1956)

$$-\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{curl}\left(\frac{1}{4\pi\sigma}\operatorname{curl}\boldsymbol{H} - \boldsymbol{v}\times\boldsymbol{H}\right) \tag{1}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\operatorname{grad})\,\boldsymbol{v} - \frac{1}{4\pi\rho}\operatorname{curl}\boldsymbol{H}\times\boldsymbol{H} = -\operatorname{grad}\left(\frac{p}{\rho} + \mathfrak{B}\right), \tag{2}$$

where $\mathfrak{B}$ denotes the gravitational potential, ρ the density, and p the pressure. Further,

$$\operatorname{div}\boldsymbol{H} = 0 \quad \text{and} \quad \operatorname{div}\boldsymbol{v} = 0\,. \tag{3}$$

Using the identity

$$(\boldsymbol{v}\cdot\operatorname{grad})\,\boldsymbol{v} = \operatorname{curl}\boldsymbol{v}\times\boldsymbol{v} + \operatorname{grad}\left(\tfrac{1}{2}\,|\,\boldsymbol{v}\,|^2\right), \tag{4}$$

we can rewrite equation (2) in the form

$$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{1}{4\pi\rho}\operatorname{curl}\boldsymbol{H}\times\boldsymbol{H} - \operatorname{curl}\boldsymbol{v}\times\boldsymbol{v} - \operatorname{grad}\left(\frac{p}{\rho} + \tfrac{1}{2}\,|\,\boldsymbol{v}\,|^2 + \mathfrak{B}\right). \tag{5}$$

Instead of $\boldsymbol{H}$, it is convenient to introduce the variable

$$\boldsymbol{h} = \frac{\boldsymbol{H}}{\sqrt{(4\pi\rho)}}, \tag{6}$$

which is of the dimensions of a velocity. It is also convenient to make the equations nondimensional by measuring length in a certain unit R (appropriate for the problem on hand),

$$t \text{ in the unit } 4\pi\sigma R^2$$

and

$$\boldsymbol{h} \text{ and } \boldsymbol{v} \text{ in the unit } (4\pi\sigma R)^{-1}\,. \tag{7}$$

If we are dealing with the earth's core, these units have the values (cf. Bullard 1954)

$$R = 3.47\times 10^8 \text{ cm},$$

$$4\pi\sigma R^2 = 1.44\times 10^5 \text{ years}, \tag{8}$$

and

$$(4\pi\sigma R)^{-1} = 7.65\times 10^{-5} \text{ cm/sec}\,.$$

In the units stated, the equations for $\boldsymbol{h}$ and $\boldsymbol{v}$ can be expressed in the forms

$$\frac{\partial \boldsymbol{h}}{\partial t} = -\operatorname{curl}\boldsymbol{E} \tag{9}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} = \mathfrak{L} - \operatorname{grad} \varpi , \tag{10}$$

where

$$\boldsymbol{E} = \operatorname{curl} \boldsymbol{h} - \boldsymbol{v} \times \boldsymbol{h} , \tag{11}$$

$$\mathfrak{L} = \operatorname{curl} \boldsymbol{h} \times \boldsymbol{h} - \operatorname{curl} \boldsymbol{v} \times \boldsymbol{v} \tag{12}$$

and

$$\varpi = \frac{p}{\rho} + \tfrac{1}{2} | \boldsymbol{v} |^2 + \mathfrak{V} . \tag{13}$$

We can eliminate ϖ from equation (10) by taking its curl; thus

$$\frac{\partial}{\partial t} \operatorname{curl} \boldsymbol{v} = \operatorname{curl} \mathfrak{L} . \tag{14}$$

3. THE EQUATIONS IN THE AXISYMMETRIC CASE

As Lüst and Schlüter (1954; see also Chandrasekhar 1956*a*) have recently pointed out, any axisymmetric solenoidal vector field can be expressed as a superposition of a *poloidal* and a *toroidal* field in terms of two scalars. Thus

$$\boldsymbol{h} = -\varpi \frac{\partial P}{\partial z} \mathbf{1}_\varpi + \varpi T \mathbf{1}_\varphi + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 P) \mathbf{1}_z , \tag{15}$$

and

$$\boldsymbol{v} = -\varpi \frac{\partial U}{\partial z} \mathbf{1}_\varpi + \varpi V \mathbf{1}_\varphi + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 U) \mathbf{1}_z , \tag{16}$$

where ϖ, φ, and z define a system of cylindrical polar co-ordinates (with the axis of symmetry in the z-direction), $\mathbf{1}_\varpi$, $\mathbf{1}_\varphi$, and $\mathbf{1}_z$ are unit vectors along the three principal directions, and P, T, U, and V are four scalars which are azimuth-independent.

As is evident from equations (15) and (16), the field derived from a poloidal scalar has nonvanishing components only in the meridional planes, while the field derived from a toroidal scalar has nonvanishing components only in the transverse φ-direction. Thus U defines currents which are entirely meridional, while V defines motions which are entirely rotational.

When we consider the vorticity of an axisymmetric solenoidal vector field, we find a certain reciprocity: the poloidal field gives rise to a toroidal vorticity, and, conversely, a toroidal field gives rise to a poloidal vorticity. Thus the current (which is proportional to curl $\boldsymbol{H}$) derived from a poloidal magnetic field is toroidal and conversely. Specifically, we have

$$\operatorname{curl} \boldsymbol{h} = -\varpi \frac{\partial T}{\partial z} \mathbf{1}_\varpi - \varpi \Delta_5 P \mathbf{1}_\varphi + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 T) \mathbf{1}_z \tag{17}$$

and

$$\operatorname{curl} \boldsymbol{v} = -\varpi \frac{\partial V}{\partial z} \mathbf{1}_\varpi - \varpi \Delta_5 U \mathbf{1}_\varphi + \frac{1}{\varpi} \frac{\partial}{\partial \varpi} (\varpi^2 V) \mathbf{1}_z , \tag{18}$$

where

$$\Delta_5 = \frac{\partial^2}{\partial \varpi^2} + \frac{3}{\varpi}\frac{\partial}{\partial \varpi} + \frac{\partial^2}{\partial z^2} \tag{19}$$

is the Laplacian operator for axisymmetric functions in five-dimensional Euclidean space.

We now return to equations (9) and (14), which govern $\boldsymbol{h}$ and $\boldsymbol{v}$. First, we find from equations (15), (16), and (17) that

$$\begin{aligned} \boldsymbol{E} &= \operatorname{curl} \boldsymbol{h} - \boldsymbol{v} \times \boldsymbol{h} \\ &= \left\{ -\varpi \frac{\partial T}{\partial z} - V \frac{\partial}{\partial \varpi}(\varpi^2 P) + T \frac{\partial}{\partial \varpi}(\varpi^2 U) \right\} \mathbf{1}_\varpi \\ &+ \left\{ -\varpi \Delta_5 P + \frac{\partial P}{\partial z}\frac{\partial}{\partial \varpi}(\varpi^2 U) - \frac{\partial U}{\partial z}\frac{\partial}{\partial \varpi}(\varpi^2 P) \right\} \mathbf{1}_\varphi \\ &+ \left\{ \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi^2 T) + \varpi^2 T \frac{\partial U}{\partial z} - \varpi^2 V \frac{\partial P}{\partial z} \right\} \mathbf{1}_z \\ &= E_\varpi \mathbf{1}_\varpi + E_\varphi \mathbf{1}_\varphi + E_z \mathbf{1}_z \qquad \text{(say)}; \end{aligned} \tag{20}$$

and, in terms of the components of $\boldsymbol{E}$,

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial E_\varphi}{\partial z} \mathbf{1}_\varpi + \left(\frac{\partial E_\varpi}{\partial z} - \frac{\partial E_z}{\partial \varpi} \right) \mathbf{1}_\varphi + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi E_\varphi)\, \mathbf{1}_z . \tag{21}$$

Equation (9) now gives

$$\begin{aligned} \frac{\partial}{\partial t} &\left\{ -\varpi \frac{\partial P}{\partial z} \mathbf{1}_\varpi + \varpi T \mathbf{1}_\varphi + \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi^2 P)\, \mathbf{1}_z \right\} \\ &= \frac{\partial E_\varphi}{\partial z} \mathbf{1}_\varpi - \left(\frac{\partial E_\varpi}{\partial z} - \frac{\partial E_z}{\partial \varpi} \right) \mathbf{1}_\varphi - \frac{1}{\varpi}\frac{\partial}{\partial \varpi}(\varpi E_\varphi)\, \mathbf{1}_z . \end{aligned} \tag{22}$$

For the defining scalars P and T this equation implies

$$\frac{\partial P}{\partial t} = -\frac{E_\varphi}{\varpi} \tag{23}$$

and

$$\frac{\partial T}{\partial t} = -\frac{1}{\varpi}\left(\frac{\partial E_\varpi}{\partial z} - \frac{\partial E_z}{\partial \varpi} \right). \tag{24}$$

Now, substituting for the components of $\boldsymbol{E}$ in accordance with equation (20), we find, after some reductions, that

$$\frac{\partial P}{\partial t} = \Delta_5 P - \frac{1}{\varpi^3}\frac{\partial(\varpi^2 P, \varpi^2 U)}{\partial(z, \varpi)} \tag{25}$$

and

$$\frac{\partial T}{\partial t} = \Delta_5 T + \frac{1}{\varpi}\left\{ \frac{\partial(V, \varpi^2 P)}{\partial(z, \varpi)} - \frac{\partial(T, \varpi^2 U)}{\partial(z, \varpi)} \right\}. \tag{26}$$

Next, considering equation (14), we find that

$$\begin{aligned}\mathfrak{L} &= \text{curl}\,\boldsymbol{h}\times\boldsymbol{h} - \text{curl}\,\boldsymbol{v}\times\boldsymbol{v}\\ &= \left\{-\Delta_5 P\frac{\partial}{\partial\varpi}(\varpi^2 P) - T\frac{\partial}{\partial\varpi}(\varpi^2 T) + \Delta_5 U\frac{\partial}{\partial\varpi}(\varpi^2 U) + V\frac{\partial}{\partial\varpi}(\varpi^2 V)\right\}\mathbf{1}_\varpi\\ &+\left\{-\frac{\partial P}{\partial z}\frac{\partial}{\partial\varpi}(\varpi^2 T) + \frac{\partial T}{\partial z}\frac{\partial}{\partial\varpi}(\varpi^2 P) + \frac{\partial U}{\partial z}\frac{\partial}{\partial\varpi}(\varpi^2 V) - \frac{\partial V}{\partial z}\frac{\partial}{\partial\varpi}(\varpi^2 U)\right\}\mathbf{1}_\varphi \qquad (27)\\ &+\left\{-\varpi^2 T\frac{\partial T}{\partial z} - \varpi^2\frac{\partial P}{\partial z}\Delta_5 P + \varpi^2 V\frac{\partial V}{\partial z} + \varpi^2\frac{\partial U}{\partial z}\Delta_5 U\right\}\mathbf{1}_z\\ &= \mathfrak{L}_\varpi\mathbf{1}_\varpi + \mathfrak{L}_\varphi\mathbf{1}_\varphi + \mathfrak{L}_z\mathbf{1}_z \qquad \text{(say)}.\end{aligned}$$

In terms of the components of $\mathfrak{L}$, equation (14) gives (cf. eqs. [18] and [21])

$$\begin{aligned}\frac{\partial}{\partial t}\left\{-\varpi\frac{\partial V}{\partial z}\mathbf{1}_\varpi - \varpi\Delta_5 U\mathbf{1}_\varphi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi^2 V)\mathbf{1}_z\right\}&\\ = -\frac{\partial\mathfrak{L}_\varphi}{\partial z}\mathbf{1}_\varpi + \left(\frac{\partial\mathfrak{L}_\varpi}{\partial z} - \frac{\partial\mathfrak{L}_z}{\partial\varpi}\right)\mathbf{1}_\varphi + \frac{1}{\varpi}\frac{\partial}{\partial\varpi}(\varpi\mathfrak{L}_\varphi)\mathbf{1}_z.& \qquad (28)\end{aligned}$$

From this equation we conclude that

$$\frac{\partial V}{\partial t} = \frac{\mathfrak{L}_\varphi}{\varpi} \qquad (29)$$

and

$$-\Delta_5\frac{\partial U}{\partial t} = \frac{1}{\varpi}\left(\frac{\partial\mathfrak{L}_\varpi}{\partial z} - \frac{\partial\mathfrak{L}_z}{\partial\varpi}\right). \qquad (30)$$

Now, substituting for the components of $\mathfrak{L}$ in accordance with equation (27), we find, after some reductions, that

$$\varpi^3\frac{\partial V}{\partial t} = \frac{\partial(\varpi^2 T, \varpi^2 P)}{\partial(z, \varpi)} - \frac{\partial(\varpi^2 V, \varpi^2 U)}{\partial(z, \varpi)} \qquad (31)$$

and

$$\varpi\Delta_5\frac{\partial U}{\partial t} = \frac{\partial(\Delta_5 P, \varpi^2 P)}{\partial(z, \varpi)} - \frac{\partial(\Delta_5 U, \varpi^2 U)}{\partial(z, \varpi)} + \varpi\frac{\partial T^2}{\partial z} - \varpi\frac{\partial V^2}{\partial z}. \qquad (32)$$

Equations (25), (26), (31), and (32) are the basic equations of the present theory. We shall rewrite them in the following forms:

$$\varpi^3\Delta_5 P - \varpi^3\frac{\partial P}{\partial t} = \frac{\partial(\varpi^2 P, \varpi^2 U)}{\partial(z, \varpi)}, \qquad (33)$$

$$\varpi\Delta_5 T - \varpi\frac{\partial T}{\partial t} = \frac{\partial(T, \varpi^2 U)}{\partial(z, \varpi)} - \frac{\partial(V, \varpi^2 P)}{\partial(z, \varpi)}, \qquad (34)$$

$$\varpi^3\frac{\partial V}{\partial t} = \frac{\partial(\varpi^2 T, \varpi^2 P)}{\partial(z, \varpi)} - \frac{\partial(\varpi^2 V, \varpi^2 U)}{\partial(z, \varpi)}, \qquad (35)$$

and

$$\varpi\Delta_5\frac{\partial U}{\partial t} - \frac{\partial(\Delta_5 P, \varpi^2 P)}{\partial(z, \varpi)} + \frac{\partial(\Delta_5 U, \varpi^2 U)}{\partial(z, \varpi)} = \varpi\frac{\partial T^2}{\partial z} - \varpi\frac{\partial V^2}{\partial z}. \qquad (36)$$

4. SOME GENERAL RELATIONS AND CONSEQUENCES

Some general consequences which follow from equations (33)–(36) will now be discussed. Consider, first, equations (33) and (34), which govern the magnetic field. It is apparent from these equations that the poloidal field is unaffected by the presence of either toroidal magnetic fields or rotational motions. There is thus, in Elsasser's terminology, no "feedback" from the toroidal to the poloidal field. However, there is a feedback from the poloidal to the toroidal field through a coupling with the rotational motions. It should be particularly noted in this connection that this coupling between the poloidal magnetic field and the toroidal motions exists only in case of nonuniform rotation; for, if V is constant, the Jacobian involving V in equation (34) will vanish and there will be no coupling.

Thus equations (33) and (34) explicitly reveal what Elsasser (1955) has described as the "remarkable topological asymmetry" in the feedback mechanism between the poloidal and the toroidal fields.

Solutions of equations (33) and (34) in certain special cases are obtained in the paper following this one (Chandrasekhar 1956*c*); they disclose the extent to which internal motions can influence the free decay of a magnetic field in a fluid conductor. These latter considerations are relevant to the problem of the origin of the earth's magnetic field.

a) The Limiting Case of Infinite Electrical Conductivity

We shall now consider the form of the equations in case the electrical conductivity tends to infinity. In this limiting case the particular units (given in eq. [7]) which have been used in writing the equations governing $\boldsymbol{h}$ and $\boldsymbol{v}$ in nondimensional variables lose their meanings. Nevertheless, it is clear that, by introducing the variable $\boldsymbol{h}$ as defined in equation (6) and continuing to measure the various quantities in their conventional units, we can still reduce the equations to the forms (33)–(36), with the one difference that the terms in $\Delta_5 P$ and $\Delta_5 T$ in equations (33) and (34) will be absent. Thus the equations for P and T in this limiting case are

$$-\varpi^3 \frac{\partial P}{\partial t} = \frac{\partial (\varpi^2 P, \varpi^2 U)}{\partial (z, \varpi)} \tag{37}$$

and

$$-\varpi \frac{\partial T}{\partial t} = \frac{\partial (T, \varpi^2 U)}{\partial (z, \varpi)} - \frac{\partial (V, \varpi^2 P)}{\partial (z, \varpi)}. \tag{38}$$

Equations (35) and (36) will continue to be valid.

If, in the limiting case considered, the conditions are further stationary, the relevant equations are

$$\frac{\partial (\varpi^2 P, \varpi^2 U)}{\partial (z, \varpi)} = 0, \tag{39}$$

$$\frac{\partial (T, \varpi^2 U)}{\partial (z, \varpi)} - \frac{\partial (V, \varpi^2 P)}{\partial (z, \varpi)} = 0, \tag{40}$$

$$\frac{\partial (\varpi^2 T, \varpi^2 P)}{\partial (z, \varpi)} - \frac{\partial (\varpi^2 V, \varpi^2 U)}{\partial (z, \varpi)} = 0, \tag{41}$$

and

$$-\frac{\partial (\Delta_5 P, \varpi^2 P)}{\partial (z, \varpi)} + \frac{\partial (\Delta_5 U, \varpi^2 U)}{\partial (z, \varpi)} = \varpi \frac{\partial T^2}{\partial z} - \varpi \frac{\partial V^2}{\partial z}. \tag{42}$$

One integral of these equations is obvious: equation (39) implies that

$$\varpi^2 P = \text{Function}\,(\varpi^2 U)\,. \tag{43}$$

b) The Law of Isorotation and Its Generalization

An interesting special case of equations (39)–(42) arises when

$$U = 0\ , \tag{44}$$

i.e., when there are no meridional currents. In this case equation (39) is clearly satisfied, and equations (40) and (41) would require that

$$V = \text{Function}\,(\varpi^2 P) \tag{45}$$

and

$$\varpi^2 T = \text{Function}\,(\varpi^2 P)\,. \tag{46}$$

Relation (45) is an expression of the law of isorotation of Ferraro (1937) and Alfvén (1943). Relation (46) was derived earlier by Chandrasekhar and Prendergast (1956) as one of the conditions for hydrostatic equilibrium of a magnetic star; but we now see that the relation is more general and requires for its validity only the absence of meridional motions.

It should be noted here that the law of isorotation would follow from equation (40) equally if, instead of setting $U = 0$, we set $T = 0$. However, in this latter case, equation (41) will require

$$\varpi^2 V = \text{Function}\,(\varpi^2 U) \qquad (T = 0)\,; \tag{47}$$

and the elimination of $\varpi^2 U$ between equations (43) and (47) will make $\varpi^2 V$ a function of $\varpi^2 P$, contradicting the law of isorotation. Consequently, *in the framework of equations (39)–(42) we cannot suppose that $T = 0$ unless $U = 0$.*

Returning to the case $U = 0$, we have yet to consider equation (42). When $U = 0$, this equation is

$$\frac{\partial\,(\Delta_5 P,\ \varpi^2 P)}{\partial\,(z,\ \varpi)} + \varpi\,\frac{\partial T^2}{\partial z} - \varpi\,\frac{\partial V^2}{\partial z} = 0\,. \tag{48}$$

The general solution of this equation can be found as follows:

In view of equations (45) and (46), we can define two functions $G(\varpi^2 P)$ and $g(\varpi^2 P)$ such that

$$2G\,(\varpi^2 P) = \frac{d}{d\,(\varpi^2 P)}\,\varpi^4 T^2 \tag{49}$$

and

$$2\,g\,(\varpi^2 P) = \frac{d}{d\,(\varpi^2 P)}\,V^2\,. \tag{50}$$

With these definitions, it can be readily verified that

$$\frac{\partial\,(G/\varpi^2,\ \varpi^2 P)}{\partial\,(z,\ \varpi)} = +\varpi\,\frac{\partial T^2}{\partial z} \tag{51}$$

and

$$\frac{\partial\,(\varpi^2 g,\ \varpi^2 P)}{\partial\,(z,\ \varpi)} = -\varpi\,\frac{\partial V^2}{\partial z}\,. \tag{52}$$

We can now combine equations (48), (51), and (52) to give

$$\frac{\partial\,(\Delta_5 P + G/\varpi^2 + \varpi^2 g,\ \varpi^2 P)}{\partial\,(z,\ \varpi)} = 0 . \tag{53}$$

Hence

$$\Delta_5 P + \frac{1}{\varpi^2} G(\varpi^2 P) + \varpi^2 g(\varpi^2 P) = \Phi(\varpi^2 P) , \tag{54}$$

where Φ is an arbitrary function of the argument. Using equations (49) and (50), we can rewrite equation (54) in the form

$$\Delta_5 P = -T \frac{d}{d(\varpi^2 P)}(\varpi^2 T) - \varpi^2 V \frac{dV}{d(\varpi^2 P)} + \Phi(\varpi^2 P) , \tag{55}$$

which makes the dependence of P on T and V explicit.

It may be recalled here that in deriving equation (55) from equations (39)–(42) we have made only one assumption, namely, that $U = 0$. Equation (55) therefore represents the general integral of the equations in case $U = 0$.

Equation (55) includes as special cases the theorems on force-free fields (Lüst and Schlüter 1954; Chandrasekhar 1956*a*); Ferraro's (1954) condition for hydrostatic equilibrium in case $T = V = U = 0$; the law of isorotation; and, finally, the theorems of Chandrasekhar and Prendergast (1956). It therefore represents the complete generalization of these different theorems.

c) A Stationary Solution of Equations (35)–(38)

It is apparent from equations (39)–(42) that

$$P = U \text{ and } T = V \tag{56}$$

is a solution. It corresponds to a stationary solution of equations (35)–(38).

Equation (56) represents, under conditions of axisymmetry, the special solution

$$\boldsymbol{v} = \frac{\boldsymbol{H}}{\sqrt{(4\pi\rho)}} \text{ and } \frac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{v}|^2 + \mathfrak{B} = \text{Constant} \tag{57}$$

of the general equations

$$\frac{\partial \boldsymbol{H}}{\partial t} = \text{curl}\,(\boldsymbol{v} \times \boldsymbol{H}) \tag{58}$$

and

$$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{1}{4\pi\rho}\,\text{curl}\,\boldsymbol{H} \times \boldsymbol{H} - \text{curl}\,\boldsymbol{v} \times \boldsymbol{v} - \text{grad}\left(\frac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{v}|^2 + \mathfrak{B}\right). \tag{59}$$

It has been shown (Chandrasekhar 1956*b*) that equation (57) represents a *stable* solution of equations (58) and (59). In case the fields and motions are axisymmetric, the periods of oscillation about the stationary solution (56) can be discussed by linearizing equations (35)–(38) in the neighborhood of this solution. The analysis reveals some novel aspects of hydromagnetic oscillations; they will be considered in a separate paper.

5. THE BOUNDARY CONDITIONS

For the sake of definiteness we shall suppose that the fluid is confined in a sphere of radius R. Let this radius R be the unit of length chosen in § 2. Boundary conditions are then required on the sphere $|\boldsymbol{r}| = 1$.

In spherical polar co-ordinates the expressions for $\boldsymbol{h}$ and $\boldsymbol{v}$ are

$$\boldsymbol{h} = -\frac{\partial}{\partial \mu}[(1-\mu^2)P]\,\mathbf{1}_r - \frac{(1-\mu^2)^{1/2}}{r}\frac{\partial}{\partial r}(r^2 P)\,\mathbf{1}_\vartheta + r(1-\mu^2)^{1/2} T\,\mathbf{1}_\varphi \tag{60}$$

and

$$\boldsymbol{v} = -\frac{\partial}{\partial \mu}[(1-\mu^2)U]\,\mathbf{1}_r - \frac{(1-\mu^2)^{1/2}}{r}\frac{\partial}{\partial r}(r^2 U)\,\mathbf{1}_\vartheta + r(1-\mu^2)^{1/2} V\,\mathbf{1}_\varphi, \tag{61}$$

where $\mathbf{1}_r$, $\mathbf{1}_\vartheta$, and $\mathbf{1}_\varphi$ are unit vectors along the arcs dr, $r d\vartheta$, and $r \sin \vartheta d\varphi$ in the three principal directions. The corresponding expression for the current $\boldsymbol{j}$ is

$$4\pi \boldsymbol{j} = \text{curl}\, \boldsymbol{h}$$

$$= -\frac{\partial}{\partial \mu}[(1-\mu^2)T]\,\mathbf{1}_r - \frac{(1-\mu^2)^{1/2}}{r}\frac{\partial}{\partial r}(r^2 T)\,\mathbf{1}_\vartheta - r(1-\mu^2)^{1/2}\Delta_5 P\,\mathbf{1}_\varphi, \tag{62}$$

where now

$$\Delta_5 = \frac{\partial^2}{\partial r^2} + \frac{4}{r}\frac{\partial}{\partial r} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2} - \frac{4\mu}{r^2}\frac{\partial}{\partial \mu}. \tag{63}$$

The boundary conditions on the magnetic field are

$$\boldsymbol{h} \text{ and } \boldsymbol{j}\cdot\mathbf{1}_r \text{ are continuous on } r = 1\,. \tag{64}$$

If no current flows in the volume exterior to $r = 1$, the continuity of the normal component of $\boldsymbol{j}$ requires

$$\boldsymbol{j}\cdot\mathbf{1}_r = 0 \text{ for } r = 1 \qquad (\boldsymbol{j} \equiv 0 \text{ for } r > 1)\,. \tag{65}$$

According to equation (62), this latter condition implies that

$$T = 0 \text{ on } r = 1. \tag{66}$$

On the other hand, the continuity of $\boldsymbol{h}$ on $r = 1$ requires

$$P \text{ and } \frac{\partial P}{\partial r} \text{ to be continuous on } r = 1\,. \tag{67}$$

To apply the boundary conditions (67), we must know the equation governing P outside $r = 1$. On the same assumption as before regarding $\boldsymbol{j}$ (namely, $\boldsymbol{j} \equiv 0$ outside $r = 1$), the equation is

$$\Delta_5 P = 0\,. \tag{68}$$

The resulting conditions on P are explicitly derived in the paper following this one (Chandrasekhar 1956*c*, § 4).

It should be pointed out here that, in many astrophysical connections, the assumption that no current flows in the external regions is not justifiable on physical grounds. In these latter connections it appears more reasonable to suppose that the field outside is force-free (cf. Lüst and Schlüter 1954); if this assumption is made, it can be shown that (cf. Chandrasekhar 1956*a*)

$$P = T = 0 \text{ on } r = 1 \qquad (\text{force-free case})\,. \tag{69}$$

As for the boundary conditions on $\boldsymbol{v}$, it is clear that we must in all cases require that its normal component vanish on the boundary. According to equation (61), this requires that

$$U=0 \text{ on } r=1 \; . \tag{70}$$

In the absence of viscosity there are no other conditions on $\boldsymbol{v}$.

6. INTEGRAL RELATIONS

Letting

$$[\phi, \psi] = \frac{\partial(\phi, \psi)}{\partial(z, \varpi)} = \frac{\partial \phi}{\partial z}\frac{\partial \psi}{\partial \varpi} - \frac{\partial \phi}{\partial \varpi}\frac{\partial \psi}{\partial z}, \tag{71}$$

we can rewrite equations (33)–(36) more compactly in the forms

$$\varpi^3 \Delta_5 P - \varpi^3 \frac{\partial P}{\partial t} = [\varpi^2 P, \varpi^2 U] \; , \tag{72}$$

$$\varpi \Delta_5 T - \varpi \frac{\partial T}{\partial t} = [T, \varpi^2 U] - [V, \varpi^2 P] \; , \tag{73}$$

$$\varpi^3 \frac{\partial V}{\partial t} = [\varpi^2 T, \varpi^2 P] - [\varpi^2 V, \varpi^2 U] \; , \tag{74}$$

and

$$\varpi \Delta_5 \frac{\partial U}{\partial t} = [\Delta_5 P, \varpi^2 P] - [\Delta_5 U, \varpi^2 U] + \varpi \frac{\partial T^2}{\partial z} - \varpi \frac{\partial V^2}{\partial z} . \tag{75}$$

Quite generally, we may supplement these equations by the boundary conditions (cf. eqs. [66], [67], and [70])

$$T=U=0 \text{ on } r=1 \; . \tag{76}$$

The following integral relations will now be derived from equations (72)–(76):

$$\begin{aligned} \int\int \varpi^3 \{ (\Delta_5 P)^2 - T \Delta_5 T \} d\varpi \, dz = & -\frac{1}{2} \frac{d}{dt} \int\int \varpi^3 (T^2 + V^2) \, d\varpi \, dz \\ & + \int\int \varpi^3 \left(U \Delta_5 \frac{\partial U}{\partial t} + \frac{\partial P}{\partial t} \Delta_5 P \right) d\varpi \, dz \, , \end{aligned} \tag{77}$$

$$\int\int \varpi^3 (U \Delta_5 T + V \Delta_5 P) \, d\varpi \, dz = \int\int \varpi^3 \left(U \frac{\partial T}{\partial t} + V \frac{\partial P}{\partial t} \right) d\varpi \, dz \, , \tag{78}$$

$$\int\int \varpi^5 P \Delta_5 P \, d\varpi \, dz = \frac{1}{2} \frac{d}{dt} \int\int \varpi^5 P^2 \, d\varpi \, dz \, , \tag{79}$$

$$\int\int \varpi^5 U \Delta_5 P \, d\varpi \, dz = \int\int \varpi^5 U \frac{\partial P}{\partial t} \, d\varpi \, dz \, , \tag{80}$$

and

$$\int\int \varpi^5 T \Delta_5 P \, d\varpi \, dz = \int\int \varpi^5 \left(T \frac{\partial P}{\partial t} + U \frac{\partial V}{\partial t} \right) d\varpi \, dz \, , \tag{81}$$

where the integrations are over the whole volume.

First, we state the following elementary lemma:

$$\iint f\,[\phi, \psi]\, d\varpi\, dz = -\iint \phi\,[f, \psi]\, d\varpi\, dz\,, \tag{82}$$

provided, at least, that one of the two functions, f or ϕ, vanishes on the boundary. This lemma is needed in the proofs of relations (77)–(81) given below.

Multiplying equation (73) by $\varpi^2 T$, integrating over the range of the variables, and successively transforming the integrals by making use of the lemma (eq. [82]) and using one or another of the remaining equations, we find

$$\begin{aligned}
&\iint \varpi^3 T \Delta_5 T\, d\varpi\, dz - \frac{1}{2}\frac{d}{dt}\iint \varpi^3 T^2 d\varpi\, dz \\
&= \iint \varpi^2 T\,[T, \varpi^2 U]\, d\varpi\, dz - \iint \varpi^2 T\,[V, \varpi^2 P]\, d\varpi\, dz \\
&= -\iint \varpi^2 U\,[T, \varpi^2 T]\, d\varpi\, dz + \iint V\,[\varpi^2 T, \varpi^2 P]\, d\varpi\, dz \\
&= -\iint \varpi^3 U \frac{\partial T^2}{\partial z}\, d\varpi\, dz + \iint V \left\{\varpi^3 \frac{\partial V}{\partial t} + [\varpi^2 V, \varpi^2 U]\right\} d\varpi\, dz \\
&= -\iint \varpi^3 U \frac{\partial T^2}{\partial z}\, d\varpi\, dz + \frac{1}{2}\frac{d}{dt}\iint \varpi^3 V^2 d\varpi\, dz - \iint \varpi^2 U\,[\varpi^2 V, V]\, d\varpi\, dz \\
&= -\iint \varpi^3 U \left(\frac{\partial T^2}{\partial z} - \frac{\partial V^2}{\partial z}\right) d\varpi\, dz + \frac{1}{2}\frac{d}{dt}\iint \varpi^3 V^2 d\varpi\, dz \\
&= -\iint \varpi^2 U \left\{\varpi \Delta_5 \frac{\partial U}{\partial t} - [\Delta_5 P, \varpi^2 P] + [\Delta_5 U, \varpi^2 U]\right\} d\varpi\, dz + \frac{1}{2}\frac{d}{dt}\iint \varpi^3 V^2 d\varpi\, dz \\
&= -\iint \varpi^3 U \Delta_5 \frac{\partial U}{\partial t}\, d\varpi\, dz - \iint \Delta_5 P\,[\varpi^2 U, \varpi^2 P]\, d\varpi\, dz + \frac{1}{2}\frac{d}{dt}\iint \varpi^3 V^2 d\varpi\, dz \\
&= -\iint \varpi^3 U \Delta_5 \frac{\partial U}{\partial t}\, d\varpi\, dz + \iint \varpi^3 \Delta_5 P \left(\Delta_5 P - \frac{\partial P}{\partial t}\right) d\varpi\, dz + \frac{1}{2}\frac{d}{dt}\iint \varpi^3 V^2 d\varpi\, dz\,.
\end{aligned} \tag{83}$$

Rearranging, we have equation (77).

Similarly, multiplying equation (73) by $\varpi^2 U$ and integrating, we obtain

$$\begin{aligned}
\iint \varpi^3 U \Delta_5 T\, d\varpi\, dz - \iint \varpi^3 U \frac{\partial T}{\partial t}\, d\varpi\, dz &= \iint \varpi^2 U \{[T, \varpi^2 U] - [V, \varpi^2 P]\}\, d\varpi\, dz \\
&= \iint V\,[\varpi^2 U, \varpi^2 P]\, d\varpi\, dz = -\iint \varpi^3 V \left(\Delta_5 P - \frac{\partial P}{\partial t}\right) d\varpi\, dz\,.
\end{aligned} \tag{84}$$

This proves equation (78).

Next, multiplying equation (72) by $\varpi^2 P$ and integrating, we obtain

$$\iint \varpi^5 P \Delta_5 P\, d\varpi\, dz - \frac{1}{2}\frac{d}{dt}\iint \varpi^5 P^2 d\varpi\, dz = \iint \varpi^2 P\,[\varpi^2 P, \varpi^2 U]\, d\varpi\, dz = 0\,, \tag{85}$$

which is equation (79). Equation (80) similarly follows from equation (72) after multiplication by $\varpi^2 U$ and integration. Finally, multiplying equation (72) by $\varpi^2 T$ and integrating, we obtain the last of the required relations; thus

$$\int\int \varpi^5 T \Delta_5 P d\varpi d z - \int\int \varpi^5 T \frac{\partial P}{\partial t} d\varpi d z$$

$$= \int\int \varpi^2 T [\varpi^2 P, \varpi^2 U] d\varpi d z = - \int\int \varpi^2 U [\varpi^2 P, \varpi^2 T] d\varpi d z \qquad (86)$$

$$= \int\int \varpi^2 U \left\{ \varpi^3 \frac{\partial V}{\partial t} + [\varpi^2 V, \varpi^2 U] \right\} d\varpi d z = \int\int \varpi^5 U \frac{\partial V}{\partial t} d\varpi d z .$$

Applications of the integral relations derived in this section will be given in a later paper.

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