COVERING BY STAR DOMAINS

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If S is a symmetrical convex domain in R_2 then it is known that the best general covering density θ (S) of S is equal to its best lattice covering density θ_L (S). Examples of symmetrical as well as non-symmetrical star domains S in R_2 are given for which θ (S) $<$ θ_L (S). For cylinders $\mathcal{C} = S \times [-\frac{1}{2}, \frac{1}{2}]$ based on symmetrical convex domains S in R_2 one has θ_L $(S) = \theta_L$ (C). It is shown that this result need not hold if S is a star domain.

§ 1. Let S be a set in the n-dimensional Euclidean space R_n. Let $\theta_L(S)$, $\theta(S)$ denote the densities of its best lattice coverings and best general coverings respectively. If S is a symmetrical convex domain in R_2 it was proved in the 1950's that $\theta(S) = \theta_L(S)$ (Toth 1950, Bambah and Rogers 1952, Bambah *et al.* 1964, Bambah and Woods 1968). The question whether $\theta(S) = \theta_L(S)$ in R_2 for all convex sets S or for all star domains S (symmetrical or not) seems to have been still open. Stein (1972), however, has produced examples of symmetrical star bodies in R_{10} and non-symmetrical star bodies in R_5 for which $\theta(S) < \theta_L(S)$ (His sets have other nice properties too). In § 4 we give examples of two dimensional star domains S (symmetrical as well as non-symmetrical) for which $\theta(S)$ $< \theta_L(S)$.

If $\mathcal{C}$ is the cylinder $S \times [-\frac{1}{2}, \frac{1}{2}]$, then, since $\theta(S) = \theta_L(S)$ implies $\theta_L(S)$ $= \theta_L(C)$, it follows that if S is a symmetrical convex two dimensional set then $\theta_L(S) = \theta_L(C)$ (for an independent proof see Bramah and Woods 1968). That the result $\theta_L(S) = \theta_L(C)$ is not true if S is a star domain (symmetrical or not) is shown in § 5. This result is in an obvious sense an analogue of a result of Davenport and Rogers (1950) on critical determinants of cylinders based on star sets.

§ 2. Let $S \subset R_n$ and Λ be a discrete subset of R_n. We say that (S, Λ) is a covering if $R_n \subset \bigcup_{A \in \Lambda} (S + A)$ where $S + A = \{X + A \colon X \in S\}$. If Λ is a lattice, then (S, Λ) is called a lattice covering. Let S have finite volume $V(S)$. For any covering (S, Λ) we define its density $\theta(S, \Lambda)$ as follows:

Let $\mathfrak{B}$ be the box $|x_i| \leq s$ $(i = 1, 2, \ldots, n)$. Let $N(\mathfrak{B})$ denote the number of sets $S + A$ which have a point in common with $\mathfrak{B}$. Then

$$\theta(S, \Lambda) = \liminf_{s \to \infty} \frac{N(\mathfrak{B}) \, V(S)}{(2s)^n} .$$

If Λ is a lattice then $\theta(S, \Lambda) = \dfrac{V(S)}{d(\Lambda)}$ where $d(\Lambda)$ is the determinant of Λ. We define

$$\theta(S) = \inf\{\theta(S)\Lambda) : (S, \Lambda) \text{ is a covering}\}$$

and

$$\theta_L(S) = \inf\{\theta(S, \Lambda) : (S, \Lambda) \text{ is a lattice covering}\}.$$

$\theta(S)$ is called the density of the thinnest covering of S and $\theta_L(S)$ is called the density of the thinnest lattice covering of S. If we define the covering constant $C(S)$ of S by

$$C(S) = \sup\{d(\Lambda) : (S, \Lambda) \text{ is a lattice covering}\},$$

then

$$\theta_L(S) = \frac{V(S)}{C(S)}.$$

The theorems we prove are:

Theorem 1—There exist in R_2 star domains S (symmetrical as well as non-symmetrical) for which $\theta(S) < \theta_L(S)$.

Theorem 2—For a set S in R_2 let $\mathcal{C}$ be the cylinder $S \times [-\frac{1}{2}, \frac{1}{2}]$ of height 1 and base S. There exist star domains S, symmetrical as well as non-symmetrical for which $C(\mathcal{C}) > C(S)$, so that $\theta_L(\mathcal{C}) < \theta_L(S)$.

We recollect here the definitions of a star body, a star domain and a star set. A set $S \subset R_n$ is star body if 0 is an interior point of S and every half ray through 0 meets the boundary of S in atmost one point. A two dimensional star body is called a star domain. S is called a star set if it is closed and $\lambda X \in S$ whenever $0 \leq \lambda \leq 1$ and $X \in S$.

We would like to remark that just as Theorem 2 is in some sense an analogue of Theorem of Davenport and Rogers, Theorem 1 can be looked upon as an analogue of a result of Wolf (1962) and Groemer (1964) on admissible sets.

§ 3. In this section we shall construct a symmetrical star set T for which $\theta(T) < \theta_L(T)$. Let $P_1, \ldots, P_8$ be the points $(0, 1)$, $(-1, 1)$, $(-1, \frac{1}{4})$, $(-2, 0)$, $(-1, 0)$, $(-2, -\frac{1}{4})$, $(-2, -1)$ and $(-1, -1)$ respectively. Let π be the polygon $0 P_1, P_2, \ldots, P_8$ and $T = \pi \cup (-\pi)$, where $-\pi$ is the image of π in 0. Then T is a star set with centre 0 (see Fig. 1) and area $a(T) = 5$. We shall prove

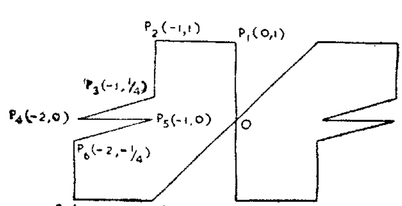

FIG. 1.

Theorem 3—$\theta(T) \leq 5/4 < \theta_L(T)$.

PROOF: Let Γ be the lattice generated by (4, 0) and (0, 2), then $d(\Gamma) = 8$. Let $\Lambda = \Gamma \cup (\Gamma + (1, 0))$. Then (T, Λ) is a covering with density $\theta(T, \Lambda) = 5/4$. Therefore $\theta(T) \leq 5/4$.

Now we shall prove that $\theta_L(T) > 5/4$.

Since T has interior points, its best lattice covering density is attained by some lattice covering (T, Λ) (e.g. Bambah 1953, Lekkerkerker). Therefore, it is enough to show that for every lattice covering (T, Λ), $d(\Lambda) < 4$.

The following Lemma is easy to prove (see, e.g. Hans-Gill 1971).

Lemma—Let K be a bounded set in the plane with area α. Let Λ be a covering lattice for K. Let $A \in \Lambda$, $A \neq 0$ and let α' be the area of $K \cap (K + A)$. Then $d(\Lambda) \leq \alpha - \alpha'$.

Now we proceed to prove that if (T, Λ) is a lattice covering then $d(\Lambda) < 4$. Since T is closed and bounded and 0 belongs to its boundary, there exists a point $A = (a_1, a_2) \neq 0$ in Λ such that $0 \in T + A$ or $A \in T$. Since T is a star set we can suppose A is primitive and replacing A by $-A$ if necessary we can suppose that $a_2 \geq 0$. Now we deal with different cases depending upon the position of A (see Fig. 2).

Case I: When A lies in one of the following regions:

$$\mathcal{K}_1 = \{|x| \leq 1,\ 0 \leq y < \tfrac{2}{3}\}.$$

$$\mathcal{K}_2 = \{-2 \leq x \leq -1,\ 0 \leq y \leq \tfrac{1}{4}(x + 2)\}$$

$$\mathcal{K}_3 = \{\tfrac{2}{3} \leq y \leq x + 1,\ x \leq 0\}.$$

Let K_1 be the rectangle $|x| \leq 2$, $|y| \leq 1$.

Then Λ is a covering for K_1, because $T \subset K_1$.

For $A \in \mathcal{R}_1$,

$$a(K_1 \cap K_1 + A) = (2 - a_2)(4 - |a_1|) > (2 - \tfrac{2}{3})(4 - 1) = 4.$$

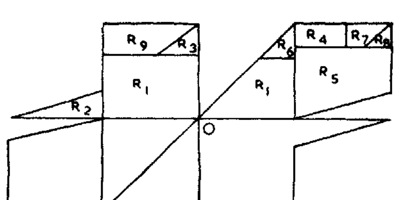

FIG. 2. (R in figure be read as $\mathcal{R}$)

For $A \in \mathfrak{P}_2$,

$$a(K_1 \cap K_1 + A) = (2 - a_2)(4 + a_1)$$
$$\geq [2 - \tfrac{1}{4}(a_1 + 2)](4 + a_1)$$
$$= \tfrac{1}{4}(24 + 2a_1 - a_1^2) \geq 4$$

and equality holds only if $a_1 = -2$ and $a_2 = 0$.

For $A \in \mathfrak{P}_3$

$$a(K_1 \cap K_1 + A) = (2 - a_2)(4 + a_1)$$
$$\geq [2 - (a_1 + 1)](4 + a_1)$$
$$= 4 - 3a_1 - a_1^2 \geq 4.$$

and equality is needed if and only if $a_1 = 0$, $a_2 = 1$. Since Λ is a covering lattice for K_1,

$$d(\Lambda) \leq a(K_1) - a(K_1 \cap K_1 + A) \leq 4.$$

with strict inequality except when $A = (-2, 0)$ or $(0, 1)$. In this case one can verify that $a(T \cap (T + A)) > 1$ so that

$$d(\Lambda) \leq a(T) - a((T + A) \cap T) < 4.$$

Case II: When A lies in one of the following regions:

$$\mathfrak{P}_4 = \{1 < x < 3/2,\ 3/4 \leq y \leq 1\}$$
$$\mathfrak{P}_5 = \{1 < x < 2,\ \ 0 \leq y \leq 3/4\} \cap T$$
$$\mathfrak{P}_6 = (y \leq x \leq 1,\ \ \ 2/3 \leq y \leq 1\}.$$

Let K_2 be the union of the rectangles

$$T_1 = \{-1 \leq x \leq 2,\ \ \ -\tfrac{1}{4} \leq y \leq 1\}$$

and

$$-T_1 = \{-2 \leq x \leq 1,\ \ \ -1 \leq y \leq \tfrac{1}{4}\}.$$

Since $K_2 \supset T$, Λ is a covering lattice for K_2. $a(K_2) = 13/2$. For $A \in \mathfrak{P}_4$,

$$K_2 \cap (K_2 + A) \supset T_1 \cap (-T_1 + A) = T_1 \cap \{T_1 + (a_1 - 1,\ a_2 - 3/4)\}$$

Therefore,

$$a(K_2 \cap (K_2 + A)) \geq (3 - a_1 + 1)(5/4 - a_2 + 3/4) = (4 - a_1)(2 - a_2) > 5/2.$$

For $A \in \mathfrak{P}_5$, $K_2 \cap (K_2 + A)$ contains the non-overlapping rectangles.

$$\{1 \leq x \leq 2,\ -\tfrac{1}{4} \leq y \leq 1\} \text{ and } \{a_1 - 2 \leq x \leq 1,\ a_2 - 1 \leq y \leq a_2 + \tfrac{1}{4}\},$$

so that

$$a(K_2 \cap K_2 + A) \geq 5/4(3 - a_1) + 5/4 > 5/2.$$

Let $A \in \mathfrak{P}_6$. If $a_2 \geq \frac{3}{4}$, $K_2 \cap (K_2 + A)$ contains the rectangle $\{-1 \leq x \leq 7/4, 0 \leq y \leq 1\}$ of area $11/4 > 5/2$. If $2/3 \leq a_2 \leq \frac{3}{4}$, $K_2 \cap (K_2 + A)$ contains the rectangle $\{-1 \leq x \leq 3/2, -\frac{1}{4} \leq y \leq \frac{2}{3} + \frac{1}{4}\}$ of area $35/12 > 5/2$. Hence in all these cases $d(\Lambda) < 4$.

Case III: Let $\mathfrak{P}_7 = \{3/2 \leqslant x \leqslant y + 1, \ 3/4 \leqslant y \leqslant 1\}$. If $A \in \mathfrak{P}_7$ then the point $-P_5 = (1, 0)$ does not lie in $T + kA$ for $k \neq 0, 1$. Also it lies on the boundary of $T \cup (T + A)$. Therefore to cover the points near it we need a point $B \in \Lambda$ linearly independent of A such that $(1, 0) \in T + B$. Since the slope of OA is positive and less than 1, the triangle OAB lies between the lines through $(2, -1)$ and $(0, 1)$ parallel to OA. Therefore, $a(OAB)$ is maximum when $B = (2, -1)$ or $(0, 1)$. Hence

$$d(\Lambda) \leq \max(2a_2 + a_1, a_1) = 2a_2 + a_1 \leqslant 4.$$

There is strict inequality unless $B = (2, -1)$ and $A = (2, 1)$ and A, B generate Λ. But this lattice is easily seen to be non-covering for T, because the points near $(1, 0)$ are not covered.

Case IV: $A \in \mathfrak{P}_8 = \{-\frac{3}{4} \leqslant y < x - 1, x \leqslant 2\} \cup \{x = 2, \frac{1}{4} \leqslant y \leq \frac{3}{4}\}$.

The point $Q = (a_1 - a_2, 0)$ does not lie in $T + kA$ for $k \neq 0$, 1 and is on the boundary of $T \cup (T + A)$. To cover the points near Q one gets $B \in \Lambda$, B linearly independent of A such that $Q \in T + B$, i.e. $B \in T + Q$. Since the slope of OA is positive and less than 1, as before

$$d(\Lambda) \leq \max |\det(A, B)|$$

where

$$B = (a_1 - a_2 + 1, -1) \quad \text{or} \quad (a_1 - a_2 - 1, 1).$$

Therefore,

$$d(\Lambda) \leq \max\{(a_1 - a_2 + 1)\, a_2 + a_1, \quad a_1 - (a_1 - a_2 - 1)\, a_2\}.$$
$$= a_1 + (a_1 - a_2 + 1)\, a_2.$$

The right hand side for fixed a_1 is an increasing function of a_2, so that

$$d(\Lambda) \leq 2a_1 \leq 4.$$

with strict inequality unless $a_2 = 1$, $a_1 = 2$. But $(2, 1) \notin \mathcal{K}_8$.

Case V: $A \in \mathfrak{P}_9 = \{-1 \leq x < y - 1, \frac{2}{3} \leq y \leq 1\}$.

The point $(a_1 + 1, a_2)$, which lies on the boundary of $T + A$, does not lie in $T + kA$ for integers $k \neq 1$. Therefore, to cover points near this point we need $B \in \Lambda$, linearly independent of A such that $B \in T + (a_1 + 1, a_2)$. As before, we see that

$$d(\Lambda) \leq \max |\det(A, B)|,$$

where

$$B = (a_1 + 3, a_2 + 1) \quad \text{or} \quad (a_1 - 1, a_2 - 1).$$

Therefore

$$d(\Lambda) \leq 3a_2 - a_1 \leq 4$$

with strict inequality unless $A = (a_1, a_2) = (-1, 1)$, $B = (2, 2)$ and A, B generate Λ. But then Λ is not a covering for T because the points near the origin are not covered.

This completes the proof of Theorem 3.

§ 4. Now we prove Theorem 1, i.e. construct star domains S for which $\theta(S) < \theta_L(S)$. Let $C_n = \left(\frac{1}{n}, \frac{2}{n}\right)$, $D_n = \left(1 + \frac{1}{n}, \frac{1}{4n}\right)$. Let S_n be the union of the polygonal region with vertices $C_n, P_1, P_2, P_3, P_4, -D_n, P_6, P_7, P_8, -C_n$ and its reflection in 0. Then for each natural number n, S_n is a star domain symmetrical about 0 which contains T. Each S_n has a maximal covering lattice Λ_n (see e.g. Bambah 1953). Then $d(\Lambda_n) = C(S_n) \geq C(T)$. Since Λ_n is a covering lattice for the rectangle $|x| \leq 2, |y| \leq 1$, $d(\Lambda_n) \leq 8$. It follows easily as in Bambah (1953) that the sequence Λ_n is bounded in the sense of Mahler and so has a convergent subsequence Λ_{n_k} converging to a lattice Λ which can be shown to be a covering lattice for T. From this we see that $d(\Lambda_{n_k}) = C(S_{n_k}) \rightarrow C(T) < 4$. Therefore for k large enough, $C(S_{n_k}) < 4$. and

$$\theta_L(S_{n_k}) = \frac{a(S_{n_k})}{C(S_{n_k})} > \frac{a(S_{n_k})}{4}.$$

Since the lattice Γ generated by (4, 0) and (0, 2) has determined 8 and $\Gamma \cup (\Gamma + (1, 0))$ is a covering for S_{n_k}, therefore

$$\theta(S_{n_k}) \leq \frac{a(S_{n_k})}{4}.$$

Hence $\theta(S_{n_k}) < \theta_L(S_{n_k})$ for large k.

Remark 1: Since the sequence S_n is descending $C(S_n) \geq C(S_{n+1})$ for al n and so it follows that $\theta(S_n) < \theta_L(S_n)$ for all large n.

Remark 2: If we require S_n to be non-symmetric we can for example replace the vertex $-D_n$ of S_n by $-D_{2n}$.

§ 5. To prove Theorem 2, it is enough to observe that if S is one of the star domains determined in the last section and $\mathcal{C}$ is the cylinder $S \times [-\frac{1}{2}, \frac{1}{2}]$, then the lattice generated by $(4, 0, 0)$, $(0, 2, 0)$ and $(1, 0, \frac{1}{2})$ is a covering lattice for $\mathcal{C}$ of determinant 4, so that

$$C(\mathcal{C}) \geq 4 > C(S).$$

Hence

$$\theta_L(\mathcal{C}) < \theta_L(S).$$

REFERENCES

Bambah, R. P., and Rogers, C. A. (1952). Covering the plane with convex sets. *J. Lond. math. Soc.*, **27**, 304–14.

Bambah, R. P. (1953). On lattice coverings. *Proc. natn. Inst. Sci. India*, **19**, 447–59.

Bambah, R. P., Rogers, C. A., and Zassenhaus, H. (1964). On coverings with convex domains. *Acta Arith.*, **9**, 191–207.

Bambah, R. P., and Woods, A. C. (1968). The covering constant for a cylinder. *Monat. Math.* **72**, 107–17.

Davenport, H., and Rogers, C. A. (1950). On the critical determinants of cylinders. *Quart. J. Math.*, (2), **1**, 215–18.

Groemer, H. (1964). Uber gewisse dichteste Anordnungen von Punkten in der Ebene. *Arch. Math.*, **15**, 385–87.

Hans, R. J. (1971). Maximal Covering sets II. *J. Indian math. Soc.*, **35**.

Lekkerkerker, C. G. Geometry of Numbers. North-Holland Publishing Company, Amsterdam.

Stein, S. K. (1972). A symmetric star body that tiles but not as a lattice. *Proc. Am. math. Soc.*, **36**, 543–48.

Toth Fejes, L. (1950). Some packing and covering theorems. *Acta Sci. Math. (Szeged)*, **12A**, 152–61.

Wolff, M. R. Von (1962). A star domain with densest admissible point set not a lattice. *Acta Math.*, **108**, 53–60.