POTENTIAL PROBLEMS CONCERNING CURVED BOUNDARIES

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The solution of two dimensional potential problems connected with rectilinear boundaries has been discussed by me in a number of papers[1]. The boundary condition satisfied by the potential function ϕ is of the form

$$\left[\frac{\partial\phi}{\partial x} - \sum_{r=0}^{n} f_r\right]\cos(x\nu) + \left[\frac{\partial\phi}{\partial y} + \sum_{r=0}^{n} \chi_n\right]\cos(y\nu) = 0, \tag{1}$$

where f_n and χ_n are homogeneous integral polynomials of the nth degree in x and y and ν denotes the direction of the normal drawn to the boundary.

If a_p is the argument of dz when the pth side of the boundary is described in the positive sense by the point whose affix is z ($=x+iy$), we know[2] that (1) can be put in the form

$$\begin{aligned} X_{n+1}\sin(n+1)\,a_p + Y_{n+1}\cos(n+1)\,a_p = |\underline{n}\,[&\sin a_p\, f_n(\cos a_p, \sin a_p) \\ &+ \cos a_p\, \chi_n(\cos a_p, \sin a_p)], \\ &(p = 1, 2, 3, \cdots m), \end{aligned} \tag{2}$$

m being the number of different sides present in the boundary, and X_{n+1} Y_{n+1} being given by

$$X_{n+1} + i\,Y_{n+1} = \frac{d^{n+1}}{dz^{n+1}}(\phi + i\,\psi). \tag{3}$$

If the z-area is externally bounded by a polygon the solution of the problem is given by the relations

$$\frac{dz}{d\zeta} = A \prod_{r=1}^{m} (\xi - \xi_r)^{a_r - 1}, \tag{4}$$

$$\frac{d\Omega_{n+1}}{d\zeta} = C\,R(\zeta) \prod_{r=1}^{m} (\zeta - \xi_r)^{-(n+1)a_r}, \tag{5}$$

where, $\xi_1, \xi_2, \cdots \xi_m$ are the points on the real axis of ξ in the ζ-plane that correspond to the angular points of the z-figure, $\pi a_1, \pi a_2, \cdots, \pi a_m$ are the internal angles at these points, $R(\zeta)$ is a rational integral function of the

$[(n+1)\,m - 2\,(n+2)]$th degree in ζ with real coefficients, and A, C are in general complex constants.

The object of this paper is to extend the solution given by (4) and (5) to some curved boundaries.

Let the transformation

$$z = f(t) \tag{6}$$

change an area with curved boundaries in the t-plane into a rectilinear area in the z-plane, and let us assume that the altered boundary condition is of the same form as (1). The problem can be solved for the z-figure with the help of (4) and (5). Putting $z = f(t)$ in the solution we get the result for the corresponding problem with curved boundaries.

To give an example let us take the motion of liquid contained in a cylinder rotating with an angular velocity ω. If ψ is the stream function, the boundary condition is

$$\psi = \tfrac{1}{2}\,\omega\, t\, t' + a \text{ constant.} \tag{7}$$

Let

$$t = \sum_0^n A_n z^n, \tag{8}$$

where A's are constants and n is a positive integer. The new boundary condition is

$$\psi = \tfrac{1}{2}\,\omega \Big[\sum_0^n A_n z^n\Big]\Big[\sum_0^n A_n z'^n\Big] + \text{a constant.} \tag{9}$$

Both (7) and (9) give rise to boundary conditions in terms of ϕ which are of the same form as (1). The solution for the rectilinear boundary in the z-plane is given by (4) and (5). These combined with (8) give the result for boundaries made up of arcs of a family of orthogonal curves.

Parabolic Boundaries

In the simple case of

$$t = z^2 \tag{10}$$

we get

$$x = \rho^{\frac{1}{2}} \cos \tfrac{1}{2}\alpha, \quad y = \rho^{\frac{1}{2}} \sin \tfrac{1}{2}\alpha,$$

ρ, α being the polar co-ordinates in the t-plane. Thus a rectilinear area in the z-plane corresponds to one bounded by confocal parabolas in the t-plane.

The boundary condition is now

$$\psi = \tfrac{1}{2}\,\omega\,(x^2 + y^2)^2 + a \text{ constant} \tag{11}$$

which, with the help of (2) can be written as

$$X_4 \sin 4\alpha_p + Y_4 \cos 4\alpha_p = 12\,\omega. \tag{11·1}$$

Area bounded by two Confocal Parabolas

Let a rectangular area in the z-plane be given by $x = \pm a$, $y = \pm b$. The area in the t-plane is bounded by the two confocal parabolas

$$\rho(1 + \cos\alpha) = 2a^2, \quad \rho(1 - \cos\alpha) = 2b^2, \tag{12}$$

the corresponding areas being as shown in Fig. 1.

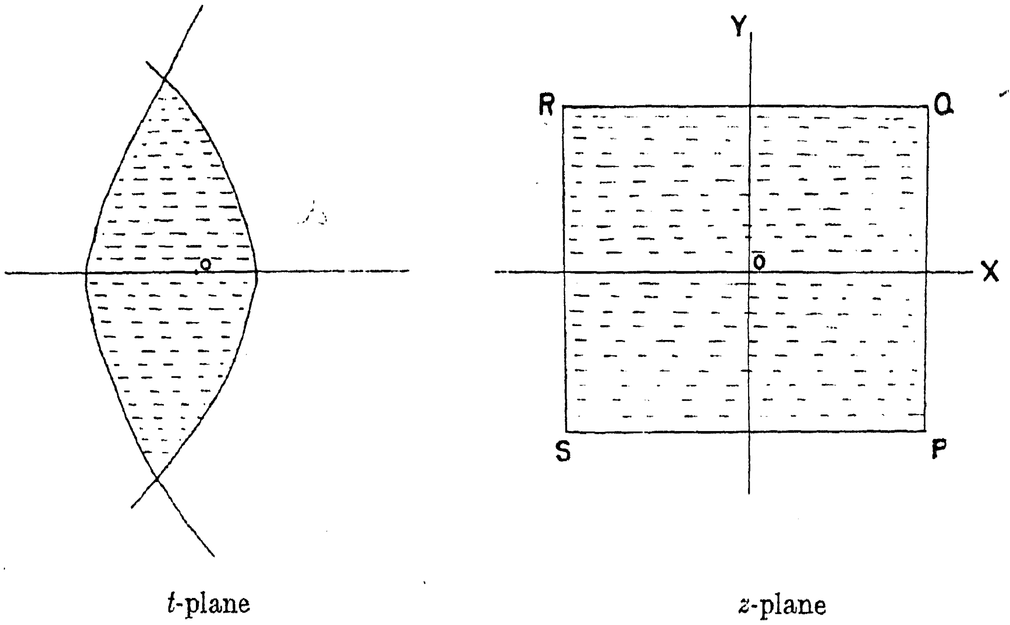


FIG. 1

The boundary condition (11·1) shows that the area in the Ω_4-plane reduces to the point $Y_4 = 12\omega$. Hence

$$\Omega_4 = 12i\omega. \tag{13}$$

If Ω_{40} is the value of the Ω_4 for a rotating rectangular cylinder in the z-plane, the boundary condition in this case is

$$\psi_0 = \tfrac{1}{2}\,\omega\,(x^2 + y^2) + a \text{ constant}, \tag{14}$$

and it is obvious an addition of this to (11) leaves (11·1) unaltered. Hence Ω_4 can contain a term of the type-$H_1\,\Omega_{40}$. That the constant H_1 is finite or zero will be determined from the boundary condition (11).

Integrating (13) thrice in succession we get

$$\Omega_1 = 2i\omega z^3 + C_1 z^2 + C_2 z + C. \tag{15}$$

From (11) and (14) we get at an angular point in the z-plane

$$\left.\begin{aligned} \Omega_1 &= X_1 + iY_1 = 2\,\omega\,(x^2 + y^2)\,(y + ix), \\ \Omega_{10} &= X_{10} + iY_{10} = y + ix, \end{aligned}\right\} \tag{16·1}$$

and hence

$$[2\,\omega\,(x^2 + y^2) - H_1]\,(y + ix) = 2i\omega z^3 + C_1 z^2 + C_2 z + C_3. \tag{16·2}$$

This condition at the four angular points P, Q, R, S, gives $C_1 = 0, \quad C_3 = 0$.

Thus $$H_1 = 4\omega (a^2 + b^2),\ C_2 = -4i\omega (a^2 - b^2),$$

$$\phi + i\psi = \tfrac{1}{2} i\omega z^4 - 2i\omega (a^2 - b^2) z^2 + 4\omega (a^2 + b^2)\Omega_0. \tag{17}$$

But we know that

$$\Omega_0 = \tfrac{1}{2} i z^2 - 4ib^2 \left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \cdot \frac{\cosh\{\frac{1}{2}(2n+1)\pi z\}/b}{\cosh\{\frac{1}{2}(2n+1)\pi a\}/b}. \tag{18}$$

Hence

$$\psi = \tfrac{1}{2}\omega \rho^2 \cos 2\alpha + 4\omega b^2 \rho \cos \alpha$$
$$- 16 b^2 \omega (a^2 + b^2) \left(\frac{2}{\pi}\right)^3 \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)^3} \cdot \frac{\cosh\{\frac{1}{2}(2n+1)\rho^{\frac{1}{2}} \cos \frac{1}{2}\alpha\}/b}{\cosh\{\frac{1}{2}(2n+1\ \pi a\}/b} \right.$$
$$\left. \cos\left\{\frac{2n+1)\pi}{2b} \rho^{\frac{1}{2}} \sin \tfrac{1}{2}\alpha\right\}\right] \tag{19}$$

a result which has been laboriously obtained by Ferrers.[3]

Area bounded by four Confocal Parabolas

If, instead of taking the centre of the z-figure as the origin, we take any other point as the origin, we get the corresponding area in the z-plane bounded by four confocal parabolas. The corresponding result can be written down with the help of (15) and (18). In the particular case when the origin coincides with one of the angular points the area in the t-plane is bounded by two confocal parabolas and the initial line.

Area bounded by three Confocal Parabolas

In this case we take a triangle in the z-plane. For an isosceles triangle (5) gives

$$\frac{d\Omega_4}{d\zeta} = \frac{A_2\zeta^2 + A_1\zeta + A_0}{(\zeta - \xi_1)^{4a_1}(\zeta - \xi_2)^{4a_2}(\zeta - \xi_3)^{4a_1}}. \tag{20}$$

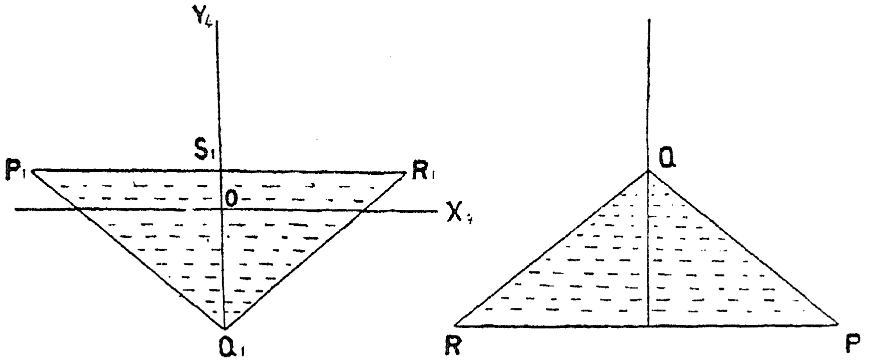
From (11·1) we see that the zeros of R (ξ) in general coincide with ξ_4. The simplest case occurs when $a_1 = \frac{1}{5}$, and now we get

$$\frac{d\Omega_4}{d\zeta} = \frac{C}{(\zeta - \xi_1)^{\frac{4}{5}}(\zeta - \xi_2)^{\frac{2}{5}}(\zeta - \xi_3)^{\frac{4}{5}}}, \tag{21·1}$$

$$\frac{dz}{d\zeta} = \frac{A}{(\zeta - \xi_1)^{\frac{4}{5}}(\zeta - \xi_2)^{\frac{2}{5}}(\zeta - \xi_2)^{\frac{4}{5}}}, \tag{21·2}$$

which shows that $\Omega_4 = Cz + D$. (22)

The corresponding areas are shown shaded in Fig. 2.



Ω_4-plane $\qquad$ z-plane

$OS_1 = 12\omega$, $MQ_1 = 12\omega \text{ Sec } \frac{\Pi}{5}$

FIG. 2

Taking Q in the z-plane as the origin we get

$$C = -\frac{12\omega}{d}\cot\frac{\pi}{10}, \quad D = -\,12i\omega\sec\frac{\pi}{5}.$$

Integrating (22) thrice in succession we have

$$\Omega_1 = -\tfrac{1}{2}\,\frac{\omega}{d}\cot\frac{\pi}{10}\,z^4 - 2i\omega\sec\frac{\pi}{5}z^3 + \tfrac{1}{2}C_1 z^2 + C_2 z + C_3 + H_1\Omega_{10}. \qquad (23)$$

Using (16·1) we can get the constants C_1, C_2, C_3, H_1.

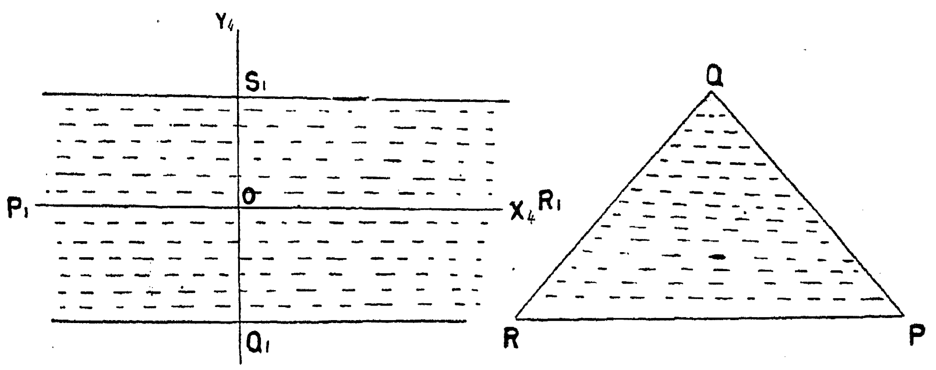
The corresponding t-figure is bounded by a parabola and the radii $\alpha = \pm\frac{3\pi}{5}$. By shifting the origin in the z-plane to a point inside the triangle we get a space in the t-plane bounded by three confocal parabolas. The drawback with (23) is that Ω_{10} not known. In the following cases Ω_{10} is known[4], and we know that (13) can be always integrated[5]. In all these cases we shall put $\xi_1 = -1$, $\xi_2 = 0$, $\xi_3 = 1$.

Isosceles Triangle containing an angle of 45°

In this case

$$\frac{d\Omega_4}{d\zeta} = \frac{C}{1-\xi^2}, \quad \frac{dz}{d\zeta} = \frac{A}{\zeta^{\frac{1}{2}}(\zeta^2-1)^{\frac{3}{4}}}.$$

and the areas that correspond are shown shaded in Fig. 3.



Ω_4-plane $\qquad$ z-plane

$OQ_1 = OS_1 = 12\omega$

FIG. 3

By considering the increment in Ω_4 as ξ passes through the vertex P or R we get

$$\Omega_4 = \frac{24\omega}{\pi} \log \frac{1+\zeta}{1-\zeta} - 12i\omega.$$

Also, if R is taken as the origin and RP as the x-axis, we get

$$\zeta = \wp(z) \quad [\wp^2(z) - 1]^{-\frac{1}{2}},$$

the angular points being $(2\omega_1, 0)$ (ω_1, ω_1), $(0, 0)$ and ω_1 being ω_1 the real half period.

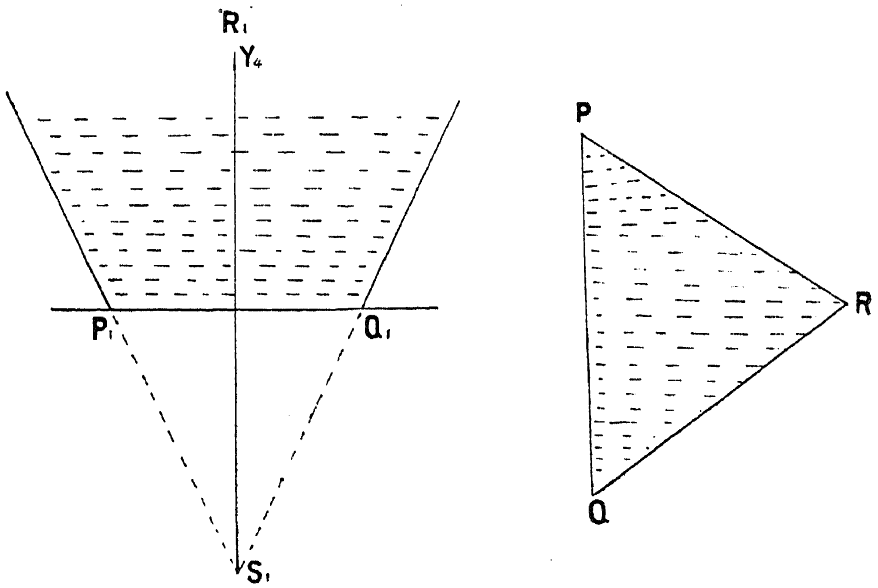
Equilateral triangle

Now

$$\frac{d\Omega_4}{d\zeta} = \frac{C}{\zeta^{\frac{4}{3}}(\zeta^2-1)^{\frac{1}{3}}},$$

$$\frac{dz}{d\zeta} = \frac{A}{[\zeta(\zeta^2-1)]^{\frac{2}{3}}},$$

and the areas that correspond are shown shaded in Fig. 4. In this case $P_1S_1Q_1$ is also an equilateral triangle.



Ω_4-plane z-plane

FIG. 4

If Q is taken as the origin and QP as the Y-axis,

$$\zeta = \frac{2i}{\wp'(z)},$$

the angular points being $(0, 2\omega_1/\sqrt{3})$, $(0, 0)$, $(\omega_1, \omega_1/\sqrt{3})$.

Thus we get in the usual notation for elliptic functions

$$\Omega_4 = -\frac{18\sqrt{3}\,\omega_1}{\eta_1}(1 - \tfrac{1}{3}i\sqrt{3})\,\zeta(z) - 24i\omega.$$

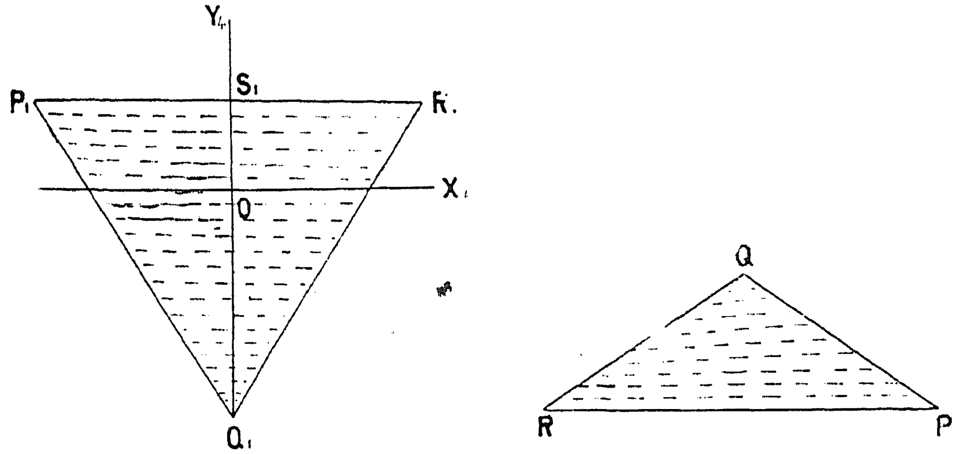
Isosceles triangle containing an angle of 120°

Now

$$\frac{d\Omega_4}{d\zeta} = \frac{C}{[\zeta\,(\zeta^2 - 1)]^{\frac{2}{3}}},$$

$$\frac{dz}{d\zeta} = \frac{A}{\zeta^{\frac{1}{3}}\,(\zeta^2 - 1)^{\frac{5}{6}}},$$

and the areas that correspond are shown shaded in Fig. 5. $P_1Q_1R_1$ is an equilateral triangle.



t-plane
$OS_1 = 12\omega$, $OQ_1 = 24\ \omega$

z-plane

FIG. 5

If R is taken as the origin and RP as the *x*–axis,

$$\left(\frac{\zeta^2}{\zeta^2 - 1}\right)^{\frac{1}{3}} = \wp\,(z),$$

the angular points are $(2\ \omega_1,\ 0)$, $(\omega_1,\ \omega_1/\sqrt{3})$, $(0,\ 0)$, and

$$\wp^3\,(z)\,\wp^3\,(C\,\Omega_4 + D) = 1,$$

the constants C and D being determined from Fig. 5.

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