ON THE SUFFICIENCY OF THE CONSISTENCY-EQUATIONS.

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When there are no body forces we know that for any stress-system to be possible in an isotropic solid body the stress-components must satisfy besides the three body-stress equations

$$\frac{\partial \widehat{xx}}{\partial x} + \frac{\partial \widehat{xy}}{\partial y} + \frac{\partial \widehat{xz}}{\partial z} = 0, \tag{1. 1}$$

$$\frac{\partial \widehat{xy}}{\partial x} + \frac{\partial \widehat{yy}}{\partial y} + \frac{\partial \widehat{yz}}{\partial z} = 0, \tag{1. 2}$$

$$\frac{\partial \widehat{xz}}{\partial x} + \frac{\partial \widehat{yz}}{\partial y} + \frac{\partial \widehat{zz}}{\partial z} = 0, \tag{1. 3}$$

the six consistency-equations given by

$$(1 + \eta) \nabla^2 \widehat{xx} + \frac{\partial^2 \theta}{\partial x^2} = 0, \tag{2. 1}$$

$$(1 + \eta) \nabla^2 \widehat{yy} + \frac{\partial^2 \theta}{\partial y^2} = 0, \tag{2. 2}$$

$$(1 + \eta) \nabla^2 \widehat{zz} + \frac{\partial^2 \theta}{\partial z^2} = 0, \tag{2. 3}$$

$$(1 + \eta) \nabla^2 \widehat{yz} + \frac{\partial^2 \theta}{\partial y \, \partial z} = 0, \tag{2. 4}$$

$$(1 + \eta) \nabla^2 \widehat{zx} + \frac{\partial^2 \theta}{\partial z \, \partial x} = 0, \tag{2, 5}$$

$$(1 + \eta) \nabla^2 \widehat{xy} + \frac{\partial^2 \theta}{\partial x \, \partial y} = 0, \tag{2. 6}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, η is the Poisson's ratio, and $\theta = \widehat{xx} + \widehat{yy} + \widehat{zz}$.

In many treatises* on Applied Mechanics a mistake has been made in obtaining a stress-system from the body-stress equations without reference to the consistency-equations. The stress-components thus determined do not correspond to any possible displacement in an isotropic body. In view

* See Love, *Mathematical Theory of Elasticity*, p. 22 (4th Ed.), where references to such books by Grashof and others will be found.

of this mistake the proving of the sufficiency of the consistency-equations whose necessity is well known becomes a matter of some interest.

We shall shew that, if a stress-system satisfies the nine equations given by (1) and (2), the corresponding components of displacement can be uniquely determined but for a rigid body displacement which, as we know, does not affect the state of strain in the body.

Differentiating (1. 1) with respect to x, (1. 2) with respect to y, (1. 3) with respect to z, and subtracting the last from the sum of the first two results, we have

$$2\frac{\partial^2 \widehat{xy}}{\partial x \partial y} = \frac{\partial^2 \widehat{zz}}{\partial z^2} - \left[\frac{\partial^2 \widehat{xx}}{\partial x^2} + \frac{\partial^2 \widehat{yy}}{\partial y^2}\right] \tag{3. 1}$$

If E is the Young's modulus and μ the rigidity, we can re-write (3. 1) as

$$\frac{1}{\mu}\frac{\partial^2 \widehat{xy}}{\partial x \partial y} = \frac{1+\eta}{\mathrm{E}}\frac{\partial^2 \theta}{\partial z^2} - \frac{1+\eta}{\mathrm{E}}\left[\frac{\partial^2 \widehat{xx}}{\partial x^2} + \frac{\partial^2 \widehat{yy}}{\partial y^2} + \frac{\partial^2}{\partial z^2}(\widehat{xx} + \widehat{yy})\right],$$

which, if we notice that

$$\nabla^2 \theta = 0,$$

becomes on using (2. 1), (2. 2), and (2. 3)

$$\begin{aligned}\frac{1}{\mu}\frac{\partial^2 \widehat{xy}}{\partial x \partial y} &= \frac{1+\eta}{\mathrm{E}}\left[\frac{\partial^2 \widehat{xx}}{\partial y^2} + \frac{\partial^2 \widehat{yy}}{\partial x^2}\right] - \frac{\eta}{\mathrm{E}}\left[\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right] \\ &= \frac{1}{\mathrm{E}}\frac{\partial^2}{\partial y^2}\left[\widehat{xx} - \eta(\widehat{yy} + \widehat{zz})\right] + \frac{1}{\mathrm{E}}\frac{\partial^2}{\partial x^2}\left[\widehat{yy} - \eta(\widehat{zz} + \widehat{xx})\right]\end{aligned} \tag{3. 2}$$

Hence,

$$\begin{aligned}\frac{1}{\mu}\widehat{xy} = \frac{1}{\mathrm{E}}\frac{\partial}{\partial y}\int\left[\widehat{xx} - \eta(\widehat{yy} + \widehat{zz})\right]dx &+ \frac{1}{\mathrm{E}}\frac{\partial}{\partial x}\int\left[\widehat{yy} - \eta(\widehat{zz} + \widehat{xx})\right]dy \\ &+ \frac{\partial f_1(y, z)}{\partial y} + \frac{\partial f_2(z, x)}{\partial x},\end{aligned} \tag{3. 3}$$

where f_1 and f_2 are respectively functions of y, z and z, x only.

Similarly we have

$$\begin{aligned}\frac{1}{\mu}\widehat{yz} = \frac{1}{\mathrm{E}}\frac{\partial}{\partial z}\int\left[\widehat{yy} - \eta(\widehat{zz} + \widehat{xx})\right]dy &+ \frac{1}{\mathrm{E}}\frac{\partial}{\partial y}\int\left[\widehat{zz} - \eta(\widehat{xx} + \widehat{yy})\right]dz \\ &+ \frac{\partial f_2'(z, x)}{\partial z} + \frac{\partial f_3(x, y)}{\partial y},\end{aligned} \tag{3. 4}$$

$$\begin{aligned}\frac{1}{\mu}\widehat{zx} = \frac{1}{\mathrm{E}}\frac{\partial}{\partial x}\int\left[\widehat{zz} - \eta(\widehat{xx} + \widehat{yy})\right]dz &+ \frac{1}{\mathrm{E}}\frac{\partial}{\partial z}\int\left[\widehat{xx} - \eta(\widehat{yy} + \widehat{zz})\right]dx \\ &+ \frac{\partial f_3'(x, y)}{\partial x} + \frac{\partial f_1'(y, z)}{\partial z},\end{aligned} \tag{3, 5}$$

where f_1', f_2', f_3' are functions which may be different from f_1, f_2, f_3.

Substituting the values of $\widehat{xy}$, $\widehat{yz}$, $\widehat{zx}$ from (3. 3), (3. 4), (3. 5) in (1) and (2. 4), (2. 5), (2. 6), we get

$$\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1'}{\partial z^2} = 0, \quad \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2'}{\partial z^2} = 0, \quad \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3'}{\partial x^2} = 0, \tag{4. 1}$$

$$\frac{\partial}{\partial y} \nabla^2 f_1 + \frac{\partial}{\partial x} \nabla^2 f_2 = 0, \quad \frac{\partial}{\partial z} \nabla^2 f_2' + \frac{\partial}{\partial y} \nabla^2 f_3 = 0, \quad \frac{\partial}{\partial x} \nabla^2 f_3' + \frac{\partial}{\partial z} \nabla^2 f_1' = 0 \tag{4. 2}$$

The equations in (4) are all consistent if $f_1 = f_1'$, $f_2 = f_2'$, $f_3 = f'_3$, and all the f's are harmonic. Let us assume that the general value of f''s is given by

$$f_r' = f_r + \phi_r, \quad r = 1, 2, 3.$$

Substituting these values in (4) we see that all the f''s in (3. 3), (3. 4) and (3. 5) can be replaced by the corresponding f's provided we add the term

$$A_1 x + A_0 \text{ and } B_1 y + B_0$$

to the right-hand side of (3. 4) and (3. 5), A's and B's being all constants. Putting

$$\psi_1 = f_1 + \frac{1}{2\mu} yz\,(B_1 - A_1) + \frac{1}{2\mu} B_0 z,$$

$$\psi_2 = f_2 + \frac{1}{2\mu} zx\,(A_1 - B_1) + \frac{1}{2\mu} A_0 z,$$

$$\psi_3 = f_3 + \frac{1}{2\mu} xy\,(A_1 + B_1) + \frac{1}{2\mu} (A_0 y + B_0 x),$$

we see that $\widehat{xy}$, $\widehat{yz}$, $\widehat{zx}$ now take the form

$$\frac{1}{\mu}\widehat{xy} = \frac{1}{E}\frac{\partial}{\partial y}\int\left[\widehat{xx} - \eta\,(\widehat{yy} + \widehat{zz})\right] dx + \frac{1}{E}\frac{\partial}{\partial x}\int\left[\widehat{yy} - \eta\,(\widehat{zz} + \widehat{xx})\right] dy + \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x}, \tag{5. 1}$$

$$\frac{1}{\mu}\widehat{yz} = \frac{1}{E}\frac{\partial}{\partial z}\int\left[\widehat{yy} - \eta\,(\widehat{zz} + \widehat{xx})\right] dy + \frac{1}{E}\frac{\partial}{\partial y}\int\left[\widehat{zz} - \eta\,(\widehat{xx} + \widehat{yy})\right] dz + \frac{\partial \psi_2}{\partial z} + \frac{\partial \psi_3}{\partial y}, \tag{5. 2}$$

$$\frac{1}{\mu}\widehat{xz} = \frac{1}{E}\frac{\partial}{\partial x}\int\left[\widehat{zz} - \eta\,(\widehat{xx} + \widehat{yy})\right] dz + \frac{1}{E}\frac{\partial}{\partial z}\int\left[\widehat{xx} - \eta\,(\widehat{yy} + \widehat{zz})\right] dx + \frac{\partial \psi_3}{\partial x} + \frac{\partial \psi_1}{\partial z}, \tag{5. 3}$$

where ψ's are all harmonic and are respectively functions of (y, z), (z, x) and (x, y).†

† That harmonic values for ψ's can be found we can see by operating (5) by ∇^2. The resulting equations are merely those given in (2. 4), (2. 5) and (2. 6).

Now the strain-components are connected with the stress-components by relations of the type

$$S_x = \frac{1}{E}\left[\widehat{xx} - \eta\,(\widehat{yy} + \widehat{zz})\right], \quad \sigma_{yz} = \frac{1}{\mu}\widehat{yz} \tag{6}$$

From (5) and (6) we see that quantities u, v, w, given by

$$u = \frac{1}{E}\int\left[\widehat{xx} - \eta\,(\widehat{yy} + \widehat{zz})\right] dx + \psi_1\,(y, z), \tag{7. 1}$$

$$v = \frac{1}{E}\int\left[\widehat{yy} - \eta\,(\widehat{zz} + \widehat{xx})\right] dy + \psi_2\,(z, x), \tag{7. 2}$$

$$w = \frac{1}{E}\int\left[\widehat{zz} - \eta\,(\widehat{xx} + \widehat{yy})\right] dz + \psi_3\,(x, y) \tag{7. 3}$$

can be determined which are connected with the strain-components by the well-known formulæ

$$S_x = \frac{\partial u}{\partial x}, \;\cdots\cdot\,, \quad \sigma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}.$$

We shall now shew that, if $\widehat{xx}$, $\widehat{yz}$, etc., are all given, the ψ's can be uniquely determined but for a rigid body displacement. If possible, let there be two solutions, one given by ψ's and the other by ψ''s. From (6) and (7) we see that

$$\frac{\partial}{\partial x}(\psi_1 - \psi_1') = 0, \quad \frac{\partial}{\partial y}(\psi_2 - \psi_2') = 0, \quad \frac{\partial}{\partial z}(\psi_3 - \psi_3') = 0, \tag{8. 1}$$

$$\frac{\partial}{\partial y}(\psi_3 - \psi_3') + \frac{\partial}{\partial z}(\psi_2 - \psi_2') = 0, \quad \frac{\partial}{\partial z}(\psi_1 - \psi_1') + \frac{\partial}{\partial x}(\psi_3 - \psi_3') = 0,$$

$$\frac{\partial}{\partial x}(\psi_2 - \psi_2') + \frac{\partial}{\partial y}(\psi_1 - \psi_1') = 0 \tag{8. 2}$$

If we differentiate the left-hand members of (8) with respect to x, y, z, we shall obtain eighteen linear equations connecting the eighteen second differential coefficients of $(\psi_1 - \psi_1')$, $(\psi_2 - \psi_2')$, $(\psi_3 - \psi_3')$, from which it follows that all these second differential coefficients vanish. Hence we must have

$$\psi_1 - \psi_1' = u_0 - w_3 y + w_2 z, \quad \psi_2 - \psi_2' = v_0 - w_1 z + w_3 x,$$
$$\psi_3 - \psi_3' = w_0 - w_2 x + w_1 y, \tag{9}$$

where u_0, v_0, w_0 and w's are all constants. From (9) we see that ψ's and ψ''s can only differ by a rigid body displacement, and hence, subject to this restriction, they can be uniquely determined.

The consistency-equations in stress-components are, therefore, both necessary and sufficient.